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Complex Banach Spaces with Valdivia Dual Unit Ball *

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1. Introduction

Valdivia compacta and associated real Banach spaces were studied in [12, 13, 3] and later by the author [6, 7, 8]. In the present paper we introduce associated classes of complex Banach spaces and discuss their properties in relation to those of real spaces. We start by defining the relevant classes of compact spaces and Banach spaces.

If Γ is any set we set

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

This space is considered with the pointwise convergence topology inherited from $\mathbb{R}^\Gamma$ and is called a Σ-*product of real lines*. Compact spaces which are homeomorphic to a subset of $\Sigma(\Gamma)$ are called *Corson*.

A compact space K is *Valdivia* if it is, for a set Γ, homeomorphic to a subset K' of $\mathbb{R}^\Gamma$ with $K' \cap \Sigma(\Gamma)$ dense in K'. A subset $A \subset K$ is called a Σ-*subset* of Γ if there is a homeomorphic injection $h : K \to \mathbb{R}^\Gamma$ with $A = h^{-1}(\Sigma(\Gamma))$. Hence a compact space is Valdivia if and only if it admits a dense Σ-subset.

Valdivia compact spaces are a natural generalization of Corson compact spaces. For example, the ordinal interval $[0, \omega_1]$ and the Tychonoff cube $[0, 1]^I$ for I uncountable are Valdivia compact spaces which are not Corson.

We continue by the associated classes of Banach spaces. By a Banach space we mean either a real or a complex Banach space, unless one of these possibilities is explicitly chosen. If X is a Banach space and $A \subset X$, then $\operatorname{span} A$

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denotes the set of all linear combinations (complex ones if X is complex, real ones if X is real) of elements of A.

Let X be a Banach space. A subspace $S \subset X^*$ is a Σ-*subspace* of X^* if there is $M \subset X$ with $\operatorname{span} M$ dense in X such that

$$S = \{\xi \in X^* : \{x \in M : \xi(x) \neq 0\} \text{ is countable}\}.$$

If X^* is a Σ-subspace of itself, the space X is called *weakly Lindelöf determined* (shortly *WLD*). A Banach space X is called *Plichko* (1-*Plichko*) if X^* admits a norming (1-norming, respectively) Σ-subspace. Recall that $S \subset X^*$ is *norming* if

$$|x| = \sup\{|\xi(x)| : \xi \in S \cap B_{X^*}\}, \qquad x \in X,$$

defines an equivalent norm on X. If this norm is equal to the original one, S is called 1-*norming*. Note that a subspace $S \subset X^*$ is 1-norming if and only if $S \cap B_{X^*}$ is weak* dense in B_{X^*}.

These classes have been previously defined for real spaces (see [2, 7]). It is easy to check that our definitions are in the real case equivalent to the original ones. For basic properties of Valdivia compacta and related real Banach spaces we refer to [7]. In Proposition 2.3 below we recall only few properties which will be constantly used.

If X is a complex Banach space, we denote by X_R the space X considered as a real space. For any topological space K we denote by $C(K, \mathbb{R})$ the space of real-valued continuous functions on K and by $C(K, \mathbb{C})$ the space of all complex-valued continuous functions on K. If K is compact, we consider on these spaces the supremum norm making $C(K, \mathbb{R})$ a real Banach space and $C(K, \mathbb{C})$ a complex Banach space.

In the present paper we study namely complex 1-Plichko spaces. The main results are contained in Section 3. They include 'complex analogues' of some known results on real spaces, together with formulations which work in both cases. Another part is the study of the relationship of the complex and real spaces. This relationship is not obvious as, for example, it is not clear whether a complex Banach space X is 1-Plichko whenever X_R is 1-Plichko (see Section 4 for reformulations and related problems). We show that the answer is positive in some special cases – for the space $C(K, \mathbb{C})$ and for spaces X such that X^* is the weak* closed convex hull of its weak* G_δ points.

On the other hand, it is easy to show that a complex Banach space X is WLD if and only if X_R is WLD and that for any compact space K the space $C(K, \mathbb{C})$ is WLD if and only if $C(K, \mathbb{R})$ is WLD. Similar equivalences hold for several smaller classes of Banach spaces (separable, weakly compactly generated, subspaces of weakly compactly generated spaces, weakly K-analytic, weakly countably determined). The definitions, basic properties and applications of these classes can be found for example in the book [4]. A detailed list of characterizations of these classes, together with the results on duality of these classes of Banach spaces and the respective classes of compact spaces which are scattered in the literature, is contained in an unpublished note [9].

2. Preliminaries

We will need the following easy proposition on the relationship of X and X_R.

Proposition 2.1. *Let X be a complex Banach space.*

- *The identity X onto X_R is a real-linear, isometric and weak-to-weak homeomorphic map.*
- *Define the mapping $\phi : X^* \to X_R^*$ by $\phi(\xi)(x) = \operatorname{Re} \xi(x)$, $x \in X$, $\xi \in X^*$. Then ϕ is a real-linear, isometric and weak*-to-weak* homeomorphic map. Moreover, for each $\xi \in X^*$ and $x \in X$ we have $\xi(x) = \phi(\xi)(x) - i\phi(\xi)(ix)$.*

The 'isometric part' of the first point is obvious. The 'isometric part' of the second assertion, together with the representation formula, is a standard part of the proof of complex Hahn-Banach theorem (see e.g. [5, p. 28-29]). The 'weak' and 'weak*' parts are easy consequences of the representation formula.

