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Ball Proximinality of Closed * Subalgebras in $C(Q)$

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Abstract: The notion of ball proximinality and the strong ball proximinality were recently introduced in [2]. We prove that a closed * subalgebra $\mathcal{A}$ of $C(Q)$ is strongly ball proximinal in $C(Q)$ and the metric projection from $C(Q)$, onto the closed unit ball of $\mathcal{A}$, is Hausdorff metric continuous and hence has continuous selection.

Key words: Proximinal, ball proximinal, strongly ball proximinal, metric projection, lower Hausdorff semi-continuity, upper Hausdorff semi-continuity, continuous selection.

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1. Introduction

If X is a normed linear space, let $X_1 = \{x \in X : \|x\| \leq 1\}$, the closed unit ball of X. For x in X and $r > 0$, we set

$$B(x, r) = \{y \in X : \|x - y\| \ < r\}$$

and if A is a subset of X then the distance of x from the set A is denoted by $d(x, A)$. That is,

$$d(x, A) = \inf\{\|x - z\| : z \in A\}.$$

If A and B are bounded, nonempty subsets of a Banach space, we denote by $d_H(A, B)$ the Hausdorff metric distance between A and B, given by

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

By $C(Q)$, we denote the classical Banach space of all complex valued, continuous functions, defined on a compact, Hausdorff space Q, endowed with the

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sup norm. By a closed * subalgebra of $C(Q)$ we mean a closed subalgebra A of $C(Q)$ such that f is in A then $\bar{f}$, the complex conjugate of f, is also in A.

If C is a closed subset of X, we say C is proximinal in X if for every x in X, the set

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}$$

is a non-empty set. For any $\delta > 0$ we set

$$P_C(x, \delta) = \{z \in C : \|x - z\| < d(x, C) + \delta\}.$$

Motivated by a result of Saidi [9], the notion of ball proximinality was introduced in [2].

Definition 1.1. A subspace Y of a normed linear space X is ball proximinal in X if the closed, convex set Y_1 is proximinal in X.

It is easily verified (see [9] and [2]) that if Y is ball proximinal in X, then Y is proximinal in X. That the converse is not true, was shown relatively recently by a counterexample of Saidi, in [9]. Thus, ball proximinality implies proximinality, while the converse is not true.

Following [3], we say a proximinal set C of a normed linear space X is *strongly proximinal* if for each x in X and $\epsilon > 0$, there exists $\delta > 0$ such that

$$s(x, \delta) = \sup\{d(z, P_C(x)) : z \in P_C(x, \delta)\} \ < \ \epsilon. \tag{1.1}$$

Definition 1.2. A ball proximinal subspace Y of X is called strongly ball proximinal if Y_1 is strongly proximinal in X.

Let X be Banach space and x_0 be in X. We say the metric projection P_Y is *lower semi-continuous* at x_0 if, given $\epsilon > 0$ and z in $P_Y(x_0)$, there exists $\delta = \delta(\epsilon, z) > 0$ such that the set $B(z, \epsilon) \cap P_Y(x)$ is non-empty, for any x in $B(x_0, \delta)$. If δ can be chosen to be independent of z in $P_Y(x_0)$ in the above definition, that is, given $\epsilon > 0$, there exists $\delta > 0$ such that the set $B(z, \epsilon) \cap P_Y(x)$ is non-empty, for any x in $B(x_0, \delta)$ and any z in $P_Y(x_0)$, then we say P_Y is *lower Hausdorff semi-continuous* at x_0 . The map P_Y is *upper semi-continuous* at x_0 if given any open neighborhood U of zero in X, there exists $\delta > 0$ such that

$$P_Y(x) \subseteq P_Y(x_0) + U$$

for each x in $B(x_0, \delta)$. Replacing the arbitrary open set U by an open ball in the above, yields the notion of upper Hausdorff semi-continuity. More precisely, the map P_Y is *upper Hausdorff semi-continuous* at x_0, if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$P_Y(x) \subseteq P_Y(x_0) + \epsilon X_1$$

for each x in $B(x_0, \delta)$. We say P_Y is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous on X if it is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous at each point of X. The set valued map P_Y is said to be *Hausdorff metric continuous* if it is both lower and upper Hausdorff semi-continuous.

Remark 1.1. It immediately follows from the above definitions that if a subspace Y of a Banach space X is strongly proximinal, then the metric projection P_Y is upper Hausdorff semi-continuous on X. Also, we observe that while upper semi-continuity implies upper Hausdorff semi-continuity, the implication is the other way round for the corresponding lower semi-continuity concepts. We observe that for a proximinal subspace Y, upper semi-continuity of the metric projection does not imply the existence of a continuous selection for the set valued map P_Y but (by the Michael's selection theorem), lower semi-continuity of P_Y guarantees the existence of a continuous selection for P_Y.

