EXTRACTA MATHEMATICAE Vol. **21**, Núm. 3, 211 – 220 (2006)

Tensor Product and Local Interior G-Algebras

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(Presented by A. Turull)

AMS *Subject Class.* (2000): 20C20 *Received December* 12, 2006

0. Introduction

G-algebra is a very important working object in the modern representation theory of finite groups, on which Puig set up his well-known theory of pointed groups, especially, many working objects in the representation theory of finite groups can be regarded as G-algebras, or more explicitly, as interior G-algebras (e.g. [11]).

Tensor product is a long story in mathematics, by which authors have discovered so many interesting results in many branches of mathematics.

In [10], K*ü*lshammer obtained some properties on the tensor product of some indecomposable modules, in [5] Harris generalized [10, Proposition 2.1] to the lattices over the complete discrete valuation ring. In [1], Aglhamdi and Khammash studied the tensor module of G-algebras and have achieved some important results on tensor product of Brauer homomorphism; moreover, Khammash analyzed the points, pointed groups and their defect groups in tensor algebra of G-algebras in [9].

In this paper, we devote to studying the (outer) tensor algebra of G-algebras (see §2) and the (inner) tensor algebra of G-algebras (see §3), especially, we concentrate on the defect groups of them and the relationship between block algebras and interior G-algebras under tensor product; additionally, we have discussed the centralizer of the subalgebra consisting of G-fixed elements in an interior G-algebra (see §4).

1. Preliminaries

In this paper, F is always an algebraically closed field of characteristic p, where p is a prime integer. We assume all groups are finite, all algebras and modules are finitely dimensional ones. By a G-algebra (A, ϕ) we mean a F-algebra A with a group homomorphism $\phi : G \to Aut(A)$, where $Aut(A)$ denotes the group of F-algebra automorphisms of A, hence G acts naturally on A by ${}^g a := \phi(g)(a)$, $a \in A$. An interior G-algebra (A, ρ) is a F-algebra A with a F-algebra homomorphism $\rho : FG \to A$ such that $\rho(1_{FG}) = 1_A$, which becomes a G-algebra with the G-action defined by ${}^g a{=}\rho(g)a\rho(g)^{-1}$ for any $a \in A$, $g \in G$, and we also denote this G-algebra with (A, ρ); moreover, it becomes a FG-module by extending linearly this G-action, we denote this FG-module with ${}_G A$. Sometimes we also simply say A is an interior G-algebra instead of (A, ρ) whenever there exists no confusion; we do the same for G-algebras.

For any $H \le G$, we always write A^H for the subalgebra consisting of H-fixed elements of G-algebra (A, ϕ); moreover, $Tr_H^G(A^H)$, also denoted by A_H^G, means the relative trace ideal of A^G, where $Tr_H^G(\cdot)$ is the well-known relative trace functor mapping a in A^H on $\sum_{g \in G/H}({}^g a)$ (e.g. [4]). The same holds for FG-modules (e.g. [3]).

An interior G-algebra (A, ρ) is called an epimorphic one if ρ is epimorphic, and is called a local one (in [11], which is also called a primitive one) if A^G is a local algebra; in the latter case, we define the minimal subgroups H of G such that $1_A \in A_H^G$ as the defect groups of A, which are p-subgroups of G and unique in the sense of G-conjugation, hence we denote any one of them with $D(A)$. Particularly, every p-block algebra $B(= FGb$, where b is a central primitive idempotent of FG.) is an epimorphic local interior G-algebra in a canonical way.

Let V be a FG-module. For a p-subgroup P of G, the P-relative Brauer map of V is the natural map $Br_P^V : V^P \to V(P) := V^P \big/ \sum_{Q \lneq P} V_Q^P$, where $V_Q^P = Tr_Q^P(V^Q)$ (e.g. [3]). For some subgroup H of G, we say V being H-projective if V is a direct summand of $Ind_H^G Res_H^G V$, and then $V(P) = 0$ unless $P \le_G H$ ([3, Proposition 1.3]); moreover, if V is indecomposable, we say the minimal subgroups H such that V is H-projective, which are mutually conjugate p-subgroups in G, the vertices of V, and we denote any one of them with $Vtx_G(V)$ ([11, Theorem 18.3]).