While any complex Banach space can be considered as a real one, any real Banach space has a natural complex counterpart - the complexification (see e.g. [5, p. 29]). If Y is a real linear space, its complexification Y_C is the complex linear space of formal expressions $x + iy$, $x, y \in Y$, with the obvious addition and scalar multiplication. If Y is, moreover, a normed space, one can define a norm on Y_C by the formula

$$\|x + iy\| = \max\{\|\alpha x + \beta y\| : \alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \leq 1\}$$

(note that the maximum is attained). It is easy to verify that this is really a norm on the complex space Y_C. Further, clearly $(Y_C)_R$ is isomorphic to $Y \times Y$.

This definition of complexification is quite natural as it can be (due to the following proposition) described alternatively: If Y is a real Banach space, we have a natural isometric inclusion of Y into $C((B_{Y^*}, w^*), \mathbb{R})$. Then the norm of Y_C is the subspace norm on $Y + iY$ inherited from $C((B_{Y^*}, w^*), \mathbb{C})$. Note however, that the complexification of some classical real spaces does not give their usual complex version (e.g. the complexification of the real space ℓ_1 is not isometric to the complex ℓ_1).

PROPOSITION 2.2. *Let K be a compact space. Then $C(K,\mathbb{C})$ is isometric to $C(K,\mathbb{R})_C$.*

Proof. The mapping $f \mapsto \operatorname{Re} f + i \operatorname{Im} f$ is clearly a linear bijection of $C(K,\mathbb{C})$ onto $C(K,\mathbb{R})_C$. If $x \in K$ is arbitrary, then

$$\begin{aligned} |f(x)| &= \sqrt{(\operatorname{Re} f(x))^2 + (\operatorname{Im} f(x))^2} \\ &= \max\{|\alpha \operatorname{Re} f(x) + \beta \operatorname{Im} f(x)| : \alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \le 1\} \\ &\le \| \operatorname{Re} f + i \operatorname{Im} f\|, \end{aligned}$$

and hence $\|f\| \le \| \operatorname{Re} f + i \operatorname{Im} f\|$. The inverse inequality follows immediately from the following one:

$$\|\alpha \operatorname{Re} f(x) + \beta \operatorname{Im} f(x)\| \le \sqrt{\alpha^2 + \beta^2} \cdot |f(x)|.$$

∎

In the following proposition we collect some basic properties of Valdivia compacta which are proved in [7, Chapter 1].

PROPOSITION 2.3. *Let K be a compact space.*

(1) *Any Σ-subset of K is countably compact and Fréchet-Urysohn* (*i.e., closures are described by limits of converging sequences*).
(2) *If A is a Σ-subset of K and $C \subset A$ is countable, then $\overline{C} \subset A$.*
(3) *If $A \subset K$ is a dense countably compact set, then $G \cap A$ is dense in G for each G_δ set $G \subset K$.*
(4) *Let A, B be two subsets of K which are countably compact and Fréchet-Urysohn. If $A \cap B$ is dense in K, then $A = B$.*
(5) *If $A \subset K$ is a dense Σ-subset of K, then K is the Čech-Stone compactification of A.*

A powerful tool to deal with Valdivia compacta is provided by their characterization using the notion of primarily Lindelöf space. Let us recall the definition. If Γ is any set, L_Γ denotes the one-point lindelöfication of the discrete space Γ. I.e., $L_\Gamma = \Gamma \cup \{\infty\}$ where points of Γ are isolated and neighborhoods of ∞ are complements of countable subsets of Γ. A topological space is *primarily Lindelöf* if it is a continuous image of a closed subset of $L_\Gamma^{\mathbb{N}}$ for a set Γ.

Note that any primarily Lindelöf space is Lindelöf and the class of primarily Lindelöf spaces is stable to closed subsets, continuous images, countable products and countable unions (see [1, Section IV.3]).

3. Valdivia compacta and 1-Plichko Banach spaces

We start this section by a characterization of Valdivia compacta or, more precisely, of dense Σ-subsets generalizing Pol's theorem from [10] (see [1, Section IV.3]). By $\tau_p(A)$ we denote the topology of pointwise convergence on A.

Theorem 3.1. *Let K be a compact space and $A \subset K$ a dense subset. The following assertions are equivalent.*

(1) *A is a Σ-subset of K.*
(2) *A is countably compact and $(C(K,\mathbb{R}), \tau_p(A))$ is primarily Lindelöf.*
(3) *A is countably compact and $(C(K,\mathbb{C}), \tau_p(A))$ is primarily Lindelöf.*

Proof. The equivalence 1$\Longleftrightarrow$2 is proved in [6, Theorem 2.1] (see also [7, Theorem 2.5]). To show 2$\Longleftrightarrow$3, just note that $(C(K,\mathbb{C}), \tau_p(A))$ is canonically homeomorphic to $(C(K,\mathbb{R}), \tau_p(A))^2$. ∎

A Banach space counterpart of the previous theorem is the following one. Recall that a subset A of a Banach space is *absolutely convex* if $\sum_{i=1}^n \lambda_i x_i \in A$ whenever $x_1, \dots, x_n \in A$ and $\lambda_1, \dots, \lambda_n$ are scalars satisfying $\sum_{i=1}^n |\lambda_i| = 1$.

