A Note on the Range of Generalized Derivation

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1. Introduction

Let \( \mathcal{L}(H) \) be the algebra of all bounded linear operators acting on a complex separable and infinite dimensional Hilbert space \( H \). For operators \( A, B \in \mathcal{L}(H) \) we define the generalized derivation \( \delta_{A,B} \) associated with \( (A, B) \) by

\[
\delta_{A,B}(X) = AX - XB \quad \text{for} \quad X \in \mathcal{L}(H).
\]

If \( A = B \), then \( \delta_{A,A} = \delta_A \) is called the inner derivation. The theory of derivations has been extensively dealt with in the literature (see for example [1, 2, 3, 5, 6, 7, 8, 9, 10, 16, 17, 18, 20] and [21]).

For a linear operator \( T \) acting on a Banach space \( X \), we denote by \( T^* \), \( \text{Ker } T \) and \( \text{R}(T) \) respectively the adjoint, the kernel and the range of \( T \). Also we denote by \( \text{R}(T) \) and \( \text{R}(T)^\sigma \) respectively the closure of the range of \( T \) respect to the norm topology and the weak operator topology.

In this work we give the extension of the results showed by Williams [21, p. 301] and Ho [13, p. 511] to \( \delta_{A,B} \). We will give some conditions for \( A, B \in \mathcal{L}(H) \) under which

\[
\text{R}(\delta_{A,B})^\tau \cap \text{Ker } \delta^*_A, B^* = \{0\},
\]

where \( \text{R}(\delta_{A,B})^\tau \) denotes closure of \( \text{R}(\delta_{A,B}) \) respect to the norm topology or the weak operator topology.

In section 1, we prove that if \( A \) and \( B \) are isometries (resp. co-isometries) or if \( P(A) \) and \( P(B) \) are normal for some non-trivial polynomial \( P \) with degree \( \leq 2 \), then

\[
\text{R}(\delta_{A,B}) \cap \text{Ker } \delta^*_A, B^* = \{0\}.
\]
Recall [12] that $A \in \mathcal{L}(H)$ is bloc-diagonal if there exists an increasing sequence $\{P_n\}_n$ of self-adjoint projectors of finite rank in $\mathcal{L}(H)$ such that $\lim_{sot} P_n = I$ and $P_nA = AP_n$ for all $n \in \mathbb{N}$, where $\lim$ is the limit respect to the strong operator topology in $\mathcal{L}(H)$.

In section 2, we prove that if $A$ is bloc-diagonal then every positive operator in $\overline{R(\delta A)}$ vanishes. As a consequence of this we obtain that if $A$ is bloc-diagonal then $\overline{R(\delta A,B)} \cap \ker \delta A^*,B^* = \{0\}$ for every $B \in \mathcal{L}(H)$.

2. Conditions under which $\overline{R(\delta A,B)} \cap \ker \delta A^*,B^* = \{0\}$

Let $\mathcal{C}_1(H)$ be the ideal of trace class operators, that is, the set of all compact operators $T \in \mathcal{L}(H)$ for which the eigenvalues of $(T^*T)^{1/2}$ counted according to their multiplicity are summable. The trace function is defined by $\text{Tr}(T) = \sum_{n} \langle Te_n, e_n \rangle$, where $(e_n)$ is any complete orthonormal sequence in $H$. Recall that the ultraweak continuous linear functionals on $\mathcal{L}(H)$ are those of the form $f_T$ for some $T \in \mathcal{C}_1(H)$ and the weak continuous linear functionals on $\mathcal{L}(H)$ are those of the form $f_T$, where $T$ is of finite rank.

**Lemma 2.1.** Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ on $H \oplus H$, where $A, B \in \mathcal{L}(H)$. Then we have the following assertions:

i) If $\overline{R(\delta T)} \cap \ker \delta T^* = \{0\}$, then $\overline{R(\delta A,B)} \cap \ker \delta A^*,B^* = \{0\}$;

ii) If $R(\delta T) \cap \ker \delta T^* = \{0\}$, then $R(\delta A,B) \cap \ker \delta A^*,B^* = \{0\}$.