Definition 1.1. Let (A, ρ) be an interior G-algebra, we call (A, ρ) belongs to some block $B(= FGb)$ of G if $\rho(b) = 1_A$.

Remark 1.2. In the case of local interior G-algebras, this definition is just the one in [8]. It is easy to see that if (A, ρ) belongs to some block of G, it belongs to a unique block, and every block algebra of group algebra FG belongs to itself, hence this definition is reasonable; moreover, this generalization makes sense (see §4, Proposition 4.2).

EXAMPLE 1.3. If M is a FG-module, then the endomorphism ring $E(M) := End_F(M)$ is an interior G-algebra by a natural way and hence a G-algebra defined by ${}^g f = g \cdot f \cdot g^{-1}$ for all $g \in G$ and $f \in E(M)$. Obviously, $E(M)$ is a local interior G-algebra if and only if M is an indecomposable FG-module, in this case $E(M)$ belongs to a block B of G if and only if M belongs to B, and by Higman's criterion $D(E(M)) = Vtx_G(M)$ under G-conjugation.

2. (OUTER) TENSOR G-ALGEBRA

(A_i, ϕ_i) is a G_i-algebra, $i = 1, 2$; in this paper, their (outer) tensor $G_1 \times G_2$-algebra, which is denoted by $(A_1 \otimes A_2, \phi_1 \otimes \phi_2)$, is defined by the following way

$$(\phi_1 \otimes \phi_2)(g_1, g_2) = \phi_1(g_1) \otimes \phi_2(g_2)$$

for any $(g_1, g_2) \in G_1 \times G_2$, where and in the sequal, $\otimes$ always means $\otimes_F$, that is, the tensor product of F-algebras or that of F-modules; more generally, we define similarly the (outer) tensor product of modules over group algebras. Obviously, the (outer) tensor algebra of interior G_i-algebra, $i = 1, 2$, is also an interior $G_1 \times G_2$-algebra.

LEMMA 2.1. *Let i, j be idempotents in F-algebras A and B, respectively. Then $i \otimes j$ is an idempotent in $A \otimes B$; furthermore, $i \otimes j$ is primitive if and only if i and j are primitive in A and B, respectively.*

Proof. See [5, Corollary 3.3]. ∎

LEMMA 2.2. *Let (A_i, ρ_i) be an interior G_i-algebra and B_i a block of G_i, where G_i is a finite group, $i = 1, 2$. Then $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ belongs to $B_1 \otimes B_2$ if and only if (A_i, ρ_i) belongs to B_i, $i = 1, 2$.*

Proof. First of all, we recall that $B_1 \otimes B_2$ is a block of $F(G_1 \times G_2) \cong FG_1 \otimes FG_2$. Let $B_i = FG_i b_i$, where b_i is the block idempotent of B_i, $i = 1, 2$, we have $B_1 \otimes B_2 = (FG_1 \otimes FG_2)(b_1 \otimes b_2)$ with $1_{B_1 \otimes B_2} = b_1 \otimes b_2$.

If (A_i, ρ_i) belongs to B_i, $i = 1, 2$, we obtain $\rho_i(b_i) = 1_{A_i}$, $i = 1, 2$, hence $(\rho_1 \otimes \rho_2)(b_1 \otimes b_2) = \rho(b_1) \otimes \rho(b_2) = 1_{A_1 \otimes A_2}$, that is, $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ belongs to $B_1 \otimes B_2$.

