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On p-Summable Sequences in Locally Convex Spaces †

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p-summable sequences in $E[\tau]$

We follow the notation of [7,10]. In particular, if $E[\tau]$ is a locally convex space (in short l.c.s.), $\sigma(E,E^*), \mu(E,E^*)$ and $\beta(E,E^*)$ will denote, respectively, the weak, Mackey and strong topology corresponding to the dual pair $\langle E,E^*\rangle$. If U is a neighbourhood of 0 in $E[\tau]$, E_U denotes the Banach space associated to U and $\phi_U \colon E \to E_U$ denotes the corresponding quotient map.

Definition 1. Let $E[\tau]$ be an l.c.s. and let $1 \leq p < +\infty$. A sequence (x_n) in E is said to be p-summable if, for each τ-continuous seminorm q, $\sum_n q(x_n)^p < +\infty$.

The space of p-summable sequences shall be denoted by $l_p(E[\tau])$. Elementary properties of these spaces can be found in [6, 7, 9]. The conjugate number of p is the number p^* such that $p + p^* = pp^*$; if $p = 1$ we agree that $p^* = \infty$; and, in this case, l_{p^*} has always to be understood as c_0.

The choice $\tau = \sigma(E,E^*)$ is especially interesting. In this case we speak of weakly-p-summable sequences. The notion of weakly-p-summable sequence only depends upon the dual pair $\langle E,E^*\rangle$.

Lemma 2. *A sequence* $(x_n) \in l_p(E[\sigma(E,E^*)])$ *if and only if there is a*

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$\beta(E^*,E)$-neighborhood U of 0 such that

$$\sup_{x^*\in U}\sum_n |\langle x^*, x_n\rangle|^p < +\infty.$$

Proof. The sufficiency is evident. To see the necessity, define the applications

$$\begin{aligned} T_n\colon E^* &\longrightarrow l_p \\ x^* &\longrightarrow (\langle x^*, x_1\rangle, \langle x^*, x_2\rangle, \dots, \langle x^*, x_n\rangle, 0, \dots), \end{aligned}$$

which are clearly $\sigma(E^*,E)$-continuous. Since (x_n) is weakly-p-summable, the sequence $(T_n x^*)$ is bounded for all $x^*\in E^*$. Thus, being the strong topology barrelled, $\cap_{n\in\mathbb{N}} T_n^{-1}(B_{l_p})$ is a $\beta(E^*,E)$-neighborhood of 0. ∎

A weaker form of Lemma 2 is usually established replacing "β-neighborhood" by "τ- equicontinuous set" in Lemma 2 (see, e.g.,[6]). A simple characterization of weakly-p-summable sequences of $E[\tau]$ is given by the following

LEMMA 3. *Let $1\le p<+\infty$. Let (x_n) be a sequence in a locally convex space $E[\tau]$. The following statements are equivalent:*

1. *The sequence (x_n) belongs to $l_p(E[\sigma(E,E^*)])$;*
2. *For each $\varphi\in l_p^*$, the series $\sum_n \varphi_n x_n$ is τ-Cauchy;*
3. *For each τ-continuous seminorm q there exists a constant $C_q\ge 0$ such that*
$$q\Big(\sum_{n\in\Delta}\varphi_n x_n\Big) \le C_q\|\varphi\|_{l_{p^*}},$$
for any finite set $\Delta\subset\mathbb{N}$;
4. *For each τ-neighborhood U of 0, the sequence $(\phi_U(x_n))_n$ belongs to the space $l_p(E_U[\sigma(E_U,E_U^*)])$.*

Proof. $1\Rightarrow 3$. If q denotes a τ-continuous seminorm,

$$\begin{aligned} q\Big(\sum_{n\in\Delta}\varphi_n x_n\Big) &= \sup_{x^*\in U_q^\circ}\Big|\langle x^*, \sum_{n\in\Delta}\varphi_n x_n\rangle\Big| \\ &\le \sup_{x^*\in U_q^\circ}\Big(\sum_{n\in\Delta}|\langle x^*, x_n\rangle|^p\Big)^{1/p}\|\varphi\|_{l_{p^*}}, \\ &\le C_q\|\varphi\|_{l_{p^*}}. \end{aligned}$$