Theorem 3.2. *Let X be a Banach space and $A \subset B_{X^*}$ be a weak* dense subset. The following assertions are equivalent.*

(1) *There is a Σ-subspace of X^* with $A = S \cap B_{X^*}$. (Note that S is necessarily 1-norming.)*
(2) *A is an absolutely convex Σ-subset of (B_{X^*}, w^*).*
(3) *A is weak* countably compact and $(X, \sigma(X,A))$ is primarily Lindelöf.*

The topology $\sigma(X,A)$ is the weakest topology on X making all functionals from A continuous.

Proof. For real spaces the theorem is proved in [6, Theorem 2.3] (see also [7, Theorem 2.7]). One needs only to observe that an absolutely convex subset of a real space is just a convex symmetric set.

Let us prove the theorem for X complex.

The implication $1 \Longrightarrow 2$ is obvious.

For the proof of $2 \Longrightarrow 3$ we follow the proof of the respective implication of [7, Theorem 2.7]. Consider the canonical embedding $e : X \to C((B_{X^*}, w^*), \mathbb{C})$ defined by $e(x)(\xi) = \xi(x)$. As $(C((B_{X^*}, w^*), \mathbb{C}), \tau_p(A))$ is primarily Lindelöf by Theorem 3.1 and e is $\sigma(X, A) \to \tau_p(A)$ homeomorphism, it is enough to show that $e(X)$ is $\tau_p(A)$-closed in $C((B_{X^*}, w^*), \mathbb{C})$. Let Ξ be in the $\tau_p(A)$-closure of $e(X)$ in $C((B_{X^*}, w^*), \mathbb{C})$. Then clearly:

- $\Xi(0) = 0$;
- $\Xi|A$ is affine;
- $\Xi(\alpha\xi) = \alpha\Xi(\xi)$ for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and $\xi \in A$.

As A is weak* dense in B_{X^*}, we get that Ξ is the restriction of a linear functional. Hence $\Xi \in e(X)$ by the Banach-Dieudonné theorem [5, Corollary 224].

Also the proof of $3 \Longrightarrow 1$ follows the proof of [7, Theorem 2.7]. By a result of Gul'ko (see [10, Proposition 1.4] or [1, Proposition IV.3.10]) there is a continuous one-to-one linear map $T_0' : (C((X, \sigma(X, A)), \mathbb{R}), \tau_p) \to \Sigma(\Gamma)$ for a set Γ. Define $T_0 : (C((X, \sigma(X, A)), \mathbb{C}), \tau_p) \to \mathbb{C}^\Gamma$ by $T_0(f) = T_0'(\operatorname{Re} f) + iT_0'(\operatorname{Im} f)$. Then T_0 is a continuous one-to-one linear map with range in

$$\Sigma_C(\Gamma) = \{x \in \mathbb{C}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

Clearly $\operatorname{span} A \subset C((X, \sigma(X, A)), \mathbb{C})$ and the weak* topology on $\operatorname{span} A$ coincide with the topology of pointwise convergence on X. Hence $T_0(A)$ is dense in $T_0(\operatorname{span} A \cap B_{X^*})$. However, $T_0(A)$ is a countably compact subset of $\Sigma_C(\Gamma)$ and hence it is closed in this space (see [7, Lemma 1.8]). It follows that $A = \operatorname{span} A \cap B_{X^*}$. It remains to show that $\operatorname{span} A$ is a Σ-subspace of X^*.

By [6, Lemma 2.18] applied to the space X_R we get that (B_{X^*}, w^*) is the Čech-Stone compactification of A. It follows that T_0 can be extended to a linear map $T : X^* \to \mathbb{C}^\Gamma$ such that $T_{|B_{X^*}}$ is weak* continuous. By Proposition 2.3(5) the space $T(B_{X^*})$ is the Čech-Stone compactification of $T(A)$ and hence T is one-to-one. By Banach-Dieudonné theorem [5, Corollary 224] the map T is weak* continuous. Hence for each $\gamma \in \Gamma$ there is $x_\gamma \in X$

with $T(\xi)(\gamma) = \xi(x_\gamma)$ (see [5, Theorem 55]). Set $M = \{x_\gamma : \gamma \in \Gamma\}$. Then $\operatorname{span} M$ is dense in X (as T is one-to-one). Moreover,

$$\operatorname{span} A = \{\xi \in X^* : \{x \in M : \xi(x) \neq 0\} \text{ is countable}\}.$$

Indeed, the inclusion $\subset$ is clear, the inverse one follows from Proposition 2.3(4). This completes the proof. ∎

In a similar way we can characterize norming Σ-subspaces.

THEOREM 3.3. *Let X be a Banach space and $S \subset X^*$ a norm-closed norming subspace. Then the following assertions are equivalent.*

(1) *S is a Σ-subspace of X^*.*
(2) *S is a countable union of weak* countably compact sets and $(X, \sigma(X, S))$ is primarily Lindelöf.*

Proof. Up to changing the norm on X by an equivalent one we can suppose that S is 1-norming.

Then $1 \Longrightarrow 2$ follows from Theorem 3.2.