**Proof.** i) Let $C \in \overline{R(\delta A,B)} \cap \ker \delta A^*,B^*$. Then there exists a sequence $\{X_\alpha\}_\alpha$ of elements of $\mathcal{L}(H)$ such that $\lim_{\tau} AX_\alpha - X_\alpha B = C$ and $A^*C = CB^*$. Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $Y_\alpha = \begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ on $H \oplus H$. Then

$$\lim_{\tau} TY_\alpha - Y_\alpha T = \lim_{\tau} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \lim_{\tau} \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix}.$$

If $\lim_{\omega} \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ on $H \oplus H$. Then

$$\left| \left\langle \begin{pmatrix} L_{11} & L_{12} - (AX_\alpha - X_\alpha B) \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle \right|$$
converges to 0, hence \( | < L_{12} - (AX_\alpha - X_\alpha B) | x, y > | \) converges to 0 for all \( x, y \in H \), which implies that

\[
\lim_\omega AX_\alpha - X_\alpha B = L_{12}.
\]

As the same we prove that

\[
L_{11} = L_{21} = L_{22} = 0.
\]

This implies that

\[
\lim_\omega \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lim_\omega AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}.
\]

If \( \lim \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \) on \( H \oplus H \), then

\[
\left\| \begin{pmatrix} L_{11} & L_{12} - [AX_\alpha - X_\alpha B] \\ L_{21} & L_{22} \end{pmatrix} \right\| \text{ converges to 0,}
\]

hence

\[
\| L_{12} - [AX_\alpha - X_\alpha B] \| \text{ converges to 0 and } L_{11} = L_{21} = L_{22} = 0.
\]

This implies that

\[
\lim \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lim AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}.
\]

Hence, \( \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} = S \in \overline{R(\delta_T)}^\tau \). Since \( ST^* = T^*S \), then \( S \in \overline{R(\delta_T)}^\tau \cap Ker \delta_T^* = \{0\} \). So \( C = 0 \). This completes the proof of i). To prove ii) it suffices to replace \( \overline{R(\delta_T)}^\tau \) with \( R(\delta_T) \).

In the following theorem we give an extension of the result of [21, p. 301] and [13, p. 511] to \( \delta_{A,B} \).
Theorem 2.1. Let $A, B \in \mathcal{L}(H)$. If $A$ and $B$ are isometries (resp. co-isometries) or $P(A)$ and $P(B)$ are normal for some non-trivial polynomial $P$ with degree $\leq 2$ then

$$\overline{R(\delta_{A,B})} \cap \text{Ker} \delta_{A^*,B^*} = \{0\}.$$ 

Proof. i) If $A$ and $B$ are isometries (resp. co-isometries), then $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is also an isometry (resp. co-isometry) on $\mathcal{L}(H \oplus H)$. By [21, p. 301], we have $\overline{R(\delta_T)} \cap \text{Ker} \delta_{T^*} = \{0\}$. Hence from Lemma 2.1, we conclude that $\overline{R(\delta_{A,B})} \cap \text{Ker} \delta_{A^*,B^*} = \{0\}$.

ii) The result of [13, Theorem 3 (1)] asserts that if $T \in \mathcal{L}(H)$ is such that $P(T)$ is normal for some non-trivial polynomial $P$ with degree $\leq 2$, then $\overline{R(\delta_T)} \cap \text{Ker} \delta_{T^*} = \{0\}$. Indeed, suppose that $T^2 - 2\alpha T - \beta = N$ is a normal operator. Let $\lim TX_n - X_nT = S^* \in \overline{R(\delta_T)} \cap \text{Ker} \delta_{T^*}$. Then

$$\lim(N + 2\alpha T)X_n - X_n(N + 2\alpha T) = \lim T^2X_n - X_nT^2 = TS^* + S^*T.$$ 