The equivalence 3 $\Leftrightarrow$ 4 is evident. That 3 $\Rightarrow$ 2 is very easy. That 2 $\Rightarrow$ 1 is clear: since 2 implies that, for all $x^* \in X^*$ and all $(\varphi_n) \in l_{p^*}$, the series $\sum_n \varphi_n \langle x^*, x_n\rangle$ converges, hence $(\langle x^*, x_n\rangle) \in l_p$. ∎

Remark 1. Therefore, a sequence (x_n) belongs to $l_p(E[\sigma(E, E^*)])$ if and only if the set

$$\mathrm{conv}_{\mathrm{p}^*}(x_n) = \left\{ \sum_{n\in\Delta} \theta_n x_n \colon \Delta \subset \mathbb{N}\,,\text{ finite and } (\theta_n) \in B_{l_p^*} \right\},$$

that shall be called the p^*-convex hull of (x_n), is bounded. Thus, all topologies having the same bounded sets as $\sigma(E, E^*)$ have the same weakly-p-summable sequences. Notice that $\mathrm{conv}_{\mathrm{p}^*}(x_n)$ is an absolutely convex set, hence its closure is the same no matter which compatible topology one uses. When $p = 1$, then we set

$$\mathrm{conv}_\infty(x_n) = \left\{ \sum_{n\in\Delta} \theta_n x_n \colon \Delta \subset \mathbb{N}\,,\text{ finite and } |(\theta_n)| \le 1 \right\}.$$

Remark 2. Since the notion of weakly-p-summable sequence depends only upon the dual pair, it is possible to replace "τ-Cauchy" in 2 by "$\sigma(E, E^*)$-Cauchy". Moreover, if $E[\tau]$ is quasi-complete, then condition 2 can be replaced by

2'. For each $\varphi \in l_{p^*}$, $\sum_n \varphi_n x_n$ is τ-convergent,

obtaining thus an identification

$$L(l_{p^*}, E[\tau]) = l_p(E[\sigma(E, E^*)]$$

in the form $x_n = Te_n$. The dependence upon the dual pair makes it perhaps more correct refer properties to the Mackey topology. Thus, if $E[\mu]$ is quasi-complete, one has

$$L(l_{p^*}, E[\mu]) = l_p(E[\sigma(E, E^*)] = L(l_{p^*}, E[\tau]).$$

Remark 3. A question of some interest by its own finds an answer in this context: recall that if (x_n) is a bounded sequence in a Banach space such that for all x^* of a dense subset of x^*, $\lim\langle x^*, x_n\rangle = 0$, then (x_n) is weakly null. Will this be still true for weakly-p-summable sequences? The answer is no: just consider, for instance, the space l_2; the sequence (e_n) is weakly-1-summable for the topology $\sigma(l_2, l_1)$, l_1 is dense in l_2, but the sequence (e_n) is not weakly-1-summable in l_2 with respect to the norm topology.

Remark 4. On a dual space E^*, weakly-p-summable and weakly*-p-summable sequences do not necessarily coincide. Weakly-p-summable sequences are obviously weakly*-p-summable; to obtain the converse one would need, following Lemma 2, that $\beta(E^{**}, E)_{|E} = \beta(E, E^*)$. Hence, the two notions coincide on duals of barrelled, quasi-complete or semi-reflexive spaces.

DEFINITION 4. We say that an application $T\colon E[\tau] \to F[\rho]$ is p-convergent, or briefly $T \in C_p(\tau, \rho)$, if it induces an application between the sequence spaces

$$T_{\text{ind}}\colon l_p(E[\tau]) \to c_0(F[\rho]).$$

When $p = +\infty$ then it is understood that the induced application acts from $c_0(E[\tau])$ into $c_0(F[\rho])$.

Obviously, when T is (τ, ρ)-continuous then it belongs to $C_p(\tau, \rho)$. Thus, it is only interesting to know what happens when T is not continuous. The main example is the action of the identity $\text{id}\colon E[\sigma(E, E^*)] \to E[\tau]$, when τ is another topology compatible with the duality $\langle E, E^* \rangle$.

DEFINITION 5. We shall say that $E[\tau] \in C_p$, $1 \le p \le +\infty$, or that $E[\tau]$ is a C_p-space, if the formal identity $\text{id}\colon E[\sigma(E, E^*)] \to E[\tau]$ is p-convergent.

