Substructures of Algebras with Weakly non-Negative Tits Form

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Abstract: Let \( A = kQ/I \) be a finite dimensional basic algebra over an algebraically closed field \( k \) presented by its quiver \( Q \) with relations \( I \). A fundamental problem in the representation theory of algebras is to decide whether or not \( A \) is of tame or wild type. In this paper we consider triangular algebras \( A \) whose quiver \( Q \) has no oriented paths. We say that \( A \) is essentially sincere if there is an indecomposable (finite dimensional) \( A \)-module whose support contains all extreme vertices of \( Q \). We prove that if \( A \) is an essentially sincere strongly simply connected algebra with weakly non-negative Tits form and not accepting a convex subcategory which is either representation-infinite tilted algebra of type \( \tilde{E}_p \) or a tubular algebra, then \( A \) is of polynomial growth (hence of tame type).

Key words: tame representation type, essentially sincere module, Tits form, strongly simply connected algebra.

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Let \( A \) be a finite dimensional algebra (associative with unity) over an algebraically closed field \( k \). We may assume that \( A \) has a presentation \( A \cong kQ/I \) where \( kQ \) is the path algebra of the Gabriel quiver \( Q = Q_A \) of \( A \) and \( I \) is an admissible ideal of \( kQ \). Equivalently, \( A = kQ/I \) may be considered as a \( k \)-category with objects the vertices of \( Q \) and the space of morphism \( A(x, y) \) from \( x \) to \( y \) as the quotient of the space \( kQ(x, y) \), generated by the paths from \( x \) to \( y \), by the subspace \( I(x, y) = kQ(x, y) \cap I \). We denote by \( \mod A \) the category of finite dimensional right \( A \)-modules. For basic background from representation theory of algebras we refer to [1, 4, 22, 23, 24].

From Drozd’s Tame and Wild Dichotomy Theorem [10], algebras may be divided into two disjoint classes: the tame algebras for which indecomposable modules in each dimension occur (up to isomorphism) in a finite number of one-parametric families, and the wild algebras for which the representation

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theory comprises the representation theories of all algebras. One central question in the modern representation theory of algebras is the determination of the representation type.

Let $A = kQ/I$ be a triangular algebra, that is, $Q$ has no oriented cycles. The Tits form $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is the quadratic form defined by

$$q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i, j} r(i, j)v(i)v(j),$$

where $r(i, j)$ is the cardinality of $R \cap I(i, j)$ for a minimal set of generators $R \subset \bigcup_{i, j} I(i, j)$ of $I$. The Tits form plays an important role in the problem of determining the representation type of $A$. Indeed, if $A$ is representation-finite (that is, $A$ accepts, up to isomorphism, only finitely many indecomposable modules), then $q_A$ is weakly positive, that is, $q_A(v) > 0$ for $0 \neq v \in \mathbb{N}^{Q_0}$ [5]. More generally, if $A$ is tame, then $q_A$ is weakly non-negative, that is, $q_A(v) \geq 0$ for $v \in \mathbb{N}^{Q_0}$ [15]. The converse implications have been shown for important families of algebras, satisfying some rigidity conditions (see for example [5, 6]), or algebras of small homological dimensions [3, 9, 11, 12, 15, 21, 28].

A thoroughly studied class of tame algebras are the strongly simply connected algebras. We recall that $A$ is said to be strongly simply connected if, for every convex subcategory $B$ of $A$, the first Hochschild cohomology group $H^1(B)$ vanishes, [26]. The modules over polynomial growth strongly simply connected algebras have been completely described [27] (see also [13] and [16]) and the critical tame strongly simply connected algebras of non-polynomial type have been classified [14]. It is a long standing conjecture that a strongly simply connected algebra $A$ is tame if and only if $q_A$ is weakly non-negative. The present paper answers positively the conjecture in a special case, generalizing previous results by the authors [17, 19]. This special case is shown to be essential for the solution of the conjecture as presented in [7].