Conversely, Let $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ belongs to $B_1 \otimes B_2$, that is,

$$(\rho_1 \otimes \rho_2)(b_1 \otimes b_2) = 1_{A_1 \otimes A_2},$$

hence $\rho(b_1) \otimes \rho(b_2) = 1_{A_1} \otimes 1_{A_2}$. Let $1_{FG_i} = b_{i1} + b_{i2} + \cdots + b_{i,j_i}$, a finite sum of block idempotents of FG_i with $b_{i1} = b_i$, $i = 1, 2$, we have

$$\rho(b_1) \otimes \rho(b_2) = 1_{A_1} \otimes 1_{A_2} = \rho_1(1_{FG_1}) \otimes \rho_2(1_{FG_2}) = \sum_{m=1}^{j_1} \sum_{n=1}^{j_2} \rho(b_{1m}) \otimes \rho(b_{2n}),$$

hence $\rho(b_{1m}) \otimes \rho(b_{2n}) = 0$ for all $m > 1$ or $n > 1$, and hence $\rho(b_{1m}) \otimes \rho(1_{FG_2}) = 0$ for all $m > 1$. Since $\rho(b_{1m}) \otimes a = (\rho(b_{1m}) \otimes a)(\rho(b_{1m}) \otimes \rho(1_{FG_2}))$ for all $a \in A_2$, we obtain $\rho(b_{1m}) \otimes A_2 = 0$ for all $m > 1$, and thus $\rho(b_1) = 1_{A_1}$, that is, (A_1, ρ_1) belongs to B_1; similarly, (A_2, ρ_2) belongs to B_2. ∎

THEOREM 2.3. *Let (A_i, ρ_i) be a local interior G_i-algebra with a defect group $D(A_i)$, and block $B_i(= FG_i b_i)$ of G_i with a defect group D_i, $i = 1, 2$. Then $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$, which is a local interior $G_1 \times G_2$-algebra with a defect group $D(A_1) \times D(A_2)$, belongs to $B_1 \otimes B_2$ if and only if A_i belongs to B_i, $i = 1, 2$. Additionally, if A_i is an epimorphic local interior G_i-algebra belonging to B_i, $i = 1, 2$, there exists some simple $F(G_1 \times G_2)$-module V such that*

$$Vtx_{G_1 \times G_2}(V) \leq D(A_1 \otimes A_2) \leq D_1 \times D_2,$$

under conjugation in $G_1 \times G_2$, and moreover, for any normal p-subgroup P_i of G_i, $i = 1, 2$, we have $P_1 \times P_2 \leq {}_{G_1 \times G_2} D(A_1 \otimes A_2)$.

Proof. Since (A_i, ρ_i) is a local interior G_i-algebra with a defect group $D(A_i)$, i=1, 2, we know that $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ is a local interior $G_1 \times G_2$-algebra with a defect group $D(A_1) \times D(A_2)$ and *vice versa*, by [1, Lemma 2.1], Lemma 2.1 and [9, Corollary 4.4]; hence by Lemma 2.2 we have completed the first part of Theorem 2.3.

Let's go on in the case of the additional assumptions. By [8, Lemma 2.8], there are V_1 belonging to $Irr(FG_1)$, that is, the set of all irreducible FG_1-modules, and V_2 belonging to $Irr(FG_2)$ such that $Vtx_{G_1}(V_1) \leq_{G_1} D(A_1) \leq_{G_1} D_1$ and $Vtx_{G_2}(V_2) \leq_{G_2} D(A_2) \leq_{G_2} D_2$; furthermore, by [7, Theorem 9.14] we obtain $V_1 \otimes V_2$ belongs to $Irr(F(G_1 \times G_2))$ and by [10, Proposition 1.2] we have

$$Vtx_{G_1\times G_2}(V_1 \otimes V_2) = {}_{G_1\times G_2}Vtx_{G_1}(V_1) \times Vtx_{G_2}(V_2).$$

Then

$$P_1 \times P_2 \leq Vtx_{G_1\times G_2}(V_1 \otimes V_2) \leq_{G_1\times G_2} D(A_1 \otimes A_2) \leq_{G_1\times G_2} D_1 \times D_2,$$

since it is well known that vertices of any simple FG-module contain any normal p-subgroup of G. Theorem 2.3 follows. ∎

THEOREM 2.4. *Let (A_i, ρ_i) be a local interior G_i-algebra with a defect group D_i, i=1, 2. Then $D_1 \times D_2$ is the maximal of vertices of indecomposable summands of ${}_{G_1\times G_2}A_1 \otimes A_2$, up to $G_1 \times G_2$-conjugation.*