Remark 5. The property C_∞ is usually called the Schur property.

It is clear that subspaces and arbitrary products of C_p-spaces are again C_p-spaces. Since bounded sets in locally convex sums are actually contained in convex hulls of the images of bounded sets in a finite number of factors, Remark 1 yields that a locally convex sum of C_p- spaces is a C_p-space. Quotients of C_p-spaces need not be C_p-spaces (quotients of l_1 for example). Recall that a property P is said to be a three-space property if whenever Y and X/Y have P then X has P.

PROPOSITION 6. *In the class of metrisable spaces, to be a C_p-space is a three-space property.*

Proof. Let (U_n) be a fundamental sequence of neighbourhoods of 0 for a metrisable space F; we assume that $U_m + U_m \subset U_{m-1}$. Let E and F/E be, respectively, a subspace and the corresponding quotient space, and assume that E and F/E are C_p- spaces. Let (x_n) be a weakly-p-summable sequence in F. It is clearly enough to obtain a convergent subsequence. If $Q\colon F \to F/E$ denotes the quotient map, $(Q(x_n))$ is a weakly-p-summable sequence in F/E

and, therefore, convergent to 0. This means that, for all m there exists $N(m)$ such that $Q(x_n) \subset Q(U_m)$. Thus, for each m there exists $N(m)$ so that if $n \geq N(m)$ it is possible to find $y_{n,m} \in E$ such that $x_n - y_{n,m} \in U_m$.

By diagonalisation, and relabelling the indexes if necessary, a new sequence (y_n) can be obtained such that $x_n - y_n \in 2^{-n} U_n$, for all n of some subsequence. The sequence (y_n) is as weakly-p-summable in E as (x_n) was in F: observe that the set $\mathrm{conv}_{\mathrm{p}^*}((y_n)_n)$ is bounded, since given a neighborhood U_m with $m \geq M$ and scalars θ_i with $|\theta_i| \leq 1$ one has

$$\sum_{M+1}^{M+N} \theta_i (x_i - y_i) \in \sum_{M+1}^{M+N} \frac{\theta_i}{2^i} U_i \subset \frac{1}{2^M} U_M \subset U_M.$$

Thus, the sequence (y_n) is convergent to 0, and so is (x_n). ∎

Remark 6. The ideal C_p of p-converging operators in Banach spaces, defined as those transforming weakly-p-summable sequences into convergent ones, was introduced and studied in [1, 2]. The ideal C_p is intermediate between the ideals $C_1 = U$ of unconditionally converging operators (those transforming weakly-1-summable sequences into summable sequences) and the ideal C_∞ of completely continuous or Dunford-Pettis operators (those transforming weakly convergent sequences into convergent sequences). Perhaps the most natural examples of such operators are the identities of the spaces l_p.

PROPOSITION 7. *$\mathrm{id}(l_p) \in C_r$ for all $r < p^*$.*

Proof. Let (x_n) be a weakly-r-summable sequence in l_p. If it is norm null, we have finished. If not, an application of the Bessaga-Pelczynski selection principle yields a subsequence equivalent to certain blocks of the canonical basis of l_p (recall here that two sequences (x_n) and (y_n) are called equivalent if there is an isomorphism between their closed spans $T\colon [x_n] \to [y_n]$ such that $Tx_n = y_n$). Since blocks of the canonical basis of l_p are weakly-p^*-summable, but not weakly-r-summable when $r < p^*$, the proof is complete. ∎

It is worth introducing here the Grothendieck space ideal defined by the ideals C_p: a l.c.s. $E[\tau] \in \mathrm{Groth}(C_p)$ if it can be written as a projective limit of C_p-operators between the associated Banach spaces. For instance, any projective limit of subspaces of l_p belongs, for all $r < p^*$, to $\mathrm{Groth}(C_r)$; any Schwartz space belongs, for all $1 \leq r \leq +\infty$, to $\mathrm{Groth}(C_r)$. It is also clear that if $E[\tau] \in \mathrm{Groth}(C_p)$ then $E[\tau] \in C_p$, and an interesting question is under which hypotheses the converse is also true. We shall show that this is not always the case:

EXAMPLES. Let λ_p be Köthe's example of a Fréchet-Montel non Schwartz echelon space of order p, $1 < p < +\infty$, (see [8, 10]). Since weakly convergent sequences are convergent in Fréchet- Montel spaces, $\lambda_p \in C_r$ for all r. Since λ_p is a projective limit of the spaces l_p, by Proposition 3 $\lambda_p \in \mathrm{Groth}(C_r)$ for all $r < p^*$. However, one can show (see the proof of Proposition 7) that every C_{p^*}-operator on l_p is compact; since λ_p is not a Schwartz space, we conclude that $\lambda_p \in \mathrm{Groth}(C_r)$ for all $r < p^*$, but not to $\mathrm{Groth}(C_{p^*})$. There exists a modification of this example [8]: a Fréchet-Montel non Schwartz echelon space or order 0, which yields a counterexample for p=1.

The Grothendieck space ideal $\mathrm{Groth}(C_p)$ has the same stability properties as the smallest class of C_p-spaces. Moreover,

PROPOSITION 8. $\mathrm{Groth}(C_p)$ *is a three-space class.*

Proof. Let E be a locally convex space, and let F be a closed subspace and E/F the corresponding quotient space with quotient map $Q\colon E \to E/F$. Assume that both E and E/F belong to $\mathrm{Groth}(C_p)$. The proof is easy proceeding this way: Let U be a neighborhood of 0 in E. Since E/F is in $\mathrm{Groth}(C_p)$, there exists some neighborhood $V \subset U$ such that the linking map

$$F_V \to F_U$$

is in C_p. Then, select another neighborhood $W \subset V$ such that the linking map

$$(E/F)_{Q(W)} \to (E/F)_{Q(V)}$$

is in C_p. Let now (x_n) be a weakly-p-summable sequence in E_W. Its image in $(E/F)_{Q(W)}$ is also weakly-p-summable and thus $\lim p_{Q(V)}(x_n + F) = 0$. Choosing elements (y_n) in F such that $p_V(x_n - y_n) \leq 2^{-n}$ the estimate $\sup p_W(\sum_n \theta_n x_n) < \infty$ (the supremum taken over all finite combinations with (θ_n) in the unit ball of l_{p^*}) yields that the sequence of the images of (y_n) is weakly-p-summable in E_V. Hence $\lim p_U(y_n) = 0$, and $\lim p_U(x_n) = 0$. ∎

PROPOSITION 9. *Let* $1 \leq p < +\infty$ *and let* $E[\tau]$ *be a quasi-complete l.c.s. Then* $E[\tau] \in C_p$ *if and only if all continuous operators from* l_{p^*} *into* $E[\tau]$ *are compact.*

Proof. Necessity (the case p=1 will be treated apart): Let $p > 1$ and let T be a continuous operator $T: l_{p^*} \to E[\tau]$. Given any bounded sequence (x_n) in l_{p^*}, a point x and a subsequence (x_m) exist such that $(x_m - x)$ is weakly-p-summable (as in the proof of Proposition 7). Therefore, the sequence $(T(x_m - x))_m$ is weakly-p-summable in $E[\tau]$, and thus it must be τ-convergent to zero. This means that $T(B_{l_{p^*}})$ is a relatively sequentially compact set, which in a quasi-complete space means relatively compact.

Case $p = 1$ ($l_{p^*} = c_0$): Let T be a continuous operator, $T: c_0 \to E[\tau]$. Since $E[\tau] \in C_1$, for every τ-neighborhood U of 0 the composition $\phi_U T$ belongs to $C_1(c_0, E_U)$ and must be compact. The operator T is necessarily compact.

Sufficiency (the case $p = 1$, $l_{p^*} = c_0$, is also covered by this argument): Let (x_n) be a weakly- p-summable sequence of E. Since $E[\tau]$ is quasi- complete, the spaces of operators $L(l_{p^*}, E[\tau])$ and $l_p(E[\sigma(E, E^*)])$ can be identified; thus, $x_n = Te_n$ for some continuous operator $T: l_{p^*} \to E[\tau]$ which, by hypothesis, is compact. So, (x_n) is τ- convergent. ∎

Remark 7. Regarding Remark 2 and Proposition 9, the Proposition 7 is, essentially, a reformulation of the result known as Pitt's theorem: *All operators $l_p \to l_q$ are compact if and only if $p > q$.*

The topology μ_p

It is clear that the formal identity id: $E[\sigma(E, E^*)] \to E[\tau]$ belongs to C_p for all p when $\tau = \sigma(E, E^*)$. If τ is replaced by a finer topology, the index p surely decreases. This observation originates the following definition.