We say that a strongly simply connected algebra $A = kQ/I$ is essentially sincere if there is an indecomposable (finite dimensional) $A$-module $X$ whose support $\text{supp} \ X = \{ i \in Q_0 : X(i) \neq 0 \}$ contains all extreme vertices (sinks and sources) of $Q$. Observe that a strongly simply connected algebra $A$ is tame if and only if every convex subcategory $B$ of $A$ which is essentially sincere is tame. The main result of the paper is the following:

**Theorem.** Let $A$ be a triangular algebra satisfying the following conditions:

(a) $A$ is essentially sincere strongly simply connected;
(b) $q_A$ is weakly non-negative;

(c) $A$ contains a convex subcategory which is either representation-infinite tilted algebra of type $\tilde{E}_p$ ($p = 6, 7$ or $8$) or a tubular algebra.

Then $A$ is either a tilted algebra or a coil algebra. In particular, $A$ is of polynomial growth, hence it is tame.

The paper is organized as follows. In Section 1 we present some remarks on essentially present modules, that is, indecomposable modules $X$ such that $\text{supp } X$ contains all the extreme vertices of the quiver of the algebra. In Section 2 we recall concepts and results needed for the proof of the Theorem. The proof presented in Section 3 depends heavily on the arguments given in [17, 19].

1. Essentially present modules

1.1. Let $A = kQ/I$ be a finite dimensional $k$-algebra. For each vertex $i \in Q_0$, we denote by $e_i$ the corresponding primitive idempotent of $A$, hence $P_i = e_iA$ is the projective cover of the simple module $S_i = e_iA/e_i \text{rad } A$ and $I_i = DAe_i$ the injective envelope of $S_i$. By $D = \text{Hom}_k(-, k)$ we denote the usual duality on $\text{mod } A$.

For a module $X \in \text{mod } A$, $i \in \text{supp } X$ if $\text{Hom}_A(P_i, X) \neq 0$ (equivalently, $\text{Hom}_A(X, I_i) \neq 0$). We say that $X$ is omnipresent (resp. essentially present) if $\text{supp } X = Q_0$ (resp. each source or sink in $Q$ belongs to $\text{supp } X$). Clearly, $X$ is essentially present if and only if for every simple projective $A$-module $S$ we have $\text{Hom}_A(S, X) \neq 0$ and for every simple injective $A$-module $T$ we have $\text{Hom}_A(X, T) \neq 0$.

We consider the Grothendieck group $K_0(A) = \mathbb{Z}^{Q_0}$ and the classes $\text{dim } X = (\text{dim}_k X(i))_{i \in Q_0}$ of modules $X \in \text{mod } A$. We recall that the homological form defined by Ringel [22] for algebras $A$ of finite global dimension is given by

$$\langle \text{dim } X, \text{dim } Y \rangle_A = \sum_{s=0}^{\infty} (-1)^s \text{dim}_k \text{Ext}^s_A(X, Y).$$

1.2. We denote by $\Gamma_A$ the Auslander-Reiten quiver of $A$ with translation $\tau_A = D\text{Tr}$. By a component of $\Gamma_A$ we mean a connected component. The structure of preprojective, preinjective and tubular components may be seen in [1, 22, 23, 24]. A path in $\text{mod } A$ is a sequence $X_0 \to X_1 \to \cdots \to X_t$ of non-zero non-isomorphisms between indecomposable $A$-modules; it is a cycle if $X_0$ and $X_t$ are isomorphic.
We say that an indecomposable \( A \)-module \( X \) is directing if it does not belong any cycle in \( \text{mod} \ A \).

Given a component \( C \) of \( \Gamma_A \) we say that \( C \) is convex in \( \text{mod} \ A \) if any path \( X_0 \to X_1 \to \cdots \to X_t \) in \( \text{mod} \ A \) with extremes \( X_0 \) and \( X_t \) in \( C \), has all \( X_i \in C, \ i = 1, \ldots, t \). We shall consider also the support \( \text{supp} C := \bigcup_{X \in C} \text{supp} X \) of \( C \).

**Proposition.** Let \( A = kQ/I \) be a triangular algebra and let \( X \) be an essentially present indecomposable \( A \)-module in a component \( C \) of \( \Gamma_A \).

(a) If \( i \notin Q_0 \setminus \text{supp} X \), then there is a cycle in \( \text{mod} \ A \) passing through \( X \) and \( S_i \).

(b) If \( C \) is convex in \( \text{mod} \ A \), then \( \text{supp} C = Q_0 \).