This implies that $TS^* + S^*T - 2\alpha S^* \in \overline{R(\delta_N)} \cap \text{Ker} \delta_{N^*}$ so that $TS^* + S^*T - 2\alpha S^* = 0$ by [4, Theorem 1.7]. Hence

$$(S + S^*)(T - \alpha) = (T - \alpha)(S - S^*) \text{ and } (T - \alpha)S^* = -S^*(T - \alpha).$$ 

The Putnam-Fuglede theorem then gives

$$(S^* + S)(T - \alpha) = (T - \alpha)(S^* - S) \text{ and } (T - \alpha)S = -S(T - \alpha).$$ 

Combining these equations we get

$$(T - \alpha)(S^* + S) = 0 \text{ and } (S^* + S)(T - \alpha) = 0.$$ 

Hence $S^*T = TS^*$. Therefore $S^*S \in \overline{R(\delta_T)} \cap \text{Ker} \delta_{T^*}$ so that $S = 0$ by [13, Lemma 3]. Now, if $P(A)$ and $P(B)$ are normal for some non-trivial polynomial $P$ with degree $\leq 2$, then $P(T)$ is also normal for $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, hence from the previous result $\overline{R(\delta_T)} \cap \text{Ker} \delta_{T^*} = \{0\}$. From Lemma 2.1, we conclude that $\overline{R(\delta_{A,B})} \cap \text{Ker} \delta_{A^*,B^*} = \{0\}$. 


Lemma 2.2. Let \( A, B \in \mathcal{L}(H) \). If \( T \in R(\delta_{A,B})^\tau \cap \text{Ker}\delta_{A^*,B^*} \), then \( T^*T \in R(\delta_B)^\tau \) and \( TT^* \in R(\delta_A)^\tau \).

Proof. If \( T \in R(\delta_{A,B})^\tau \cap \text{Ker}\delta_{A^*,B^*} \), then there exists a sequence \( \{X_\alpha\}_\alpha \) of elements of \( \mathcal{L}(H) \) such that 
\[
T = \lim_{\tau} AX_\alpha - X_\alpha A = 0 \quad \text{and} \quad A^*T - TB^* = 0.
\]
Hence
\[
T^*T = \lim_{\tau} T^*AX_\alpha - T^*X_\alpha B = \lim_{\tau} BT^*X_\alpha - T^*X_\alpha B,
\]
and
\[
TT^* = \lim_{\tau} AX_\alpha T^* - X_\alpha BT^* = \lim_{\tau} AX_\alpha T^* - X_\alpha T^*A,
\]
since right multiplication and left multiplication are continuous with respect to the topology \( \tau \).

The following lemma is proved in [19], we need it to prove the next theorem.

Lemma 2.3. Let \( B \in \mathcal{L}(H) \) be a normal operator and \( X \in C_2(H) \) such that \( BX - XB \in C_1(H) \), then \( \text{Tr}(BX - XB) = 0 \).

For the unilateral right shift with a non null weight, we have the following result.

Theorem 2.2. Let \( S \in \mathcal{L}(H) \) be the unilateral right shift with a non null weight \( (\alpha_n)_n \); \( \alpha_n \neq 0 \) for all \( n \in \mathbb{N} \) and let \( B \in \mathcal{L}(H) \) be normal. Then \( R(\delta_{S,B}) \cap \text{Ker}\delta_{S^*,B^*} = \{0\} \).

Proof. Let \( T \in R(\delta_{S,B}) \cap \text{Ker}\delta_{S^*,B^*} \). By the same argument as in the proof of Lemma 2.2, we get that \( TT^* \in R(\delta_S) \), hence from [13] \( TT^* \in C_1(H) \). Which is equivalent to \( T \in C_2(H) \). On the other hand \( T^*T = BT^*X - T^*XB \) with \( T^*T \in C_1(H) \), \( T^*X \in C_2(H) \) and \( B \) is normal. Hence by Lemma 2.3, we conclude that \( \text{Tr}(T^*T) = 0 \). Since \( T^*T \) is positive, then \( T = 0 \).