Definition 10. Let $E[\tau]$ be a locally convex space and let $1 \leq p < +\infty$. We shall denote by μ_p the topology of the uniform convergence on the p^*-convex hull of the weakly*-p-summable sequences of E^*.

Recall that for quasi-complete spaces E and $1 \leq p < +\infty$ it makes no difference to consider the p^*-convex hull of weakly*-p-summable sequences of E^* (see Remark 4). For $p = +\infty$ the topology μ_∞ is the topology of the uniform convergence on the absolutely convex hull of weakly* null sequences in E^*. It is clear that $\sigma(E, E^*) \leq \mu_p(E, E^*) \leq \mu_q(E, E^*) \leq \mu_\infty(E, E^*)$, for $1 \leq p \leq q < +\infty$. In barrelled or, more generally, in spaces having quasi-complete duals (in some compatible topology) the topologies μ_p are compatible: The first assertion directly follows from Lemma 2. To see that

$\mu_p(E,E^*) \le \mu(E,E^*)$, one needs to verify that the p^*-convex hull of a weakly-p-summable sequence in E^* is $\sigma(E^*,E)$-relatively compact. If the space, say, $E^*[\mu]$ is quasi-complete, then the closure of that set is the continuous image of the unit ball of l_{p^*} as described in the Remark 2. It is, therefore, weakly compact. The proof for $p=1$ is a consequence of this. In barrelled spaces the topology μ_∞ is compatible since the absolutely convex hull of weakly* null sequences is a relatively weakly* compact set.

If $E[\tau]$ does not satisfy any of the hypotheses, then the topologies could be not compatible: consider, for instance, the space l_2 endowed with the Mackey topology $\mu(l_2,\varphi)$, where φ is the countable dimensional space. It turns out that $\mu(l_2,\varphi)$ is not compatible with $\mu_2(l_2,\varphi)$ which turns out to be the norm-topology. Also, $\mu_\infty(c_0,\varphi)$, the norm topology, is not compatible with $\mu(c_0,\varphi)$.

Two interesting properties of the μ_p-topologies are contained in the following proposition.

Proposition 11. *Let $E[\tau]$ be a barrelled locally convex space and let $1<p<+\infty$. The topology μ_p is the strongest locally convex compatible topology on E such that $E[\mu_p]$ can be written as a projective limit of subspaces of l_p. Therefore $E[\mu_p] \in \mathrm{Groth}(C_r)$ for $r<p^*$.*

Proof. A weakly-p- summable sequence (x_n^*) of E^* induces a μ_p-continuous application from $E \to l_p$ in the form: $x \to (\langle x_n^*, x\rangle)$: in fact, the Banach spaces associated to μ_p are isometric to subspaces of l_p since

$$\Big(\sum_n |\langle x_n^*, x\rangle|^p\Big)^{\frac{1}{p}} = \sup_{\theta\in B_{l_{p^*}}} \Big|\sum_n \theta_n \langle x_n^*, x\rangle\Big|$$
$$= \sup_{\theta\in B_{l_{p^*}}} \big|\langle \textstyle\sum_n \theta_n x_n^*, x\rangle\big| = p_{(\mathrm{conv}_{\mathrm{p}^*}(x_n^*))^\circ}(x).$$

On the other hand, if ρ is another compatible topology on E such that $E[\rho]$ can be written as a projective limit of subspaces of l_p then $\rho \le \mu_p$, as we now show: Let U be a ρ-neighborhood of 0. Using appropriate identification it is possible to assume that E_U is a subspace of l_p, which gives a quotient map $l_{p^*} \to E^*_{U^\circ}$. This implies that $U^\circ \subset \mathrm{conv}_{\mathrm{p}^*}(x_n^*)$ for some weakly-p-summable sequence of E^*. Therefore $\{\mathrm{conv}_{\mathrm{p}^*}(x_n^*)\}^\circ \subset U$ and our assertion is proved. ∎

Remark 8. The space $E[\mu_\infty]$ can be written as a projective limit of subspaces of c_0, which does not imply that $E[\mu_\infty] \in C_1$ since the norm topology coincides with $\mu(c_0,l_1)$ on c_0.

