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Metric Ellipses in Minkowski Planes

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An ellipse in $\mathbb{R}^2$ can be defined as the locus of points for which the sum of the Euclidean distances from the two foci is constant. In this paper we will look at the sets that are obtained by considering in the above definition distances induced by arbitrary norms.

A real two-dimensional normed linear space will be referred to as a Minkowski plane. For a Minkowski plane X with norm $\|\cdot\|$, let $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \leq 1\}$ be, respectively, the unit sphere and unit ball of X. For $x \in X$, $x \neq 0$, let $\hat{x} = x/\|x\|$. The line through $x, y \in X$, $x \neq y$, will be denoted by $\langle x, y \rangle$, and the closed segment from x to y by $[x, y]$. For $A \subset X$ we denote by coA the convex hull of A. A closed convex curve in X is the boundary of a convex body. The curve is said to be strictly convex if it contains no segment.

Definition 1. Let X be a Minkowski plane, $x, y \in X$, $x \neq y$, and $c \geq \|x - y\|$. The set

$$E(x, y, c) = \{z \in X : \|x - z\| + \|y - z\| = c\}$$

will be called the metric ellipse of foci x, y and size c.

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From the identity

$$E(x,y,c) = E\Big(\frac{x-y}{2}, \frac{y-x}{2}, c\Big) + \frac{x+y}{2}$$
$$= \frac{\|x-y\|}{2} E\Big(\widehat{x-y}, \widehat{y-x}, \frac{2c}{\|x-y\|}\Big) + \frac{x+y}{2}$$

it follows that to study the structure and affine properties of any metric ellipse one just needs to study the ellipses

$$E(x,c) = \{y \in X : \|x+y\| + \|x-y\| = c\},$$

with $x \in S(X)$ and $c \geq 2$.

Throughout the paper we shall often use the following lemma from [4].

Lemma 1. [4, Proposition 1] *For any distinct points x, y, z in a Minkowski plane X, the inequality $\|x-z\| \leq \|x-y\| + \|y-z\|$ becomes an equality if and only if $[u,v] \subset S(X)$, where $u = \widehat{x-y}$ and $v = \widehat{y-z}$.*

Let us now look at the structure of metric ellipses. We begin by considering the special case where $x \in S(X)$ and $c = 2$.

Theorem 1. *Let X be a Minkowski plane and $x \in S(X)$. Then*

(i) *If x is a extreme point of $S(X)$ then $E(x,2) = [-x,x]$.*

(ii) [5, Proposition 3.10] *If $[u,v] \subset S(X)$ is the maximal segment that contains x, then $E(x,2)$ is the parallelogram (including the interior) with sides parallel to $\langle 0,u\rangle$ and $\langle 0,v\rangle$, and that has x and $-x$ as opposite vertices.*

Proof. (i) It is obvious that for any $x \in S(X)$, $[-x,x] \subset E(x,2)$. Assume that there exists $y \notin [-x,x]$ such that $\|x+y\| + \|x-y\| = 2$. Then x, $u = (x+y)/\|x+y\|$ and $v = (x-y)/\|x-y\|$ are three different points of $S(X)$ such that

$$x = \frac{\|x+y\|}{2} u + \frac{\|x-y\|}{2} v,$$

which implies that $x \in [u,v] \subset S(X)$, i.e., x is not an extreme point of $S(X)$. ∎

To study the case $c > 2$, let us first define the set

$$D(x,c) = \{y \in X : \|y+x\| + \|y-x\| \leq c\}.$$

For every $x \in S(X)$, it is obvious that if $c < 2$ then $D(x,c) = \emptyset$. On the other hand, if $c \geq 2$ then $[-x,x] \subset D(x,c)$ and it is immediate to see that $D(x,c)$ is a bounded centrally symmetric closed convex set.