**Proof.** (Following [5]) (a) Assume \( i \notin Q_0 \setminus \text{supp} X \). Since \( X \) is essentially present and \( A \) is triangular, there is a path \( \gamma \) of the form \( i_0 \to i_1 \to \cdots \to i_s \) in \( Q \) with \( i_0, i_s \in \text{supp} X \) and \( i_1, \ldots, i_{s-1} \notin \text{supp} X \) with \( i = i_t \) for some \( 1 \leq t \leq s - 1 \). Let \( \bar{A} \) be the quotient of \( A \) by all paths \( x \xrightarrow{\alpha} y \xrightarrow{\beta} z \) with exactly one arrow in \( \gamma \). Then there is a cycle in \( \text{mod} \ A \)

\[
X \to \bar{I}_s \to S_{i_s} \to \begin{pmatrix} i_s-1 \\ i_s \end{pmatrix} \to S_{i_{s-1}} \to \cdots \to \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \to S_{i_1} \to \bar{P}_0 \to X
\]

where \( \bar{P}_x \) (resp. \( \bar{I}_x \)) denotes the indecomposable projective (resp. injective) \( \bar{A} \)-module associated to \( x \) and \( \begin{pmatrix} x \\ y \end{pmatrix} \) is the indecomposable module of dimension two with socle \( S_y \) and top \( S_x \).

(b) Since \( X \in C \), by (a), for every \( i \notin Q_0 \setminus \text{supp} X \), the simple module \( S_i \) belongs to \( C \). Hence \( \text{supp} C = Q_0 \).

1.3. We recall that an algebra \( A \) is tame [10] if, for each \( d \in \mathbb{N} \), there is a finite number of \( k[t] - A \)-bimodules \( M_i, 1 \leq i \leq n_d \), which are finitely generated free as left \( k[t] \)-modules and such that all but finitely many isoclasses of indecomposable \( A \)-modules of dimension \( d \) are of the form \( k[t]/(t-\lambda) \otimes_{k[t]} M_i \) for some \( i \) and some \( \lambda \in k \). Let \( \mu_A(d) \) be the minimal \( n_d \) in the definition. Then \( A \) is said to be of polynomial growth [25] if there is a number \( m \) such that \( \mu_A(d) \leq d^m \) for every \( d \geq 1 \).

The following proposition on the behaviour of the Auslander-Reiten components of strongly simply connected algebras of polynomial growth has been proved in [27, Theorem 4.1].
Proposition. Let $A$ be a strongly simply connected algebra of polynomial growth. Then every component of $\Gamma_A$ is convex in mod $A$.

1.4. A useful construction is the one-point extension $B[M]$ of an algebra $B$ by a $B$-module $M$, given as the matrix algebra

$$B[M] = \begin{pmatrix} k & MB \\ 0 & B \end{pmatrix}.$$ 

One-point coextensions $[M]B$ are defined dually. The following extension of a result in [17] yields necessary conditions for an algebra to be essentially sincere.

Splitting Lemma. Let $A$ be a triangular algebra and $B = B_0, B_1, \ldots, B_s = A$ a family of convex subcategories of $A$ such that, for each $0 \leq i \leq s$ with $B_{i+1} = B_i[M_i]$ or $B_{i+1} = [M_i]B_i$ for some indecomposable $B_i$-module $M_i$. Assume that the category of indecomposable $B_i$-modules admits a splitting $\text{ind} B_i = P \vee J$, where $P$ and $J$ are full subcategories of $\text{ind} B_i$ satisfying the following conditions:

(S1) $\text{Hom}_{B_i}(J, P) = 0$;
(S2) for each $i$ such that $B_{i+1} = B_i[M_i]$, the restriction $M_{i,B}$ belongs to $\text{add} J$;
(S3) for each $i$ such that $B_{i+1} = [M_i]B_i$, $M_{i,B}$ belongs to $\text{add} P$;
(S4) there is an index $i$ with $B_{i+1} = B_i[M_i]$ and $M_i \in J$ and an index $j$ with $B_{j+1} = [M_j]B_j$ and $M_j \in P$.

Then $A$ is not essentially sincere.