Remark 9. An interesting question is to give conditions to guarantee that μ_p is the strongest compatible locally convex topology having the property C_r for all $r < p^*$. Note that μ_2 in l_1 is not the strongest compatible locally convex topology such that weakly-r-summable sequences, $r < 2$, are norm-null. Such topology is precisely the Mackey topology, strictly stronger than μ_2 since many absolutely convex weakly compact sets of l_∞ are not of the kind $\mathrm{conv}_2(x_n^*)$. This also shows that the topology μ_1 does not admit the nice description of the Proposition 11: μ_1 is not the strongest locally convex topology on l_1 making the space a projective limit of l_1, since $\mu_1 \leq \mu_2 < \mu$.

Spaces of p-summable sequences

Observe that the space $l_p(E[\tau])$ carries a natural topology induced by τ, that we shall call N, defined by the seminorms

$$N_U((x_n)) = \|(p_U(x_n))\|_{l_p},$$

where U is a τ-neighborhood of 0. It is easy to verify that N has a fundamental system of associated Banach spaces isometric to $l_p(E_U)$, where U runs through a fundamental system of τ-neighbourhoods of 0.

There are some operators on the space $l_p(E[\tau])$ endowed with the N-topology worth of being called natural: either operators $l_p(E[\tau]) \to E[\tau]$ projecting a finite number of coordinates or operators $l_p(E[\tau]) \to l_p$ having the form $(x_n) \to \langle f_n, x_n\rangle_n$ for some sequence (f_n) contained in the polar of some τ-neighbourhood of 0.

We shall characterise weakly-p-summable and N-null sequences in terms of their continuous images by natural operators. This characterisations can be considered extension of classical results about weakly null and norm null sequences in vector sequence Banach spaces [10].

Proposition 12. *Let $1 \leq r \leq +\infty$. A bounded sequence of $l_p(E[\tau])[N]$ is weakly-r-summable if and only if every natural operator transforms it into a weakly-r-summable sequence.*

Proof. The only if is obvious. Firstly observe that, when E is a Banach space, then $l_p(E)[N]^*$ is isometric to $l_{p^*}(E^*)$. Moreover, if h^* is an element of $l_\infty(E^*)$, it induces a projection $H\colon l_p(E)[N] \to l_p$ by means of $H(f)(k) =$

$\langle h^*(k), f(k)\rangle$. Thus, given $g^* \in l_{p^*}(E^*)$ one has

$$\begin{aligned}\sum_n |\langle g, f_n\rangle|^r &= \sum_n \big|\sum_k \|g^*(k)\|_{E^*}\langle \|g^*(k)\|^{-1} g^*(k), f_n(k)\rangle\big|^r \\ &= \sum_n \big|\big\langle (\|g(k)\|)_{l_{p^*}}, (\|g(k)\|^{-1} g(k) f_n(k))_{l_p}\big\rangle\big|^r \\ &< +\infty.\end{aligned}$$

The proof for an arbitrary space $E[\tau]$ easily follows from the representation of $l_p(E[\tau])[N]$ as a projective limit of $l_p(E_U)$. ∎

Remark 10. The space l_p plays no especial role and can be replaced by any Banach sequence space having an unconditional basis (e_k). The vector sequence space $\lambda(X)$ is then defined as the space of sequences $(x_n) \subset X$ such that $\sum_n \|x_n\| e_n$ is in λ. The norm of $\lambda(X)$ is given by $\|(x_n)\| = \|\sum_n \|x_n\| e_n\|_\lambda$.

For projective tensor products with an l_p space we have an analogous result. Since natural operators include in this case the projections onto obvious copies of either l_p or $E[\tau]$, we state the result in these terms. Once more, we make the proof for Banach spaces.