THEOREM 2. *Let X be a Minkowski plane. For every $x \in S(X)$ and $c > 2$, the metric ellipse $E(x,c)$ is a centrally symmetric closed convex curve.*

Proof. One just needs to show that $E(x,c)$ is the boundary of $D(x,c)$, i.e., $E(x,c) = D(x,c) \setminus \operatorname{Int} D(x,c)$. Let $A(x,c) = \{y \in X : \|x+y\| + \|x-y\| < c\}$. Then $E(x,c) = D(x,c) \setminus A(x,c)$. Since $A(x,c)$ is an open subset of $D(x,c)$ one has that $D(x,c) \setminus \operatorname{Int} D(x,c) \subset E(x,c)$. Now, let $y \in E(x,c)$. Since the convex function $f(t) = \|x+ty\| + \|x-ty\|$, $t \in \mathbb{R}$, satisfies $f(0) = 2 < c = f(1)$ one has that it is strictly increasing for $t \geq 1$. Any neighbourhood of y contains $(1+\delta)y$ for $\delta > 0$ small enough. Since $f(1+\delta) > f(1) = c$, then $(1+\delta)y \notin D(x,c)$. Therefore $E(x,c) \subset D(x,c) \setminus \operatorname{Int} D(x,c)$. ∎

Recall that a normed space X is said to be strictly convex if the unit sphere $S(X)$ has no segment; equivalently, if $\|x+y\| = \|x\| + \|y\|$ imply $x = \lambda y$, with $\lambda \geq 0$. The following theorem shows the relationship between the strict convexity of X and the strict convexity of the metric ellipses.

THEOREM 3. *A Minkowski plane X is strictly convex if and only if, for every $x \in S(X)$ and $c > 2$, $E(x,c)$ is a strictly convex curve.*

Proof. Let us assume that there exist $y_1, y_2 \in E(x,c)$, $y_1 \neq y_2$, such that $[y_1, y_2] \subset E(x,c)$. Then y_1 and y_2 are linearly independent and $(y_1+y_2)/2 \in E(x,c)$. Therefore,

$$\begin{aligned} 2c &= \|2x + y_1 + y_2\| + \|2x - y_1 - y_2\| \\ &\leq \|x + y_1\| + \|x + y_2\| + \|x - y_1\| + \|x - y_2\| = 2c, \end{aligned}$$

which implies $\|x+y_1\| + \|x+y_2\| = \|2x+y_1+y_2\|$ and $\|x-y_1\| + \|x-y_2\| = \|2x - y_1 - y_2\|$. Since y_1 and y_2 are linearly independent, one has that either $x+y_1$ and $x+y_2$ or $x-y_1$ and $x-y_2$ are linearly independent. Assuming the first case and taking $\bar{x} = x+y_1$, $\bar{y} = 0$, $\bar{z} = -x-y_2$, one gets from Lemma 1 that $[\widehat{x+y_1}, \widehat{x+y_2}] \subset S(X)$, i.e., X is not strictly convex.

Let us suppose now that X is not strictly convex and let $u, v \in S(X)$, $u \neq v$, be such that $[u,v] \subset S(X)$. Let $x = (u-v)/\|u-v\|$ and $\gamma > 1$. Then $c = 2(1+\gamma)/\|u-v\| > 2$. We shall show that $E(x,c)$ is not strictly convex. Let

$$y_1 = \frac{\gamma u + v}{\|u-v\|}, \quad y_2 = \frac{u + \gamma v}{\|u-v\|}.$$

Thus, $y_1 \neq y_2$. Let $y_\lambda = \lambda y_1 + (1-\lambda) y_2$, with $0 \leq \lambda \leq 1$, and let

$$\alpha_\lambda = \frac{\lambda(\gamma-1)}{1+\gamma}, \quad \beta_\lambda = \frac{\lambda(\gamma-1)+2}{1+\gamma}.$$

Then

$$y_\lambda - x = \frac{1+\gamma}{\|u-v\|}\big(\alpha_\lambda u + (1-\alpha_\lambda)v\big), \quad y_\lambda + x = \frac{1+\gamma}{\|u-v\|}\big(\beta_\lambda u + (1-\beta_\lambda)v\big),$$

with $0 \leq \alpha_\lambda < 1$ and $0 < \beta_\lambda \leq 1$. Therefore,

$$\|y_\lambda - x\| = \|y_\lambda + x\| = \frac{1+\gamma}{\|u-v\|},$$

which implies that $y_\lambda \in E(x,c)$ and then $[y_1, y_2] \subset E(x,c)$. ∎

Theorem 2 shows that if X is a Minkowski plane then, for every $x \in S(X)$ and $c > 2$, the metric ellipse $E(x,c)$ is a centrally symmetric closed convex curve. Thus, there arises naturally the question of whether any centrally symmetric closed convex curve in the plane can be a metric ellipse for some Minkowski plane. The following theorem shows that the answer in general is negative.