Proof. Let ${}_{G_i}A_i = A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{i,n_i}$ be an indecomposable decomposition of A_i as FG_i-module, $i = 1, 2$. We have

$${}_{G_1\times G_2}A_1 \otimes A_2 = \bigoplus_{s=1}^{n_1} \bigoplus_{t=1}^{n_2} (A_{1s} \otimes A_{2t})$$

be an indecomposable decomposition of $A_1 \otimes A_2$ as $F(G_1 \times G_2)$-module, by [10, Proposition 1.1]. Hence

$$\begin{aligned}({}_{G_1\times G_2}A_1 \otimes A_2)(D_1 \times D_2) &= \bigoplus_{s=1}^{n_1} \bigoplus_{t=1}^{n_2} (A_{1s} \otimes A_{2t})(D_1 \times D_2)\\ &\simeq \bigoplus_{s=1}^{n_1} \bigoplus_{t=1}^{n_2} A_{1s}(D_1) \otimes A_{2t}(D_2),\end{aligned}$$

by [1, Theorem 2.6]. Since

$$({}_{G_1\times G_2}A_1 \otimes A_2)(D_1 \times D_2) = (A_1 \otimes A_2)(D_1 \times D_2) \neq 0$$

by Theorem 2.3 and [11, Corollary 18.6], there are $A_{1,s_0}$and A_{2,t_0} such that $A_{1,s_0}(D_1) \otimes A_{2,t_0}(D_2) \neq 0$, for some s_0 and t_0; hence $A_{1,s_0}(D_1) \neq 0$ and $A_{2,t_0}(D_2) \neq 0$. Then, by [3, Proposition 1.3], we have $D_1 \leq_{G_1} Vtx_{G_1}(A_{1,s_0})$ and $D_2 \leq_{G_2} Vtx_{G_2}(A_{2,t_0})$. However, since A_1 is D_1-projective by [6, Lemma 2.9], we have $A_{1,s}$ is also D_1-projective, that is, $D_1 \geq_{G_1} Vtx_{G_1}(A_{1,s})$ for any s, hence $D_1 =_{G_1} Vtx_{G_1}(A_{1,s_0})$. Similarly, $D_2 =_{G_2} Vtx_{G_2}(A_{2,t_0})$, and then

$$D_1\times D_2 =_{G_1\times G_2} Vtx_{G_1}(A_{1,s_0})\times Vtx_{G_2}(A_{2,t_0}) =_{G_1\times G_2} Vtx_{G_1\times G_2}(A_{1,s_0}\otimes A_{2,t_0}),$$

by [10, Proposition 1.2]. we have seen that, up to $G_1 \times G_2$-conjugation, $D_1 \times D_2$ is the maximal of vertices of indecomposable summands of $A_1 \otimes A_2$ as $F(G_1 \times G_2)$-module, by Krull-Schmidt Theorem. ∎

3. (Inner) tensor G-algebra

(A_1, ϕ_1) and (A_2, ϕ_2) are two G-algebras, in this paper, their (inner) tensor G-algebra means a G-algebra $(A_1 \otimes A_2, \phi)$ with

$$\phi(g) := \phi_1(g) \otimes \phi_2(g)$$

for any $g \in G$; similarly, we define the (inner) tensor product of FG-modules. It is clear that the (inner) tensor G-algebra of two interior G-algebras remains to be an interior G-algebra.

Remark 3.1. Obviously, the tensor product of G-algebras in [11] is just the (inner) tensor G-algebra of G-algebras here; generally, not like the (outer) tensor G-algebra, the (inner) tensor G-algebra of two local interior G-algebras does not remain to be a local one, whereas the following Proposition 3.2 gives us a surprise.

(A, ϕ) is a G/N-algebra, where $N \trianglelefteq G$, the inflated G-algebra of (A, ϕ) is a G-algebra $(inf(A), inf(\phi))$, where $inf(A) = A$, $(inf(\phi))(g) = \phi(gN)$ for all $g \in G$. Obviously, if (A, ρ) is an interior G/N-algebra, $(inf(A), inf(\rho))$ is also an interior G-algebra, and moreover, if (A, ρ) is a local one, so is $(inf(A), inf(\rho))$.