PROPOSITION 13. *A bounded sequence of $l_p \hat{\otimes}_\pi E[\tau]$ (resp. $c_0 \hat{\otimes}_\pi E[\tau]$) is weakly-$r$-summable if and only if every continuous projection onto l_p (resp. c_0) and every continuous projection onto $E[\tau]$ transform it into a weakly-r-summable sequence.*

Proof. Let X be a Banach space. It is folklore that the space $l_p \hat{\otimes}_\pi X$ can be represented as a sequence space (see [3] for details) and so we will do. Let (a^n) be a non weakly-r-summable sequence in $l_p \hat{\otimes}_\pi X$ having weakly-r-summable projections. There must be some $\theta \in l_{r^*}$ and a sequence (N_i) of naturals such that, if I_i denotes the set $\{N_i + 1, ..., Ni + 1\}$ and $P_i \colon l_p \hat{\otimes}_\pi X \to l_p \hat{\otimes}_\pi X$ denotes the projection over the indices of I_i then

$$\pi\big(P_i(\sum_n \theta_n a^n)\big) > i + 1.$$

Find elements $z_i \in (l_p \hat{\otimes}_\pi X)^* = L(l_p, X^*)$ with $|z_i| \le 1$ such that

$$\pi\big(P_i(\sum_k \theta_n a^n)\big) = \big|\langle P_i\big(\sum_k \theta_n a^n\big), z_i\rangle\big|.$$

It is not difficult to see that if $Q_i\colon l_p \to l_p$ denotes the projection over the indices of I_i then $|\langle P_i(\sum_n \theta_n a^n), z_i Q_i\rangle| > i+1$. Arguments in the proof of [3, Thm. 1] show that the operator $B\colon l_p \hat{\otimes}_\pi X \to l_p$ defined by $B(y) = (\langle P_i y, z_i Q_i\rangle)$ is a continuous projection, which yields a contradiction with

$$i+1 \le \big|\langle P_i\big(\sum_n \theta_n a^n\big), z_i Q_i\rangle\big| = \big|\sum_n \theta_n \langle P_i a^n, z_i Q_i\rangle\big| \\ = \big|\sum_n \theta_n B(a^n)(i)\big| \le \big\|\sum_n \theta_n B(a^n)\big\|_{l_p} < \infty.$$

∎

Remark 11. Another characterisation of weakly-r-summable sequences in projective tensor products has been mentioned to us by J.A. López Molina. We omit the proof.

PROPOSITION 14. *A bounded sequence of $l_p \hat{\otimes}_\pi X$ is weakly-r-summable, $1 \le r < +\infty$, if and only if it is pointwise weakly-r-summable in X and every strongly p^*-summable sequence of x^* (in the sense of Cohen) transforms it into a weakly-r- summable sequence of l_1.*

Passing to convergent sequences, one has

PROPOSITION 15. *A bounded sequence of elements of $l_p(E[\tau])[N]$ is convergent to zero if and only if every natural operator transforms it into a sequence convergent to zero.*

Proof. We make once more the proof for Banach spaces. We prove that if all continuous projections of a sequence (f_n) converge to zero then

$$\lim_{N\to\infty} \sup_n \sum_{k=N}^{\infty} \|f_n(k)\|^p = 0,$$

which clearly implies the result.

Assume not. Then it is possible to find an $\varepsilon > 0$ and two sequences, (n_i) and (N_i), of integers such that

$$\sum_{k=N_i+1}^{N_{i+1}} \|f_{n_i}(k)\|^p > \varepsilon.$$

Find norm one elements $x^*(i,k) \in X^*$ such that $\langle x^*(i,k), f_{n_i}(k)\rangle = \|f_{n_i}(k)\|$, and form the element $(y^*(k)) \in l_\infty(X^*)$ defined by $y^*(k) = x^*(i,k)$, if $N_i <$

$k \leq N_{i+1}$, and $y^*(k) = 0$ otherwise. The sequence $(y^*(k))$ defines a natural operator $P\colon l_p(X) \to l_p$ by means of $P(f_n) = (\langle y^*(k), f_n(k)\rangle)_k$. Since the set $\{P(f_n)\}_n$ is relatively compact in l_p, one has:

$$\lim_{N\to\infty} \sup_n \sum_{k=N}^{\infty} |P(f_n)(k)|^p = 0,$$

which is in contradiction with

$$\sup_n \sum_{k=N}^{\infty} |P(f_n)(k)|^p \geq \sum_{k=N_i+1}^{N_{i+1}} \|f_{n_i}(k)\|^p > \varepsilon.$$