THEOREM 4. *Let X be a Minkowski plane. There is no $x \in S(X)$ and $c > 2$ such that $E(x,c)$ is a parallelogram.*

Proof. We shall prove the theorem by reductio ad absurdum. Let $x \in S(X)$ and $c > 2$ be such that $E(x,c)$ is the parallelogram of vertices $\pm y$, $\pm z$. As in the first part of the proof of Theorem 3 one gets that

$$[\widehat{x \pm y}, \widehat{x \pm z}] \subset S(X). \tag{1}$$

Now, one can assume without loss of generality that

$$x = \rho\big(\alpha y + (1-\alpha)z\big)$$

with $0 \leq \alpha \leq 1/2$ and $\rho > 0$. Since $\alpha y + (1-\alpha)z \in E(x,c)$, from the identities

$$(\rho+1)x = \rho\big(x + \alpha y + (1-\alpha)z\big), \quad (\rho-1)x = \rho\big(x - \alpha y - (1-\alpha)z\big),$$

it follows that

$$\rho + 1 + |\rho - 1| = \rho\big(\|x + \alpha y + (1-\alpha)z\| + \|x - \alpha y - (1-\alpha)z\|\big) = \rho c > 2\rho,$$

which implies $\rho < 1$.

Assume that $\alpha \neq 0$. Then, taking $\bar{x} = 2\rho\alpha(y - x)$, $\bar{y} = 0$, $\bar{z} = (1 + \rho - 2\rho\alpha)(x - z)$, and $\tilde{x} = (\rho - 1)(x + z)$, $\tilde{y} = 0$, $\tilde{z} = 2\rho\alpha(x - y)$, it follows from (1) and Lemma 1 that

$$\|2\rho\alpha(x-y)\| + \|(1+\rho-2\rho\alpha)(x-z)\| = \|(1-\rho)(x+z)\|$$

and

$$\|(\rho-1)(x+z)\|+\|2\rho\alpha(x-y)\|=\|(1+\rho-2\rho\alpha)(x-z)\|.$$

By summing the above identities one gets that $x=y$, which is impossible because $x\notin E(x,c)$. Therefore $\alpha=0$, and then $x=\rho z$.

Since $(x-z)/\|x-z\|=-x$, taking $\bar{x}=0$, $\bar{y}=-x$, $\bar{z}=y$, and $\tilde{x}=0$, $\tilde{y}=-x$, $\tilde{z}=-y$, one gets again from (1) and Lemma 1 that

$$\|y\|=1+\|x+y\|=1+\|x-y\|,$$

from which follows that

$$2+\|x+y\|+\|x-y\|=2\|y\|\leq\|x+y\|+\|x-y\|,$$

which is impossible. ∎

J. Lindenstrauss [3] introduced the modulus of smoothness $\rho_E(t)$ of a normed space E as the function

$$\rho_E(t)=\frac{1}{2}\sup\{\|x+ty\|+\|x-ty\|-2:\ \|x\|=\|y\|=1\},\quad t\geq 0.$$

A normed space E is uniformly smooth if and only if $\lim_{t\to 0}\rho_E(t)/t=0$. Other known properties (see, e.g., [3], [6]) are the following:

(i) $0\leq\rho_E(t)\leq t$, for $t\geq 0$;

(ii) $\rho_E(t)$ is a monotone increasing convex function;

(iii) $\rho_E(t)=\frac{1}{2}\sup\{\|x+ty\|+\|x-ty\|-2:\ \|x\|\leq 1,\ \|y\|\leq 1\}$, for $t\geq 0$.

Theorem 5 shows the relationship between the "size" of metric ellipses and the modulus of smoothness of a Minkowski plane X.

THEOREM 5. *Let X be a Minkowski plane and let*

$$c(t)=\sup_{x\in S(X)}\inf\{c:\ tB(X)\subset D(x,c)\},\quad t\geq 0.$$

Then

$$\rho_X(t)=\frac{c(t)}{2}-1\quad\textit{for all}\ \ t\geq 0. \tag{2}$$

Proof. It is convenient to recall that if $c<2$, then $D(x,c)=\emptyset$. From (iii) it follows that one only needs to prove that for every $t\geq 0$ and $x\in S(X)$ the identity

$$\sup\{\|x+ty\|+\|x-ty\|:\ y\in B(X)\}=\inf\{c:\ tB(X)\subset D(x,c)\}$$

holds. Taking $\bar{c} = \sup\{\|x+ty\| + \|x-ty\| : y \in B(X)\}$ one has that if $y \in B(X)$ then $\|x+ty\| + \|x-ty\| \leq \bar{c}$, which implies that $tB(X) \subset D(x, \bar{c})$. Therefore $\bar{c} \geq \inf\{c : tB(X) \subset D(x,c)\}$. On the other hand, if $c \geq 2$ is such that $tB(X) \subset D(x,c)$, then $\|x+ty\|+\|x-ty\| \leq c$ for every $y \in B(X)$, which implies $\bar{c} \leq c$. Thus one has $\bar{c} \leq \inf\{c : tB(X) \subset D(x,c)\}$. ∎