Proof. Let $x_1, \ldots, x_r$ (resp. $y_1, \ldots, y_t$) be those vertices at the quiver $Q$ of $A$ being sources (resp. targets) or arrows with target (resp. source) in $B$. For each $i$, denote by $B_i^+$ the maximal convex subcategory of $B_i$ not containing any $y_1, \ldots, y_t$ (resp. $x_1, \ldots, x_r$). Let $P_i$ (resp. $J_i$) be the full subcategory of $\text{ind} B_i^-$ (resp. of $\text{ind} B_i^+$) consisting of modules $X$ such that $X|_B \in \text{add} P_i$ (resp. $X|_B \in \text{add} J_i$). We claim that $\text{ind} B_i = P_i \vee J_i$ and $\text{Hom}_{B_i}(J_i, P_i) = 0$.

The proof of the claim follows by induction as in [17, page 1022].

We get that $\text{ind} A = P_s \vee J_s$ with $\text{Hom}_A(J_s, P_s) = 0$, $P_s$ consists of $B_s^+$-modules and $J_s$ consists of $B_s^-$-modules. Moreover, by (S4), $B \neq B_s^+$ and $B \neq B_s^-$. Let $X \in P_s$ and let $y$ be a sink in $Q$ which is a successor of $y_1$. Since $B_s^+$ is convex in $A$, then $y$ is not in $B_s^+$, hence $X(y) = 0$. That is, $X$ is not essentially present. Similarly, any module $Y \in J_s$ is not essentially present. We conclude that $A$ is not essentially sincere.
Observe that, for a strongly simply connected algebra $A$ and a convex subcategory $B$ of $A$, there exists a chain $B = B_0, B_1, \ldots, B_s = A$ of convex subcategories of $A$ such that $B_{i+1} = B_i[M_i]$ or $B_{i+1} = [M_i]B_i$ for some indecomposable $B_i$-module $M_i$ (see [17, Proposition 2.2]).

1.5. The following are typical examples of strongly simply connected algebras $A$ and essentially present (not omnipresent) indecomposable $A$-modules $X$. (We indicate, the relations in $A$ by dotted edges: given $i \rightarrow j$, the sum of all paths from $i$ to $j$ in $Q$ is zero).

(1)

\[ A : \]

\[ X : \]

We note that, in the first case, $A$ is a tame concealed algebra, and hence is of polynomial growth.

On the other hand, in the second case, $A$ is a tame algebra of non-polynomial growth and there is an infinite family of pairwise nonisomorphic indecomposable $A$-modules $(Y_\lambda)_{\lambda \in k}$ with $\dim Y_\lambda = \dim X$.

**Proposition.** Let $A$ be a strongly simply connected algebra. Assume $v \in \mathbb{N}^{Q_0}$ is an essentially present vector which is not omnipresent and such that there exists an infinite family $(Y_\lambda)_{\lambda}$ of pairwise nonisomorphic indecomposable $A$-modules with $\dim Y_\lambda = v$. Then $A$ is not of polynomial growth.
Proof. Assume that $A$ is tame of polynomial growth. Since $A$ is tame, by a result of Crawley-Boevey [8], some module $Y$ in the family $(Y_\lambda)_\lambda$ satisfies $\tau_A Y \cong Y$, and hence lies in a stable tube $C$ of rank one in $\Gamma_A$. Further, since $A$ is of polynomial growth, applying 1.3, we conclude that $C$ is convex in mod $A$. Hence, applying 1.2, we obtain $\text{supp} C = Q_0$. Finally, since every module $X \in C$ has $\dim X = qv$ for certain rational number $q > 0$, we conclude that the vector $v$ is omnipresent, a contradiction.

2. Algebras of polynomial growth

2.1. Let $C$ be a tame concealed algebra, that is, $A = \text{End}_H(T)$ for a preprojective tilting module $T$ over a tame hereditary algebra $H$, and let $(T_\lambda)_{\lambda \in P_1(k)}$ be the unique family of stable tubes in $\Gamma_C$. Let $E = (E_1, \ldots, E_s)$ be a sequence of pairwise non-isomorphic $C$-modules which are simple among the regular modules and a family $K = (K_1, \ldots, K_s)$ of branches. In [22], the tubular extension $B = C[E, K]$ is defined and has tubular type $n_B = (n_\lambda)_\lambda$ with $n_\lambda = \text{rank} T_\lambda + \sum_{E_i \in T_\lambda} |K_i|$. Since almost all $n_\lambda = 1$, we write instead of $n_B = (n_\lambda)_\lambda$ the finite sequence consisting of at least two $n_\lambda$, keeping those which are larger than 1, and arranged in non-decreasing order.