The proof for $c_0(X)$ is analogous. ∎

The same arguments of Proposition 13 used as in Proposition 15 yield:

PROPOSITION 16. *A bounded sequence of* $l_p\hat{\otimes}_\pi E[\tau]$ *(resp.* $c_0\hat{\otimes}_\pi E[\tau]$*) is convergent to zero if and only if every continuous projection onto* l_p *(resp.* c_0*) and every continuous projection onto* $E[\tau]$ *transforms it into a sequence convergent to zero.*

Propositions 12 and 15 together yield that weakly-r-summable sequences in $l_p(E[\tau])[N]$ are convergent if and only if the same happens in l_p and $E[\tau]$; thus

PROPOSITION 17. *The space* $l_p(E[\tau])[N]$ *is a* C_r*-space if and only if* $E[\tau]$ *and* l_p *are* C_r*-spaces (i.e., if and only if* $E[\tau]$ *is a* C_r*-space and* $r < p^*$*). In other words, if* $E[\tau]$ *is quasi-complete and every continuous linear operator from* l_r *into* E *and from* l_r *into* l_p *is compact, then every continuous linear operator from* l_r *into* $l_p(E[\sigma(E, E^*)])[N]$ *is compact.*

Analogously, it is not difficult to verify the following.

PROPOSITION 18. *The space* $l_p(E[\tau])[N]$ *belongs to* Groth(C_r) *if and only if* $E[\tau]$ *and* l_p *belong to* Groth(C_r) *(i.e., if and only if* $E[\tau]$ *is belongs to* Groth(C_r) *and* $r < p^*$*).*

Remark 12. For any compatible locally convex topology ω in E, a locally convex topology in $l_p(E[\sigma(E, E^*)])$ can be defined by means of the seminorms

$$q_U((x_n)) = \sup_{x^*\in U^\circ} \Big(\sum_n |\langle x^*, x_n\rangle|^p\Big)^{1/p},$$

where U is an ω-neighborhood of zero. The analogue of Proposition 18, however, could fail: if $l_2[weack]$ denotes l_2 endowed with the weak $\sigma(l_2, l_2)$ topology, consider the space $l_2(l_2[weak])$ endowed with the topology inherited from the norm topology of l_2; that topology makes $l_2(l_2[weak])$ a Banach space, usually denoted by $l_2^w(l_2)$. The identification of Remark 2 becomes isometry and $l_2^w(l_2) = L(l_2, l_2)$. Thus, one sees that although l_2 has property C_r for all $r < 2$, $l_2^w(l_2)$ has not even property C_1 since it contains copies of l_∞ (e.g., the subspace of diagonal operators). Of course, $l_2(l_2[weak])$ endowed with its own μ_2 topology is a C_r-space for all $r < 2$, but a description of the μ_p-topologies in the spaces $l_p^w(X)$ is, at this moment, unknown to us.

Propositions 14 and 16 together yield:

Proposition 19. *The space $l_p \hat{\otimes}_\pi E[\tau]$ is a C_r-space if and only if $E[\tau]$ and l_p are C_r-spaces (i.e., if and only if $E[\tau]$ is a C_r-space and $r < p^*$). In other words, if $E[\tau]$ is quasi-complete and every continuous linear operator from l_r into E and from l_r into l_p is compact, then every continuous linear operator from l_r into $l_p \hat{\otimes}_\pi E[\tau]$ is compact.*

Proposition 20. *The space $l_p \hat{\otimes}_\pi E[\tau]$ belongs to* $\mathrm{Groth}(C_r)$ *if and only if $E[\tau]$ and l_p belong to* $\mathrm{Groth}(C_r)$ *(i.e., if and only if $E[\tau]$ is belongs to* $\mathrm{Groth}(C_r)$ *and $r < p^*$).*

These results extend those in [3].

This paper is dedicated to the memory of Klaus Floret. We knew him since a long time, and will treasure the moments shared with him for longer yet.

We discussed a lot with him for years about the contents of this paper. We never became co-authors because although he liked some things here enclosed, he disliked others; and, as everybody who was fortunate enough to have meet him knows, he only accepted things done his own way.

We had nice times quarreling; but now that the story is over, we'll miss Klaus Floret.

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Reference added in proof:

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