Let us show $2 \Longrightarrow 1$. We have $S = \bigcup_{n \in \mathbb{N}} S_n$ with each S_n weak* countably compact. As S is norm closed, by Baire category theorem there is $\xi \in S$ and $r > 0$ such that $\overline{B(\xi, r)} \cap S_n$ is norm-dense in $\overline{B(\xi, r)} \cap S$. As S is a linear subspace, it follows that $S \cap B_{X^*}$ has a norm-dense weak* countably compact subset D. Then $D = S \cap B_{X^*}$. Indeed, if $\xi \in (S \cap B_{X^*}) \setminus D$, there is a sequence of $d_n \in D$ norm-converging to ξ. Then the sequence $\{d_n\}$ has no weak* cluster point in D, a contradiction with weak* countable compactness of D. We conclude by Theorem 3.2 that S is a Σ-subspace of X^*. ∎

Now we proceed to the relationships of the complex and real cases.

PROPOSITION 3.4. *Let X be a complex Banach space and $S \subset X^*$ be a linear subspace. Let $\phi : X^* \to X_R^*$ be as in Proposition 2.1.*

- *If S is 1-norming, then $\phi(S)$ is a 1-norming subspace of X_R^*.*
- *If S is norming, then $\phi(S)$ is a norming subspace of X_R^*.*
- *If S is a Σ-subspace of X^*, then $\phi(S)$ is a Σ-subspace of X_R^*.*

Proof. The first assertion follows immediately from Proposition 2.1. To see the second one use the first one together with the fact that S is norming if and only if it is 1-norming for an equivalent norm.

Let us show the last assertion. For a subset $A \subset X$ we denote by $\operatorname{span}_C A$ ($\operatorname{span}_R A$) the set of all complex (real, respectively) linear combination of the elements of A.

Let $M \subset X$ be such that $\operatorname{span}_C M$ is dense in X and $S = \{\xi \in X^* : \{x \in M : \xi(x) \neq 0\}$ is countable$\}$. Set $M' = M \cup iM$. Then $\operatorname{span}_R M'$ is dense in X. Further, if $\xi \in X^*$ and $x \in X$, then $\xi(x) = \operatorname{Re}\xi(x) - i\operatorname{Re}\xi(ix)$ (see Proposition 2.1). Hence $\xi(x) = 0$ if and only if $\operatorname{Re}\xi(x) = 0$ and $\operatorname{Re}\xi(ix) = 0$. Thus

$$S = \{\xi \in X^* : \{x \in M' : \operatorname{Re}\xi(x) \neq 0\} \text{ is countable}\},$$

therefore $\phi(S)$ is a Σ-subspace of X_R^*. ∎

THEOREM 3.5. *Let X be a complex Banach space. Consider the following assertions.*

(1) *X is 1-Plichko.*

(2) *X_R is 1-Plichko.*

(3) *(B_{X^*}, w^*) is a Valdivia compactum.*

Then $1 \implies 2 \implies 3$. If B_{X^*} is the weak closed convex hull of its weak* G_δ-points, then $1 \iff 2$.*

Proof. The implication $2 \implies 3$ easily follows from the definitions and Proposition 2.1, $1 \implies 2$ follows from Proposition 3.4.

Finally, suppose that B_{X^*} is the weak* closed convex hull of it weak* G_δ-points. We will show $2 \implies 1$. Let G denote the set of all weak* G_δ-points of B_{X^*}. Let X_R be 1-Plichko. Then (B_{X^*}, w^*) has a dense convex symmetric Σ-subset A (see Theorem 3.2). Let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$. As $x \mapsto \alpha x$ is a homeomorphism of (B_{X^*}, w^*), αA is also a (convex symmetric) Σ-subset. By Proposition 2.3 $A \cap \alpha A$ contains G, hence also $\operatorname{conv} G$, so $A \cap \alpha A$ is weak* dense in B_{X^*}. It follows from Proposition 2.3 that $A = \alpha A$. Thus A is absolutely convex and, by Theorem 3.2, X is 1-Plichko. ∎

It is natural to ask whether the converse implication are valid. It turns out that the implication $3 \implies 2$ does not hold even if we suppose that (B_{X^*}, w^*) has a dense set of G_δ points - see Example 3.9 at the end of this section. We do not know whether $2 \implies 1$ holds in general. The following example shows that a converse of Proposition 3.4 is false.

EXAMPLE 3.6. There is a complex Banach space X and $M \subset X$ with $\operatorname{span}_R M$ dense in X such that the Σ-subset S_R of X_R^* defined by M is 1-norming while the Σ-subset S of X^* defined by M is not even weak* dense. Moreover, $A = \phi^{-1}(S_R) \cap B_{X^*}$ is a convex symmetric Σ-subset of B_{X^*} which is not absolutely convex. (ϕ is the map defined in Proposition 2.1.)

Proof. Let $X = \ell_1(\Gamma)$ $(= \ell_1(\Gamma, \mathbb{C}))$ for some uncountable Γ. By e_γ, $\gamma \in \Gamma$ denote the canonical unit vectors. Choose $\gamma_0 \in \Gamma$ and set

$$M = \{e_{\gamma_0}, ie_{\gamma_0}\} \cup \{e_\gamma - e_{\gamma_0} : \gamma \in \Gamma \setminus \{\gamma_0\}\} \cup \{i(e_\gamma + e_{\gamma_0}) : \gamma \in \Gamma \setminus \{\gamma_0\}\}.$$

Then $\operatorname{span}_R M$ is clearly dense in X. Further, X^* can be canonically identified with $\ell_\infty(\Gamma)$ $(= \ell_\infty(\Gamma, \mathbb{C}))$.