We recall that $B$ is a domestic tubular (resp. tubular) algebra if $n_B$ is $(p, q)$, $1 \leq p \leq q$, (2, 2, r), $2 \leq r$, (2, 3, 3), (2, 3, 4), (2, 3, 5) (resp. (3, 3, 3), (2, 4, 4), (2, 3, 6) or (2, 2, 2, 2)).

The following fact is well known (see [15, 22]).

**Proposition.** Let $B$ be a tubular extension of a tame concealed algebra $C$. Then the following statements are equivalent:

(a) $B$ is tame;
(b) $B$ is domestic tubular or a tubular algebra;
(c) $q_B$ is weakly non-negative.

2.2. For the definitions of admissible operations and the construction of coils, we refer the reader to [2, 3].

Following [3], an algebra $B$ is said to be a coil enlargement of a tame concealed algebra $C$ if there is a finite sequence of algebras $C = B_0, B_1, \ldots, B_m = B$ such that $B_{j+1}$ is obtained from $B_j$ by an admissible operations (ad 1), (ad 2) or (ad 3) (resp. (ad 1*), (ad 2*), (ad 3*)) with a pivot (resp. a copivot) on a stable tube of $\Gamma_C$ or in a component of $\Gamma_{B_j}$ obtained from a stable tube.
of $\Gamma_C$ by a sequence of admissible operations done so far. By a coil algebra we mean a tame strongly simply connected algebra obtained as a coil enlargement of a tame concealed algebra.

The following structure result has been proved in [3].

**Proposition.** Let $B$ be a coil enlargement of a tame concealed algebra $C$. Then:

(a) There exists a unique maximal tubular extension $B^+$ of $C$ which is a convex subcategory of $B$ such that $B$ is obtained from $B^+$ as a sequence of algebras $B^+ = B_0, B_1, \ldots, B_m = B$ such that $B_{j+1}$ is obtained from $B_j$ by an admissible operation (ad 1*), (ad 2*) or (ad 3*) with a copivot on a coil component of $\Gamma_{B_j}$.

(b) There exists a unique maximal tubular coextension $B^-$ of $C$ which is a convex subcategory of $B$ such that $B$ is obtained from $B^-$ as a sequence of algebras $B^- = B_0, B_1, \ldots, B_n = B$ such that $B_{j+1}$ is obtained from $B_j$ by an admissible operation (ad 1), (ad 2) or (ad 3) with a pivot on a coil component of $\Gamma_{B_j}$.

(c) There is a splitting $\text{ind } B = \mathcal{P} \vee \mathcal{J}$, where $\mathcal{P}$ is formed by components of $\Gamma_{B^-}$ and some coils obtained by admissible operations as in (b), and $\mathcal{J}$ is formed by components of $\Gamma_{B^+}$. The splitting satisfies conditions (S1), (S2) and (S3) in 1.4. It satisfies (S4) if and only if $B^+$ is a proper subcategory of $B$ (equivalently $B^-$ is a proper subcategory of $B$).

(d) $B$ is tame if and only if $B^+$ and $B^-$ are tame.

As a consequence of the splitting of $\text{ind } B$ for a coil enlargement $B$ of a tame concealed algebra, we get the following result [27, 18].

**Proposition 2.3.** Let $A$ be a polynomial growth strongly simply connected algebra and $X$ be an essentially present indecomposable $A$-module. Then one of the following situations occur:

(a) $A$ is a tilted algebra of tame representation type, $X$ is a directing module and $q_A(\dim X) = 1$.

(b) $A$ is a coil algebra and $X$ belongs to a coil component of $\Gamma_A$. 
3. The proof of the Theorem

We start with some technical considerations.