Proposition 3.2. *Let G be a finite group with a normal subgroup N and A be a G-algebra such that $Res_N^G(A)$ is a local N-algebra; C is a local G/N-algebra. Then the (inner) tensor G-algebra $A \otimes inf(C)$ is a local G-algebra.*

Proof. Since $(Res_N^G(A))^N$ is a local algebra, we have a decomposition

$$A^N = F \cdot 1_A \oplus J(A^N)$$

as F-module, hence as FG-module since $N \trianglelefteq G$; furthermore, since N acts trivially on $inf(C)$, we obtain

$$\begin{aligned}
\big(A \otimes inf(C)\big)^N &= \big(A \otimes inf(C)\big)^{N\times N} && \text{(as (outer) tensor } G\text{-algebra)}\\
&= A^N \otimes \big(inf(C)\big)^N && \text{(by [1, Lemma 2.1])}\\
&= A^N \otimes inf(C) \\
&= F \cdot 1_A \otimes inf(C) \oplus J(A^N) \otimes inf(C) && \text{(as } FG\text{-module)}
\end{aligned}$$

and $J(A^N) \otimes inf(C)$ is a nilpotent ideal of $(A \otimes inf(C))^N$. On the other hand, since $F \cdot 1_A$ is a G-subalgebra of A, it is easy to see that

$$f : F \cdot 1_A \otimes inf(C) \to inf(C)$$

is an isomorphism of G-algebras by the following way

$$f(t \cdot 1_A \otimes c) = t \cdot c$$

for any $t \in F$ and any $c \in C$, hence

$$\big(F \cdot 1_A \otimes inf(C)\big)^G = \big(F \cdot 1_A \otimes inf(C)\big) \cap \big(A \otimes inf(C)\big)^G$$

is a local algebra. Then

$$\big(A \otimes inf(C)\big)^G = \big(F \cdot 1_A \otimes inf(C)\big)^G \oplus \big(J(A^N) \otimes inf(C)\big)^G$$

has only one idempotent since

$$\big(J(A^N) \otimes inf(C)\big)^G = \big(J(A^N) \otimes inf(C)\big) \cap \big(A \otimes inf(C)\big)^G$$

is a nilpotent ideal in $(A \otimes inf(C))^G$ and $(F \cdot 1_A \otimes inf(C))^G$ is a local algebra, that is, we have known that $A \otimes inf(C)$ is a local G-algebra. ∎

By Proposition 3.2 we see that for two F-algebras A and C, $A \otimes C$ is a local F-algebra if and only if both A and C are.

THEOREM 3.3. *In the case of Proposition* 3.2, *set D is a defect group of the local G-algebra $A \otimes inf(C)$, where D is some p-subgroup of G. Then DN/N is a defect group of C as a G/N-algebra.*

Proof. Since $A \otimes inf(C)$ is a local G-algebra with a defect group D by Proposition 3.2, let $1_{A \otimes inf(C)} = Tr_D^G(d)$, where $d \in (A \otimes inf(C))^D$, hence

$$Tr_D^{DN}(d) \in \big(A \otimes inf(C)\big)^{DN} \subseteq \big(A \otimes inf(C)\big)^N.$$

Since

$$\big(A \otimes inf(C)\big)^N = A^N \otimes inf(C) = F \cdot 1_A \otimes inf(C) \oplus J(A^N) \otimes inf(C),$$

we have $Tr_D^{DN}(d) = 1_A \otimes i + j$ for some $i \in inf(C)$ and $j \in J(A^N) \otimes inf(C)$; moreover, since $N \trianglelefteq G$ and $F \cdot 1_A \otimes inf(C) \simeq inf(C)$ as G-algebras, it is easy to see that $i \in inf(C)^{DN}$ and $j \in (A \otimes inf(C))^{DN}$. Then,