The Σ-subspace of X^* defined by M is

$$S = \{\xi = (\xi_\gamma)_{\gamma \in \Gamma} : \{x \in M : \xi(x) \neq 0\} \text{ is countable}\}.$$

Suppose that $\xi \in S$. Then there is $\gamma \in \Gamma \setminus \{\gamma_0\}$ such that $\xi(e_\gamma - e_{\gamma_0}) = \xi(i(e_\gamma + e_{\gamma_0})) = 0$. But $\xi(e_\gamma - e_{\gamma_0}) = \xi_\gamma - \xi_{\gamma_0}$ and $\xi(i(e_\gamma + e_{\gamma_0})) = i(\xi_\gamma + \xi_{\gamma_0})$. If both these numbers are 0, necessarily $\xi_{\gamma_0} = 0$. Hence

$$S \subset \{\xi \in \ell_\infty(\Gamma) : \xi(e_{\gamma_0}) = 0\}.$$

The set on the right-hand side is a weak* closed hyperplane, so S is not weak* dense.

The Σ-subspace of X_R^* defined by M is

$$S_R = \{\xi = (\alpha_\gamma + i\beta_\gamma)_{\gamma \in \Gamma} : \{x \in M : \operatorname{Re} \xi(x) \neq 0\} \text{ is countable}\}.$$

Let $\xi = (\alpha_\gamma + i\beta_\gamma)_{\gamma \in \Gamma} \in \ell_\infty(\Gamma)$ and $\gamma \in \Gamma \setminus \{\gamma_0\}$. Then $\operatorname{Re} \xi(e_\gamma - e_{\gamma_0}) = \alpha_\gamma - \alpha_{\gamma_0}$ and $\operatorname{Re} \xi(i(e_\gamma + e_{\gamma_0})) = -\beta_\gamma - \beta_{\gamma_0}$. Thus

$$S_R = \{\xi = (\xi_\gamma)_{\gamma \in \Gamma} : \{\gamma \in \Gamma : \xi_\gamma \neq \overline{\xi_{\gamma_0}}\} \text{ is countable}\}.$$

This subspace is clearly 1-norming.

That $A = \phi^{-1}(S_R) \cap B_{X^*}$ satisfies the required property is obvious. ∎

We continue by a result on the relationship of Y and Y_C.

THEOREM 3.7. *Let Y be a real Banach space. If Y is 1-Plichko, Y_C is 1-Plichko as well.*

Proof. Let M be a subset of Y such that $\operatorname{span} M$ is dense in Y and the Σ-subset S defined by M is 1-norming. We can consider Y as a subset of Y_C (identify each $y \in Y$ with $y + i0$). Then Y is a closed real-linear subspace of Y_C. Note also that the original norm on Y coincide with the subspace norm inherited from Y_C. Hence M can be also viewed as a subset of Y_C. Clearly $\operatorname{span} M$ is dense in Y_C and thus M defines a Σ-subspace of Y_C^*. We are going to show it is 1-norming.

First note, that Y_C^* is isomorphic (not necessarily isometric) to $(Y^*)_C$. Indeed, if $\xi, \eta \in Y^*$, then

$$(\xi + i\eta)(x + iy) = \xi(x) - \eta(y) + i(\xi(y) + \eta(x)), \qquad x + iy \in Y_C$$

defines an element of Y_C^*. Conversely, it is easy to see that each element of Y_C^* has this form. As $M \subset Y$, the Σ-subspace of Y_C^* defined by M is equal to

$$S + iS = \{\xi + i\eta, \xi, \eta \in S\}.$$

Now we are going to show that $S + iS$ is 1-norming.

Let $x_0 + iy_0 \in Y_C$ be arbitrary. Then there exist real numbers α, β with $\alpha^2 + \beta^2 = 1$ such that $\|x_0 + iy_0\|_{Y_C} = \|\alpha x_0 + \beta y_0\|_Y$. As S is 1-norming and $S \cap B_{Y^*}$ is weak* countably compact, there is $\xi \in S$ with $\|\xi\| = 1$ such that $|\xi(\alpha x_0 + \beta y_0)| = \|\alpha x_0 + \beta y_0\|_Y$. Note that this ξ has norm one also when considered as an element of Y_C^*. Indeed, let $x + iy \in Y_C$. Choose $\gamma, \delta \in \mathbb{R}$ with $\gamma^2 + \delta^2 = 1$ such that $|\xi(x) + i\xi(y)| = |\gamma\xi(x) + \delta\xi(y)|$. Then

$$\begin{aligned}|\xi(x + iy)| = |\xi(x) + i\xi(y)| &= |\gamma\xi(x) + \delta\xi(y)| \\ &= |\xi(\gamma x + \delta y)| \le \|\gamma x + \delta y\|_Y \le \|x + iy\|_{Y_C}.\end{aligned}$$

Set $\widetilde{\xi} = (\alpha - i\beta)\xi$. Then $\widetilde{\xi} \in S + iS$ and $\|\widetilde{\xi}\| = 1$. Moreover,

$$\begin{aligned}|\widetilde{\xi}(x_0 + iy_0)| &= |(\alpha - i\beta)\xi(x_0 + iy_0)| = |(\alpha - i\beta)(\xi(x_0) + i\xi(y_0))| \\ &= |\xi(\alpha x_0 + \beta y_0) + i\xi(\alpha y_0 - \beta x_0)| \ge |\xi(\alpha x_0 + \beta y_0)| = \|x_0 + iy_0\|.\end{aligned}$$

This completes the proof. ∎

It is not clear whether the converse is true. Another open question is whether Y_C is 1-Plichko whenever $(Y_C)_R$ is 1-Plichko. For $C(K)$ spaces both questions have positive answers and hence the conditions (1) and (2) of Theorem 3.5 are equivalent in this case. It is contained, together with other facts on $C(K)$ spaces, in the following theorem.