**Proposition 3.1.** Let $A$ be an essentially sincere strongly simply connected algebra such that $q_A$ is weakly non-negative. Let $B \subset D = [X]B$ be two convex subcategories of $A$ such that $B$ is a coil enlargement of a tame concealed algebra $C$ and $X$ is an indecomposable module lying on a coil $\Gamma$ of $\Gamma_B$ such that $\text{Hom}_B(Z, X) \neq 0$ for a non-directing $Z$ in $\Gamma$. Then $D$ is either a coil algebra or $B^-$ is a tilted algebra of type $\tilde{D}_n$ with an indecomposable $Y$ in the preprojective component of $\Gamma_B$ satisfying $\dim_k \text{Hom}_B(Y, X) = 2$.

**Proof.** Let $F = B^-$ and $N$ be the restriction of $X$ to $F$. Then $[N]F$ is a convex subcategory of $D = [X]B$. By 2.1, $F$ is a domestic tubular or a tubular algebra which is a tubular extension of $C$. Assume, in order to get a contradiction, that $F$ is a tubular algebra. Then $X$ belongs to the inserted family of coils in $\Gamma_F$. If $X$ is copivoting, then $D = [X]B$ is a coil algebra. Suppose now that $X$ is not copivoting. We distinguish two situations.

Assume first that the support of $\text{Hom}_F(-, N)|_T$ contains the $k$-linear category of a subquiver $S$ of the component $T$ of $\Gamma_F$ with $N \in T$, where $S$ has the shape (1).

\[
\begin{array}{c}
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\end{array}
\]

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\begin{array}{c}
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\end{array}
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In this case, $F$ is a tubular extension of the tame concealed algebra $C$ of type $\tilde{D}_n$. Then there is a component $T' \neq T$ of $\Gamma_F$ containing projective modules. A simple application of the Splitting Lemma implies that $A$ is not essentially sincere, a contradiction. Since $X$ is not copivoting, then $\text{supp} \text{Hom}_F(-, N)|_T$ contains a $k$-linear category of a poset of type (2). If $C$ is of type $\tilde{D}_n$ we obtain a contradiction as above. Otherwise, $\text{Hom}_F(\text{mod} F, N)$
contains a full subcategory given by a poset

\[
\begin{array}{cccc}
\text{Hom}_F(Z_5, N) & \text{Hom}_F(Z_1, N) & \text{Hom}_F(Z_2, N) & \text{Hom}_F(Z_3, N) & \text{Hom}_F(Z_4, N)
\end{array}
\]

of type \((1, 1, 1, 2)\) where \(Z_1, Z_2\) lie in \(T\) and \(Z_3, Z_4, Z_5\) lie in the preprojective component of \(\Gamma_F\). Considering the coextension vertex \(t\) of \([N]F\), and the vector

\[
v = 4e_t + 2 \sum_{i=1}^{4} \dim Z_i + \dim Z_5 \in K_0([N]F)
\]
evaluating the Tits form \(q_{[N]F}\) at \(v\) (using that \(\text{gldim } F \leq 2\)) we get

\[
q_{[N]F}(v) = \langle v, v \rangle_F + 8 \sum_{i=1}^{4} \dim_k \text{Ext}^3_{[N]F}(Z_i, S_t) + 4 \dim_k \text{Ext}^3_{[N]F}(Z_5, S_t)
\]

\[
= -1 + 8 \sum_{i=1}^{4} \dim_k \text{Ext}^2_F(Z_i, N) + 4 \dim_k \text{Ext}^2_F(Z_5, N) = -1.
\]
The last equality due to the fact that \(\text{pdim}_F Z_i \leq 1\) for \(i = 3, 4, 5\) and \(\text{Ext}^2_F(Z_i, N) = 0, i = 1, 2\), from the structure of \(T\). This contradicts the weak non-negativity of \(q_A\) and shows that \(F\) is tilted of type \(\tilde{\mathbb{D}}_n\) or \(\tilde{\mathbb{E}}_p\) \((p = 6, 7\) or \(8)\).

If \(X\) is copivoting, then the vector space category \(\text{Hom}_B(\text{mod } B, X)\) is tame. Indeed, if it is not linear, say \(\dim \text{Hom}_B(M, X) \geq 2\) for an indecomposable \(B\)-module \(M\), then every object \(Y \in \Gamma_B\) is comparable with \(X\) (that is, there is \(0 \neq f \in \text{Hom}_B(X, Y)\) or \(0 \neq f \in \text{Hom}_B(Y, X)\) with \(\text{Hom}_B(f, X) \neq 0\)). Then \(F\) is tilted of type \(\tilde{\mathbb{D}}_n\) and \(M\) is preprojective in \(\Gamma_F\). Assume \(\text{Hom}_B(\text{mod } B, X)\) is linear.