The identity (2) was obtained by Baronti, Casini and Papini [2, (2.3)], for $t = 1$. From (i), (ii), and (2) it follows that the modulus $c(t)$ is a monotone increasing convex function such that $2 \leq c(t) \leq 2(t+1)$ for all $t \geq 0$.

If H is an inner product space then $\rho_H(t) = \sqrt{1+t^2} - 1$ for every $t \geq 0$. Lindenstrauss [3] proved that for every normed space E, $\rho_E(t) \geq \rho_H(t)$, $(t \geq 0)$, with the equality for every $t \geq 0$ if and only if E is an inner product space. In [1] this result was improved by showing that if $\rho_E(t) \leq \rho_H(t)$ for some $t > 0$ not belonging to the countable and dense subset of $\mathbb{R}^+$

$$T = \{\tan \frac{k\pi}{2n} : n = 2, 3, \ldots,\ k = 1, 2, \ldots, n-1\},$$

then E is an inner product space. The above allows us to obtain the following theorem from the identity (2).

THEOREM 6. *Let X be a Minkowski plane. Then*

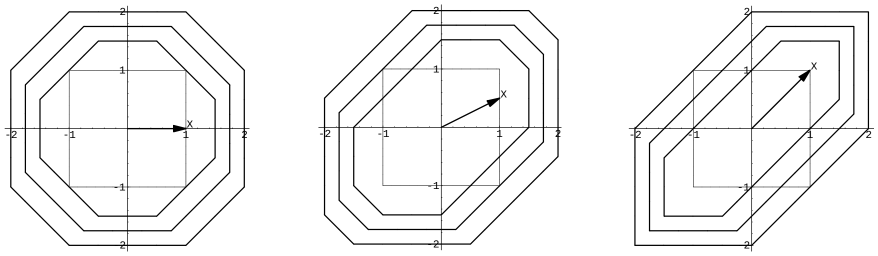
(i) $c(t) \geq 2\sqrt{1+t^2}$ *for every* $t \geq 0$.

(ii) *If* $c(t) \leq 2\sqrt{1+t^2}$ *for some* $t > 0$, $t \notin T$, *then the norm in X is induced by an inner product, i.e., the unit sphere $S(X)$ is an ellipse.*

However, if $t \in T$ then the identity $c(t) = 2\sqrt{1+t^2}$ does not force $S(X)$ to be an ellipse. For example, if $S(X)$ is a $4n$-gon then $c(t) = 2\sqrt{1+t^2}$ for $t = \tan \frac{k\pi}{2n}$, with $k = 1, 2, \ldots, n-1$.

If the norm in X is induced by an inner product, then for any $x \in S(X)$ and $c > 2$ the ellipse $E(x,c)$ is the set of points $y = \alpha x + \beta x^{\perp}$, where $x^{\perp} \in S(X)$, $x^{\perp} \perp x$ and $\sqrt{(\alpha-1)^2 + \beta^2} + \sqrt{(\alpha+1)^2+\beta^2} = c$. Theorem 7 shows that two-dimensional inner product spaces are the only Minkowski planes with such ellipses.

LEMMA 2. *A Minkowski plane X is an inner product space if and only if there exists $e_1, e_2 \in S(X)$ such that $\|e_1 + te_2\|^2 = 1 + t^2$ for every $t \in \mathbb{R}$.*



$x = (1, 0)$ $\qquad$ $x = (1, \frac{1}{2})$ $\qquad$ $x = (1, 1)$

Figure 1: Metric ellipses $E(x, c)$ for $X = (\mathbb{R}^2, \| \|_\infty)$ and $c = 3$, 3.5, 4.