THEOREM 3.8. *Let K be a compact space. Consider the following assertions:*

(1) *K is Valdivia.*

(2_C) *$C(K,\mathbb{C})$ is 1-Plichko.*

($2'_C$) *$C(K,\mathbb{C})_R$ is 1-Plichko.*

(2_R) *$C(K,\mathbb{R})$ is 1-Plichko.*

(3) *$P(K)$ has a dense convex Σ-subset.*

(4_C) *$(B_{C(K,\mathbb{C})}, w^*)$ is Valdivia.*

(4_R) *$(B_{C(K,\mathbb{R})}, w^*)$ is Valdivia.*

(5) *$P(K)$ is Valdivia.*

Then the following implications hold:

$$\begin{array}{ccccccc} 1 \implies 2_C \iff 2'_C \iff 2_R \iff & 3 & \implies & 4_C \\ & \Downarrow & & \Downarrow \\ & 4_R & \implies & 5 \end{array}$$

If K has a dense set of G_δ-points, then all these assertions are equivalent.

Proof. By [7, Theorem 5.2] we have $1 \implies 2_R \iff 3 \implies 4_R \implies 5$. Further, $2_C \implies 2'_C$ by Theorem 3.5 and $2_C \implies 4_C$ is clear.

As $P(K) = \{\xi \in C(K,\mathbb{C})^* : \|\xi\| \le 1 \ \&\ \xi(1) = 1\}$ and this set is weak* G_δ in $B_{C(K,\mathbb{C})^*}$, the implications $2'_C \implies 3$ and $4_C \implies 5$ follow from Proposition 2.3.

The implication $2_R \implies 2_C$ follows from Proposition 2.2 and Theorem 3.7.

If K has a dense set of G_δ-points, then $5 \implies 1$ by [7, Theorem 5.3]. ∎

We do not know whether all the assertions in the previous theorem are equivalent without the additional assumption.

EXAMPLE 3.9. There is a complex Banach space X isomorphic to $C([0,\omega_1],\mathbb{C})$ such that (B_{X^*}, w^*) is Valdivia but X_R is not 1-Plichko.

Proof. In [8] a real Banach space Y isomorphic to $C([0,\omega_1],\mathbb{R})$ such that (B_{Y^*}, w^*) is Valdivia but Y is not 1-Plichko is constructed. For the new example we use the same method:

Note that the dual $C([0,\omega_1],\mathbb{C})^*$ is, due to Riesz theorem, identified with the space M of all complex Radon measures on $[0,\omega_1]$. Define $f : M \to \mathbb{C}$ by

$f(\mu) = \mu(\{0\}) \cdot |\mu(\{0\})|$. Set

$$A = \{\mu \in M : \|\mu\| \le 1 \ \& \ \mu(\{\omega_1\}) = f(\mu)\},$$
$$B = \{\mu \in M : |\mu|([0, \omega_1)) + |f(\mu)| + |\mu(\{\omega_1\}) - f(\mu)| \le 1\}.$$

Then the following hold.

(a) B is convex and $\alpha B = B$ for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$.

(b) B is weak* closed.

(c) If B_M denotes the unit ball of M, there is $\delta > 0$ with $\delta B_M \subset B \subset B_M$.

(d) A is a dense Σ-subset of (B, w^*).

(e) A is not convex.

Suppose that we already know that (a)–(e) hold. It follows from (a)–(c) that there is an equivalent norm $|\cdot|$ on $C([0, \omega_1], \mathbb{C})$ such that the respective dual unit ball is B. Set $X = (C([0, \omega_1], \mathbb{C}), |\cdot|)$. The dual unit ball is Valdivia by (d). Further, B has a dense set of G_δ points (it follows from [4, Theorem 1.1.3] that $(C([0, \omega_1]), \mathbb{C})_R$ is Asplund, then use [4, Theorems 1.1.1 and 5.1.12]) and hence A is the unique dense Σ-subset of B (by Proposition 2.3). Hence, by (e) (B, w^*) has no convex dense Σ-subset and so X_R is not 1-Plichko (by Theorem 3.2).

It remains to show the assertions (a)–(e). Except for convexity of B they are either easy or they can be derived from the results of [8]. As f is clearly weak* continuous, the assertions (b) and (d) can be proved copying the proof of [8, Lemma 4] and the assertion (c) follows from the proof of [8, Lemma 3]. If $\alpha \in \mathbb{C}$, $|\alpha| = 1$, then $f(\alpha\mu) = \alpha f(\mu)$ for each $\mu \in M$ and hence $\alpha B = B$. Finally, A is not convex, as 0 and $\frac{1}{2}\delta_0 + \frac{1}{4}\delta_{\omega_1}$ belong to A but $\frac{1}{4}\delta_0 + \frac{1}{8}\delta_{\omega_1}$ does not. (Note that δ_x is the Dirac measure supported by x.)