If it is not of tame type, then it contains a full subposet \(L\) belonging to the Nazarova’s list. We identify each point \(a \in L\) with an indecomposable \(X_a\) in the preprojective component \(\mathcal{P}\) of \(\Gamma_F\). Moreover, since the orbit graph of \(\mathcal{P}\) is a tree (since \(A\) is strongly simply connected), we may choose \(L\) such that any subchain \(H\) yields a sectional path in \(\mathcal{P}\). Let \(v\) be a positive vector such that \(\chi_L(v) = -1\) for the graphical form \(\chi_L\) associated to \(L\) (see [22]). Then using that \(\text{gldim } D \leq 2\) we get

\[
q_D \left( \sum_{a \in L} v(a) \dim X_a + v(w)e_t \right) = \chi_L(v) = -1,
\]
for \( t \) the extension vertex of \( D \) such that \( I_t / \text{soc} I_t = X \). This leads to a contradiction with the weak non-negativity of \( q_A \), showing that \( \text{Hom}_B(\text{mod} B, X) \) is tame. Hence \( D \) is a tame coil enlargement of \( C \).

If \( X \) is not copivoting, then \( \text{supp} \text{Hom}_B(-, X)|_\Gamma \) contains one of the posets (1) or (2).

In the first case, as above, \( F = B^- \) is of type \( \tilde{\mathcal{D}}_n \). In the second case, if \( F \) is not of type \( \tilde{\mathcal{D}}_n \) we find a full subposet of \( \text{Hom}_F(\text{mod} F, X) \) of type \((1, 1, 1, 2)\) and, as above, we get a contradiction against the weak non-negativity of \( q_A \). In both cases, there is a preprojective module \( Y \) in \( \Gamma_F \) with \( \dim_k \text{Hom}_F(Y, X) = 2 \).

**Proposition 3.2.** Let \( A \) be an essentially sincere strongly simply connected algebra with \( q_A \) weakly non-negative. Let \( B \) be a convex subcategory of \( A \) satisfying the following conditions:

(i) \( B \) is a representation-infinite tilted algebra of type \( \tilde{\mathcal{E}}_p \) (\( p = 6, 7 \) or 8) having a complete slice in its preinjective component;

(ii) \( A \) admits not a convex subcategory of the form \([N]B\) for any indecomposable \( B \)-module \( N \);

(iii) for any convex subcategory \( B[M] \) of \( A \), \( M \) is an indecomposable preinjective \( B \)-module.

Then \( A \) is a tame tilted algebra.

**Proof.** We know that \( \Gamma_B \) consists of a preprojective component \( \mathcal{P} \), a family \( T_\lambda \) of inserted tubes and a preinjective component \( \mathcal{J} \) having a section of type \( \tilde{\mathcal{E}}_p \). We may choose \( \Sigma \) a section of \( \mathcal{J} \) such that any indecomposable \( M \) such that \( B[M] \) is a convex subcategory of \( A \), is a successor of \( \Sigma \) (in order of paths in \( \mathcal{J} \)).

Choose a sequence of categories \( B = B_0, B_1, \ldots, B_s = A \) such that \( B_{j+1} = B_j[M_j] \) or \( B_{j+1} = [M_j]B_j \) for an indecomposable \( B_j \)-module \( M_j \). We claim that for each \( j \), there is a component \( C_j \) in \( \Gamma_{B_j} \) satisfying:

(a) \( C_j \) is a directing component (that is, \( C_j \) is convex in \( \text{mod} B_j \) and without cycles);

(b) \( C_j \) has a complete slice \( \Sigma_j \) which is a tree.

In particular, this shows that \( A = B_s \) is a tilted algebra. Then, by [11], \( A \) is tame.