Proof. If X is an inner product space and $e_1, e_2 \in S(X)$ are such that $e_1 \perp e_2$ then $\|e_1 + te_2\|^2 = 1 + t^2$ for every $t \in \mathbb{R}$. Conversely, assume that $e_1, e_2 \in S(X)$ are such that $\|e_1 + te_2\|^2 = 1 + t^2$ for every $t \in \mathbb{R}$. Then e_1 and e_2 are linearly independent, and for every $s, t \in \mathbb{R}$, $\|se_1 + te_2\|^2 = s^2 + t^2$. Let $x, y \in X$ and let $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$, be such that $x = \alpha_1 e_1 + \beta_1 e_2$ and $y = \alpha_2 e_1 + \beta_2 e_2$. Then $\|x + y\|^2 + \|x - y\|^2 = (\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2 + (\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 = 2(\alpha_1^2 + \beta_1^2) + 2(\alpha_2^2 + \beta_2^2) = 2(\|x\|^2 + \|y\|^2)$, i.e., the parallelogram equality holds. ∎

Theorem 7. *Let X be a Minkowski plane. Then X is an inner product space if and only if there exist $x_0, y_0 \in S(X)$ such that for any $c > 2$*

$$E(x_0, c) = \left\{\alpha x_0 + \beta y_0 : \sqrt{(\alpha - 1)^2 + \beta^2} + \sqrt{(\alpha + 1)^2 + \beta^2} = c\right\}.$$

Proof. One just has to prove the sufficiency. Let $x_0, y_0 \in S(X)$ satisfying the hypothesis. For every $t \in \mathbb{R}$, $t \neq 0$,

$$x_0 + 2ty_0 \in \left\{\alpha x_0 + \beta y_0 : \sqrt{(\alpha - 1)^2 + \beta^2} + \sqrt{(\alpha + 1)^2 + \beta^2} = c\right\},$$

where $c = 2\big(|t| + \sqrt{1 + t^2}\big) > 2$. Thus, $x_0 + 2ty_0 \in E(x_0, c)$, which implies

$$\|x_0 + 2ty_0 + x_0\| + \|x_0 + 2ty_0 - x_0\| = 2\big(|t| + \sqrt{1 + t^2}\big),$$

and then $\|x_0 + ty_0\| = \sqrt{1 + t^2}$. Lemma 2 completes the proof. ∎

Example 1. Let $X = (\mathbb{R}^2, \| \cdot \|_\infty)$ and $c > 2$. For $0 \leq \gamma \leq 1$, $E((1, \gamma), c)$ is the convex polygon of vertices $\pm(\frac{c}{2}, \frac{c}{2} + \gamma - 1)$, $\pm(\frac{c}{2}, 1 + \gamma - \frac{c}{2})$, $\pm(1 + \gamma - \frac{c}{2}, \frac{c}{2})$, $\pm(\frac{c}{2} + \gamma - 1, \frac{c}{2})$. (See Figure 1.)

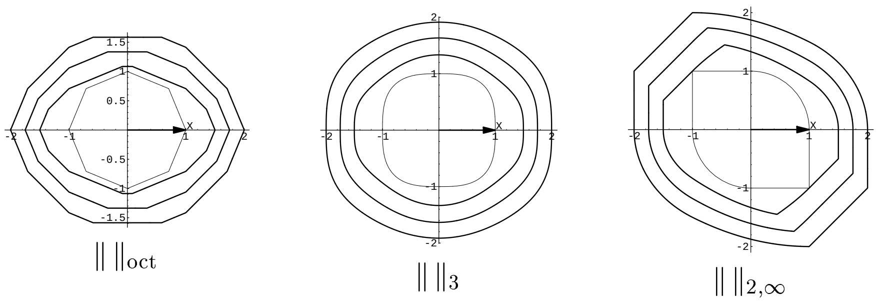


Figure 2: Metric ellipses $E((1,0),c)$ for $c = 3$, 3.5, 4 and several norms.

Example 2. Figure 2 shows the ellipses $E(x,c)$ with $x = (1,0)$, $c = 3,\ 3.5,\ 4$, and the norms

$$\|(x_1,x_2)\|_{\text{oct}} = \max\{|x_1| + (\sqrt{2}-1)|x_2|, |x_2| + (\sqrt{2}-1)|x_1|\},$$

$$\|(x_1,x_2)\|_3 = (|x_1|^3 + |x_2|^3)^{1/3},$$

$$\|(x_1,x_2)\|_{2,\infty} = \begin{cases} (x_1^2 + x_2^2)^{1/2} & \text{i}f\ x_1x_2 \geq 0, \\ \max\{|x_1|, |x_2|\} & \text{i}f\ x_1x_2 \leq 0. \end{cases}$$

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