To show that B is convex we cannot use directly [8, Lemma 1] as it heavily uses the functions in question are real. In fact, this lemma is false for complex functions. We will show it using some facts on delta-convex mappings. Let Y and Z be real normed spaces and $F : Y \to Z$ a mapping. The mapping F is said to be *delta-convex* [14] if there is a continuous convex function $f : Y \to \mathbb{R}$ such that $f + \zeta \circ F$ is a continuous convex function on Y for every $\zeta \in Z^*$, $\|\zeta\| = 1$. Such a function f is called a *control function* of F.

We will need the following result on superpositions of delta-convex mappings proved in [14, Proposition 4.1].

LEMMA 3.10. *Let X, Y, Z be real normed spaces, $F : X \to Y$ be delta-convex with a control function f, $G : Y \to Z$ be delta-convex with a control function g. Suppose further that G and g are Lipschitz on Y with constants L_G and L_g.*

Then the mapping $G \circ F$ is delta-convex on X with a control function $g \circ F + (L_G + L_g)f$.

Using this lemma we can show the following one.

LEMMA 3.11. *Let X and Y be real normed spaces and $F : X \to Y$ be a delta-convex function with a control function $f(x) = \|F(x)\|$. Then the function $H : X \times Y \to \mathbb{R}$ defined by $H(x, y) = \|F(x)\| + \|y - F(x)\|$ is convex.*

Proof. First note that the mapping $Q : X \times Y \to Y$ defined by $Q(x, y) = y - F(x)$ is delta-convex with the control function $\tilde{f}(x, y) = \|F(x)\|$.

Further, the map $G : Y \to \mathbb{R}$ defined by $G(y) = \|y\|$ is convex and 1-Lipschitz. Therefore G is a delta-convex mapping with a control function $g = G$. Using Lemma 3.10 we get that the mapping $G \circ Q(x, y) = \|y - F(x)\|$ is delta-convex with a control function $h(x, y) = \|y - F(x)\| + 2\|F(x)\|$. In particular, $2H = h + G \circ Q$ is convex, hence H is convex, too. ∎

LEMMA 3.12. *The function $\Psi : \mathbb{C} \to \mathbb{C}$ defined by $\Psi(z) = z|z|$ is delta-convex with a control function $\psi(z) = |z|^2$ ($\mathbb{C}$ is considered as the two-dimensional real Hilbert space).*

Proof. First we express Ψ and ψ as maps on $\mathbb{R}^2$. Then

$$\Psi(x, y) = \sqrt{x^2 + y^2} \cdot (x, y), \qquad (x, y) \in \mathbb{R}^2,$$
$$\psi(x, y) = x^2 + y^2, \qquad (x, y) \in \mathbb{R}^2.$$

To show that Ψ is a delta-convex mapping with a control function ψ, we will use the definition. We have to show that $\psi + \xi \circ \Psi$ is convex for each $\xi \in (\mathbb{R}^2)^*$ with $\|\xi\| = 1$. Let $\xi \in (\mathbb{R}^2)^*$ be of norm one. Then there are $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ such that $\xi(x, y) = ax + by$ for $(x, y) \in \mathbb{R}^2$. Hence we need to prove that the function

$$(x, y) \mapsto x^2 + y^2 + (ax + by)\sqrt{x^2 + y^2}$$

is convex on $\mathbb{R}^2$ whenever $a^2 + b^2 = 1$. Due to symmetry (i.e., up to a choice of another ortonormal basis) we may suppose $a = 1$ and $b = 0$. Hence, it remains to show that the function

$$g(x, y) = x^2 + y^2 + x\sqrt{x^2 + y^2}$$

is convex on $\mathbb{R}^2$. The function g is C^∞ on $\mathbb{R}^2 \setminus \{(0,0)\}$ and hence we can compute the Hess matrix for any point (x,y) except for $(0,0)$. This matrix is equal to

$$\begin{pmatrix} \dfrac{2\,x^3 + 3\,x\,y^2 + 2\,(x^2+y^2)^{(3/2)}}{(x^2+y^2)^{(3/2)}} & \dfrac{y^3}{(x^2+y^2)^{(3/2)}} \\ \dfrac{y^3}{(x^2+y^2)^{(3/2)}} & \dfrac{x^3 + 2\,(x^2+y^2)^{(3/2)}}{(x^2+y^2)^{(3/2)}} \end{pmatrix}$$

The determinant is equal to

$$3\,\frac{2\,x^2 + 2\,x\,\sqrt{x^2+y^2} + y^2}{x^2+y^2} = 3\,\frac{(x+\sqrt{x^2+y^2})^2}{x^2+y^2}.$$

This expression is nonnegative and for $y \neq 0$ it is strictly positive. Further,

$$\frac{\partial^2 g}{\partial y^2}(x,y) = \frac{x^3 + 2\,(x^2+y^2)^{(3/2)}}{(x^2+y^2)^{(3/2)}}$$

is also nonnegative and for $y \neq 0$ strictly positive. Hence the Hess matrix is (by the Sylvester rule) positive definite whenever $y \neq 0$.

If $y = 0$, then

$$\frac{\partial^2 g}{\partial x^2}(x,y) = \frac{2\,x^3 + 3\,x\,y^2 + 2\,(x^2+y^2)^{(3/2)}}{(x^2+y^2)^{(3/2)}}$$

is equal to $\dfrac{2x^3 + 2|x|^3}{|x|^3}$, and hence it is nonnegative. Thus, again by the Sylvester rule, the Hess matrix is positive semidefinite.