3. Positive operators in \( R(\delta_A)^\omega \)

Definition 3.1. [12] An operator \( A \in \mathcal{L}(H) \) is bloc-diagonal if there exists an increasing sequence \( \{P_n\}_n \) of self-adjoint projectors of finite rank in \( \mathcal{L}(H) \) such that \( \lim_{\text{sot}} P_n = I \) and \( P_nA = AP_n \) for all \( n \in \mathbb{N} \), where \( \lim_{\text{sot}} \) is the limit with respect to the strong operator topology in \( \mathcal{L}(H) \).
Example 1. [12] Let $H = \bigoplus_{n=0}^{\infty} H_n$. If $A = \bigoplus_{n=0}^{\infty} A_n$ where $A_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$, then $A$ is block-diagonal.

For block-diagonal operators we have the following result.

**Theorem 3.1.** Let $A \in \mathcal{L}(H)$. If $A$ is block-diagonal then every positive operator in $\overline{R(\delta A)}^\omega$ vanishes.

**Proof.** Suppose that $A$ is block-diagonal. Then there exists an increasing sequence $\{P_n\}_n$ of self-adjoint projectors of finite rank in $\mathcal{L}(H)$ such that $\lim_{\text{sot}} P_n = I$ and $P_n A = AP_n$ for all $n \in \mathbb{N}$.

Let $T$ a positive operator in $\overline{R(\delta A)}^\omega$, then there exists a sequence $\{X_\alpha\}_\alpha$ in $\mathcal{L}(H)$ such that $T = \lim_{\omega} AX_\alpha - X_\alpha A$. By multiplication right and left by $P_n$, we obtain

$$P_n TP_n = \lim_{\omega} P_n AX_\alpha P_n - P_n X_\alpha AP_n,$$

since $AP_n = P_n A$, then

$$(*) \quad P_n TP_n = \lim_{\omega} P_n AP_n P_n X_\alpha P_n - P_n X_\alpha P_n P_n AP_n.$$

Since $AP_n = P_n A$ and $A^* P_n = P_n A^*$, then $R(P_n) = H_n$ reduces $A$. Hence $A$ has the decomposition

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \quad \text{on } H = H_n \oplus H_n^\perp.$$

Let $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, $X_\alpha = \begin{pmatrix} X_{\alpha}^{11} \\ X_{\alpha}^{12} \\ X_{\alpha}^{21} \\ X_{\alpha}^{22} \end{pmatrix}$ and $P_n = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ on $H = H_n \oplus H_n^\perp$. It follow from (*) that

$$\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} = \lim_{\omega} \begin{pmatrix} A_{11} X_{\alpha}^{11} - X_{\alpha}^{11} A_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence for all $x, y \in H_n$,

$$\left| \left\langle \begin{pmatrix} T_{11} - A_{11} X_{\alpha}^{11} - X_{\alpha}^{11} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \left( \begin{array}{c} x \\ 0 \end{array} \right), \left( \begin{array}{c} y \\ 0 \end{array} \right) \right| \right|$$
converges to 0. This implies that \( \lim A_{11}X_{\alpha}^{11} - X_{\alpha}^{11}A_{11} = T_{11} \), that is \( T_{11} \in R(\delta_{A_{11}}) \). Since dimension of \( H_n \) is finite, then \( T_{11} \in R(\delta_{A_{11}}) \), hence there exists \( Y \in \mathcal{L}(H_n) \) such that \( T_{11} = A_{11}Y - YA_{11} \), which implies that

\[
Tr(T_{11}) = Tr(A_{11}Y) - Tr(YA_{11}) = 0.
\]

Since \( P_n \) is auto-adjoint, then \( P_nTP_n = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \) is positive, and hence \( T_{11} \) is positive. Since \( Tr(T_{11}) = 0 \), then \( T_{11} = 0 \), and hence \( P_nTP_n = 0 \) for all \( n \in \mathbb{N} \). On the other hand, since

\[
\lim_{sot} P_n = I, \quad \lim_{sot} \|P_nTP_nx - TP_nx\| = 0
\]

and

\[
\lim_{sot} \|TP_nx - Tx\| = \lim_{sot} \|T\|\|P_nx - x\| = 0,
\]

then \( \lim_{sot} P_nTP_n = \lim_{sot} TP_n = \lim_{sot} TP_n = T \). This implies that \( \lim_{sot} P_nTP_n = T \) for all \( n \in \mathbb{N} \). Finally, \( T = 0 \).