Indeed, \( C_0 = \mathcal{J} \) and \( \Sigma_0 = \Sigma \). Assume \( C_j \) is a directing component of \( \Gamma_{B_j} \) with a complete slice \( \Sigma_j \) such that any indecomposable \( M \), such that \( B[M] \)
is a convex subcategory of $A$, is a successor of $\Sigma_j$—maybe not in $C_j$ (observe that $\Sigma_j$ may be selected this way as an application of the Splitting Lemma).

Suppose $B_{j+1} = B_j[M_j]$ for an indecomposable. We claim that $M_j \in C_j$. Otherwise by the Splitting Lemma, there are no injective modules in $C_j$. Since $q_{B_j}$ is weakly non-negative, then $\Sigma_j$ is of extended Dynkin type and $j = 0$. In that case $C_0 = \mathcal{J}$ is a preinjective component, a contradiction showing that $M_j \in C_j$. By [20], $M_j$ lies in a directing component of $\Gamma_{B_{j+1}}$ with a (complete) slice $\Sigma_{j+1}$ which is a tree (extending $\Sigma_j$).

Suppose $B_{j+1} = [M_j]B_j$. By hypothesis, we have $M_j|B = 0$. If $M_j \notin C_j$, then the Splitting Lemma implies that $A$ is not essentially sincere as illustrated in the following picture:

Hence $M_j \in C_j$ and there should exists $\Sigma_j$ preceeding $M_j$ (apply Splitting Lemma again!). Then $M_j$ belongs to a directing component $C_{j+1}$ of $\Gamma_{B_{j+1}}$ with a complete slice $\Sigma_{j+1}$.

The case complementary Proposition 3.2 goes as follows:

**Proposition 3.3.** Let $A$ be an essentially sincere strongly simply connected algebra with $q_A$ weakly non-negative. Assume $A$ contains a full convex subcategory $B$ satisfying the conditions:

(i) $B$ is either a representation-infinite algebra of type $\tilde{E}_p$ ($p = 6, 7$ or $8$) with a complete slice in the preinjective component and some projective outside the preprojective component or $B$ is a tubular algebra;

(ii) there is a convex subcategory $A$ of the form $[N]B$ for some indecomposable $B$-module $N$.

Then $A$ is a coil algebra.
Proof. Choose $B$ maximal satisfying (i) and (ii). Let $D$ be a maximal coil enlargement of $B$ in $A$. We want to prove that $D = A$.

Let $\Gamma_D = P_\infty \lor C \lor J$ where $J_0$ is the preinjective component of $B$, $C = (C_\lambda)_{\lambda}$ is a family of coils such that, for certain $\lambda_0$, $C_{\lambda_0}$ contains a projective module and $P_\infty$ is formed by $D^-$-modules. By Proposition 2.3, $D^-$ is a tilted algebra or a tubular algebra.

Observe that the maximality of $B$ implies that $N \notin J_0$. Hence $N \in C$. The Splitting Lemma implies that $C_{\lambda_0}$ is the only component in $C$ that may contains projective or injective modules, and in fact contains both types (in particular, $N \in C_{\lambda_0}$). If $D$ is properly contained in $A$, then there is a convex subcategory $D'$ of $D$ of the form $D[X]$ or $[X]D$ for an indecomposable $D$-module $X$. Maximality of $B$ and the Splitting Lemma imply that $X \in C_{\lambda_0}$. Since $q_{D'}$ is weakly non-negative, $D'$ is a coil algebra by Proposition 3.1. Then $D' \subset D$ which is a contradiction. Therefore, $A = D$ is a coil algebra.

Proof of the Theorem. We may assume that $A$ admits a maximal proper convex subcategory $B$ which is a tubular extension of a tame concealed algebra $C$ and such that $B$ is either a tubular algebra or a representation-infinite tilted algebra of type $\tilde{E}_p$ ($p = 6, 7$ or $8$) having a complete slice in its preinjective component. Therefore, for any convex subcategory of $A$ of the form $B[M]$, $M$ is a preinjective $B$-module, since $q_B[M]$ is weakly non-negative, $M$ is not preprojective, and the maximality of $B$ and Proposition 3.1 imply that $M$ is not in a coil component). Hence the Splitting Lemma implies that $B$ is not a tubular algebra.

By the maximality of $B$ we may assume that either the hypothesis of Proposition 3.2 or those of Proposition 3.3 hold. Then either $A$ is a tilted algebra or a coil algebra.

References