$$1_{A \otimes inf(C)} = Tr_D^G(d) = 1_A \otimes Tr_{DN}^G(i) + Tr_{DN}^G(j),$$

where $Tr_{DN}^{G}(j) \in J((A \otimes inf(C))^{G})$ since j is a nilpotent element; hence $1_A \otimes Tr_{DN}^{G}(i) \notin J((A \otimes inf(C))^{G})$, and hence $1_A \otimes Tr_{DN}^{G}(i)$ is a unit in $(A \otimes inf(C))^{G}$ since $(A \otimes inf(C))^{G}$ is a local algebra. Then $Tr_{DN}^{G}(i)$ is a unit in $(inf(C))^{G}$, that is, $Tr_{DN/N}^{G/N}(i) = Tr_{DN}^{G}(i)$ is a unit in $C^{G/N}$, and it yields that the local G/N-algebra C is DN/N-projective; therefore $DN/N \geq H/N$, where $DN \geq H \geq N$ for some subgroup H of G such that H/N is a defect group of C as a G/N-algebra; we have that $inf(C)$ is H-projective, and $A \otimes inf(C)$ is also H-projective, by [11, Lemma 14.3]. It follows that H contains a defect group of $A \otimes inf(C)$, *i.e.* a conjugation of D, hence $H = DN$ and $H/N = DN/N$. we are done. ∎

Corollary 3.4. *Let G be a finite group with a normal subgroup N and A be a G-algebra such that $Res_N^G(A)$ is a local N-algebra. Then, if D is a defect group of A, DN/N is a Sylow p-subgroup of G/N.*

Proof. In the case of Theorem 3.3, Let $C = F$, the trivial G/N-algebra. We have DN/N is a defect group of F as a trivial G/N-algebra, hence DN/N is a Sylow p-subgroup of G/N. ∎

Remark 3.5. In the case of Example 1.3, Let M_1 be an indecomposable FG-module such that $Res_N^G(M_1)$ remains to be indecomposable, where N is a normal subgroup of G, and let M_2 be an indecomposable $F(G/N)$-module. Then $Res_N^G E(M_1)$ is a local interior N-algebra and $E(M_2)$ is a local interior G/N-algebra. We see that $E(M_1 \otimes inf(M_2))$ remains to be a local interior G-algebra by Proposition 3.2, that is, $M_1 \otimes inf(M_2)$ is an indecomposable FG-module. Moreover, if D is a vertex of $M_1 \otimes inf(M_2)$ as FG-module, that is, a defect group of $E(M_1 \otimes inf(M_2))$, DN/N is a vertex of M_2, by Theorem 3.3; especially, let $M_2 = F$, the trivial $F(G/N)$-module, we have $Vtx(M_1)N/N$ is a Sylow p-subgroup of G/N, by Corollary 3.4. Hence, we have generalized [10, Proposition 2.1, Proposition 2.2].

4. Centralizer of the G-fixed elements subalgebra

(A, ρ) is an interior G-algebra, the centralizer $C_A(A^G)$ of A^G in A is a subalgebra of A, with the same identity 1_A. It is easy to see that $\rho(FG) \subseteq C_A(A^G)$, hence we have an interior G-subalgebra $(C_A(A^G), \rho)$ of (A, ρ) by inheriting the structure of (A, ρ), with the following property

$$\big(C_A(A^G)\big)^G = Z(A^G) = Z\big(C_A(A^G)\big).$$

In this paper, we say that (A, ρ) is connected if $(C_A(A^G), \rho)$ is a local interior G-algebra, or equivalently, if $Z(A^G)$ is a local algebra; obviously, every local interior G-algebra is connected and if we restrict to the epimorphic interior G-algebras, the connected one is just the local one.

EXAMPLE 4.1. Under the notation of Example 1.3, let M be a FG-module, which can be regarded as an $E(M)$-module. We have

$$\varepsilon(M) := End_{End_{FG}(M)}(M) = C_{E(M)}\big(E(M)^G\big),$$

which is an interior G-subalgebra of $E(M)$. In [2], L. Barker calls M a connected module if $\varepsilon(M)$ is a local interior G-algebra, that is, $E(M)$ is a connected interior G-algebra. Obviously every indecomposable FG-module is connected.