It follows that the function g is convex on each line noncontaining $(0,0)$ and on each half-line starting at $(0,0)$. To complete the proof that g is convex, it remains to show that the function $u(t) = g(tx,ty)$ is convex on $\mathbb{R}$ for each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We already know that any such u is convex on $(-\infty,0]$ and on $[0,+\infty)$. Further, $u'(0) = 0$ and hence u is convex on $\mathbb{R}$. This completes the proof. ∎

Now we are ready to complete the proof of Example 3.9. It follows from Lemma 3.12 and Lemma 3.11 that the function $(w,z) \mapsto |z|^2 + |w - z|z||$ is convex on $\mathbb{C}^2$. As $\mu \mapsto (\mu(\{0\}), \mu(\{\omega_1\}))$ is a linear map, the map

$$\mu \mapsto \|f(\mu)\| + |\mu(\{\omega_1\}) - f(\mu)|$$

is convex on M and hence the set B is clearly convex. ∎

4. Final remarks and open questions

In this section we comment some open questions mentioned above and give some related problems. First one concerns Theorem 3.8 – are all the conditions equivalent? This was asked, in fact, already in [7, Question 5.10]. We can sum up the question to the following one.

Question 4.1. Let K be a compact space such that $P(K)$, the space of all Radon probability measures on K equipped with the weak* topology, is a Valdivia compactum. Is K Valdivia, too?

Another question is related to Theorem 3.5.

Question 4.2. Let X be a complex Banach space such that X_R is 1-Plichko. Is X 1-Plichko, too?

Example 3.6 shows that there may exist 1-norming Σ-subspace of X_R^* such that $\phi^{-1}(S)$ is not a Σ-subspace of X^* (ϕ is the mapping from Proposition 2.1). However, the example is $\ell_1(\Gamma)$ for a set Γ and this space is 1-Plichko – the Σ-subspace generated by the standard basis is 1-norming.

In view of Theorem 3.2 the previous question is equivalent to the following one.

Question 4.3. Let X be a complex Banach space such that (B_{X^*}, w^*) has a dense convex symmetric Σ-subset. Does (B_{X^*}, w^*) admit another dense Σ-subset A which is convex and satisfies $\alpha A = A$ for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$?

This question inspires some further questions on the algebraic structure of Valdivia compacta. Namely, let K be a Valdivia compactum and G a group of homeomorphisms of K. Is there a dense Σ-subset A of K which is G-invariant (i.e., $g(A) = A$ for each $g \in G$)? This general question has a negative answer: Let $K = \{0,1\}^I$ where I has cardinality continuum. Then K is Valdivia. By [11] there is a minimal homeomorphism h of K (i.e., all orbits of h are dense in K). Then there is no h-invariant nonempty Σ-subset of K. Indeed, if A is a nonempty h-invariant set, A contains a countable subset dense in K. If A was a Σ-subset, it would be countably closed in K and hence equal to K. However, K is not Corson, as it is not Fréchet-Urysohn.

Hence we will ask more modestly:

Question 4.4. Let K be a Valdivia compact space and G a finite abelian group of homeomorphisms of K. Is there a G-invariant dense Σ-subspace?

The positive answer to this question would not solve the previous one, as the group of homeomorphisms $x \mapsto \alpha x$, $|\alpha| = 1$ is infinite and, moreover, the previous question deals with convex sets. However, we do not know answer even to this question and it seems that a positive answer could help to better understand the previous case. In fact, we do not know answer even to the following question.

QUESTION 4.5. Let K be a Valdivia compact space and $h : K \to K$ be a homeomorphism such that $h \circ h = \mathrm{id}_K$. Is there an h-invariant dense Σ-subset?

In particular, the following question is open.

QUESTION 4.6. Let X be a Banach space such that (B_{X^*}, w^*) is Valdivia. Is there a symmetric dense Σ-subset of (B_{X^*}, w^*)?

Note, that if A is a dense Σ-subset of K and h a homeomorphism of K onto K, then $h(A)$ is a dense Σ-subset, too. Hence, if K has a unique dense Σ-subset, it must be h-invariant. It follows that the method used in [7, Example 6.8], [8] and in Theorem 3.9 above to produce convex Valdivia compacta without dense convex Σ-subsets, cannot be used to produce counterexamples to the mentioned questions, as all these examples are convex Valdivia compacta with a unique non-convex dense Σ-subsets.

The following isomorphic version of Question 4.2 seems to be open, too.

QUESTION 4.7. Let X be a complex Banach space such that X_R is Plichko. Is X Plichko, too?

Another question concerns the complexification.

QUESTION 4.8. Suppose that Y is a real space such that Y_C is 1-Plichko. Is Y 1-Plichko, too?

The converse is true by Theorem 3.7. If Y_C is 1-Plichko, then $(Y_C)_R$ is 1-Plichko by Theorem 3.5. It is clear that Y is a 1-complemented subspace of $(Y_C)_R$. Hence the last question is a particular case of [7, Question 4.45(ii)].

Added in proof. T. Banakh and W. Kubiś recently showed that Question 4.1 has a negative answer. They constructed a non-Valdivia compact space K such that the space $C(K)$ is 1-Plichko.

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