As an immediate consequence we have the following corollary:

**Corollary 3.1.** Let \( A \in \mathcal{L}(H) \). If \( A \) is bloc-diagonal, then

\[
\overline{R(\delta_{A,B})}^{\omega} \cap Ker\delta_{A^*,B^*} = \{0\}
\]

for every \( B \in \mathcal{L}(H) \).

**Proof.** If \( A \in \mathcal{L}(H) \) is bloc-diagonal and \( T \in \overline{R(\delta_{A,B})}^{\omega} \cap Ker\delta_{A^*,B^*} \), then by Lemma 2.2, \( TT^* \in \overline{R(\delta_A)}^{\omega} \). By Theorem 3.1, we conclude that \( TT^* = 0 \), and hence \( T = 0 \).

Recall [12] that \( A \in \mathcal{L}(H) \) is quasi-diagonal if there exists an increasing sequence \( \{P_n\}_n \) of self-adjoint projectors of finite rank in \( \mathcal{L}(H) \) such that \( \lim_{sot} P_n = I \) and \( \lim_{sot} \|P_nA - AP_n\| = 0 \) for all \( n \in \mathbb{N} \). Every bloc-diagonal operator is quasi-diagonal and the converse is false, see [12]. The following example show that in general Theorem 3.1 does not hold for quasi-diagonal operators.

**Example 2.** Let \( A = S + S^* \) where \( S \) is the unilateral shift defined by \( Se_n = e_{n+1} \) where \( \{e_n\}_n \) is any complete orthonormal sequence in \( H \). Since \( A \) is self-adjoint, then \( A \) is quasi diagonal [12]. Let \( T = I - SS^* \), then
\[ T = (S + S^*)S - S(S + S^*) = AS - SA. \] Hence \( T \in R(\delta_A) \). On the other hand, we have
\[
<Tx, x> = <(I - SS^*)x, x> = \|x\|^2 - \|S^*x\|^2, \quad \text{for all } x \in H.
\]
Since \( \|S^*\| \leq 1 \), then \( <Tx, x> \geq 0 \) for all \( x \in H \). Thus \( T \) is positive. Finally, \( T \) is a non null positive operator in \( R(\delta_A) \).

4. A COMMENT

In [1] (see also [15]) it is shown that every finite rank operator in \( R(\delta_{A,B})^{\ast} \cap Ker\delta_{A^*,B} \) vanishes and every trace class operator in \( R(\delta_{A,B})^{\ast} \cap Ker\delta_{A^*,B} \) vanishes, where \( R(\delta_{A,B})^{\ast} \) is the closure of \( R(\delta_{A,B}) \) with respect to the ultraweak topology \( \omega^* \).

However in [11] (see also [14]) the author ask; if every compact operator in \( R(\delta_{A})^{\ast} \cap \{A^*\} \) is quasinilpotent? A partial answer is given in [1] (see also [14]) if \( A \) or \( A^* \) is dominant and in [10] if \( A \) or \( A^* \) lies in \( U_0 \).

Recall that \( A \in \mathcal{L}(H) \) is dominant if for all \( \lambda \in \mathbb{C} \), there exists a real number \( M_\lambda \geq 1 \) such that \( \|(A - \lambda)x\| \leq M_\lambda\|(A - \lambda)x\| \) and \( A \) lies in \( U_0 \).

Recall that \( A \in \mathcal{L}(H) \) is dominant if for all \( \lambda \in \mathbb{C} \), there exists a real number \( M_\lambda \geq 1 \) such that \( \|(A - \lambda)x\| \leq M_\lambda\|(A - \lambda)x\| \) and \( A \) lies in \( U_0 \).

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