PROPOSITION 4.2. *Every connected interior G-algebra (A, ρ) belongs to some block B of G. In this case, we have $D(C_A(A^G)) \leq_G D(B)$.*

Proof. Let $B_i = FGb_i$, $i = 1, 2, \ldots, n$, be block algebras of G such that $\sum_{i=1}^n b_i = 1_{FG}$, $n \geq 1$. It is easy to see that $\rho(b_i) \in C_A(A^G) \cap A^G = Z(A^G)$, and since $Z(A^G)$ has only one idempotent 1_{FG}, there is only one b_{i_0} such that $\rho(b_{i_0}) = 1_{FG}$ for some $1 \leq i_0 \leq n$ and $\rho(b_i) = 0$ for any other i, that is, (A, ρ) belongs to b_{i_0}. Since $\rho(B_{i_0}) \subseteq C_A(A^G)$, we have $D(C_A(A^G)) \leq_G D(\rho(B_{i_0}))$ by [1, Proposition 4.2], and since $(\rho(B_{i_0}), \rho)$ is an epimorphic local interior G-algebra belonging to B_{i_0}, we have $D(\rho(B_{i_0})) \leq_G D(B_{i_0})$ by [8, Lemma 2.8], hence $D(C_A(A^G)) \leq_G D(B_{i_0})$; we are done. ∎

COROLLARY 4.3. *Let (A_i, ρ_i) be a connected interior G_i-algebra and belong to the block algebra B_i of G_i, $i = 1, 2$. Then the (outer) tensor $G_1 \times G_2$-algebra $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ is also a connected interior $G_1 \times G_2$-algebra belonging to $B_1 \otimes B_2$. Moreover,*

$$D\big(C_{A_1 \otimes A_2}\big((A_1 \otimes A_2)^{G_1 \times G_2}\big)\big) =_{G_1 \times G_2} D\big(C_{A_1}(A_1^{G_1})\big) \times D\big(C_{A_2}(A_2^{G_2})\big).$$

Proof. Since

$$\begin{aligned} C_{A_1 \otimes A_2}\big((A_1 \otimes A_2)^{G_1 \times G_2}\big) &= C_{A_1 \otimes A_2}(A_1^{G_1} \otimes A_2^{G_2}) \\ &= C_{A_1}(A_1^{G_1}) \otimes C_{A_2}(A_2^{G_2}), \end{aligned}$$

we have

$$Z\big(C_{A_1\otimes A_2}\big((A_1\otimes A_2)^{G_1\times G_2}\big)\big) = Z\big(C_{A_1}(A_1^{G_1})\otimes C_{A_2}(A_2^{G_2})\big)$$
$$= Z\big(C_{A_1}(A_1^{G_1})\big)\otimes Z\big(C_{A_2}(A_2^{G_2})\big),$$

and since $Z(C_{A_1}(A_1^{G_1}))$ and $Z(C_{A_2}(A_2^{G_2}))$ are local algebras,

$$Z\big(C_{A_1\otimes A_2}\big((A_1\otimes A_2)^{G_1\times G_2}\big)\big)$$

is also a local algebra, by Lemma 2.1. Hence $(A_1\otimes A_2, \rho_1\otimes\rho_2)$ is a connected interior $G_1\times G_2$-algebra belonging to some block of $G_1\times G_2$ by Proposition 4.2, and moreover, it is clear that $(A_1\otimes A_2, \rho_1\otimes\rho_2)$ belongs to $B_1\otimes B_2$, by Lemma 2.2. Then

$$D\big(C_{A_1\otimes A_2}\big((A_1\otimes A_2)^{G_1\times G_2}\big)\big) = D\big(C_{A_1}(A_1^{G_1})\otimes C_{A_2}(A_2^{G_2})\big)$$
$$=_{G_1\times G_2} D\big(C_{A_1}(A_1^{G_1})\big)\times D\big(C_{A_2}(A_2^{G_2})\big),$$

by Theorem 2.3. ∎

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