There are many examples of Banach spaces which are proximinal in their second dual. For instance, it is known for long that (see [1], [4], [6] and [11]), $C(Q)$ is proximinal in its bidual. In this paper we show that, techniques used in [4] and [7], can be adapted to prove stronger proximinality and continuity properties of the corresponding metric projections, from $C(Q)$ onto the closed unit ball of a closed * subalgebra. More precisely, we prove that every closed * subalgebra $\mathcal{A}$ of $C(Q)$ is strongly ball proximinal (and hence ball proximinal). Further, we show that the metric projection from $C(Q)$ onto the closed unit ball of $\mathcal{A}$ is Hausdorff metric continuous and hence has a continuous selection. In particular this would imply that $C(Q)$ is strongly ball proximinal in its bidual and the metric projection from $(C(Q))^{**}$ onto $(C(Q))_1$ is Hausdorff metric continuous.

2. Preliminaries

We need some notations and definitions in the sequel, which are given below. Some of these definitions are from [7] and some others are slight modifications of definitions in [7].

We denote the space of complex scalars by $\mathbb{C}$ and the closed unit disc of $\mathbb{C}$ by D. That is, $D = \{z \in \mathbb{C} : |z| \le 1\}$. By $D(\beta, R)$, we denote the closed disc in $\mathbb{C}$ with center β and radius R.

Suppose F is a map from Q into compact subsets of $\mathbb{C}$. Define the distance of an $f \in C(Q)$ from F by the relation

$$\varrho(f, F) = \sup_{t\in Q} \sup_{y\in F(t)} |f(t) - y|. \tag{2.1}$$

Let $t \in Q$. By $r_1(t, F)$, we denote the restricted Chebychev radius of the set $F(t)$, with respect to D. That is,

$$r_1(t, F) = \inf_{z\in D} \sup_{y\in F(t)} |z - y|. \tag{2.2}$$

Finally we set,

$$R_1(F) = \sup\{r_1(t, F) : t \in Q\}. \tag{2.3}$$

We observe that for each $f \in C(Q)_1$, $\sup_{y\in F(t)} |f(t) - y| \ge r_1(t, F)$, for each $t \in Q$, therefore by (2.2) and (2.3) we get

$$\varrho(f, F) \ge r_1(t, F) \tag{2.4}$$

for each $f \in C(Q)_1$. Hence,

$$d(F, C(Q)_1) = \inf_{f\in C(Q)_1} \varrho(f, F) \ge R_1(F). \tag{2.5}$$

We define a set valued map H_F from Q into the class of closed convex subsets of D by

$$H_F(t) = \{z \in D : F(t) \subset D(z, R_1(F))\},\ t \in Q, \tag{2.6}$$

where F is a map from Q into compact subsets of $\mathbb{C}$. Here after we assume that the set valued map F is upper semi-continuous on Q.

The following property of uniformly convex spaces, given in Proposition 2 in [7], turns out to be relevant for our discussion. A simpler proof of this property for the space $\mathbb{C}$ is given below. Also, the selection of η turns out to be independent of the choice of α and β (See Lemma 2.1 below) in this case. This intersection ball property of $\mathbb{C}$ is repeatedly used in the proof of our main results.

LEMMA 2.1. *For fixed $R > 0$, and $\epsilon > 0$, let $\eta = (\epsilon(2R + \epsilon))^{\frac{1}{2}}$. Then for any α, β in $\mathbb{C}$, there exists $\gamma \in \mathbb{C}$ such that $|\alpha - \gamma| \le \eta$ and*

$$D(\alpha, R + \epsilon) \cap D(\beta, R) \subset D(\gamma, R).$$

Proof. We may assume, without loss of generality, that $\alpha = 0$, β is real. We further assume β is positive, the arguments being similar when β is negative. If $\beta \leq \eta$, we take $\gamma = \beta$. So we only consider the case, where $\beta > \eta$.

For any z in $\mathbb{C}$ and $r > 0$, let $S(z, r) = \{w \in \mathbb{C} : |z - w| = r\}$. Note that $S(\beta, R) \cap S(0, R + \epsilon) = \{z, \overline{z}\}$ for some z in $\mathbb{C}$ and that x, the real part of z, is positive and attains its minimum when $\beta = \eta$. It is easily verified, using a diagram, that

$$D(\beta, R) \cap D(0, R + \epsilon) \subseteq D(\eta, R) \cap D(0, R + \epsilon)$$

and γ can be chosen to be η. ∎

We now list some facts about the set valued map H_F. Lemma 2.2 was proved in [7], in a more general context. However, the proof for the complex valued maps that we deal with is simpler, and we present it here.

LEMMA 2.2. *For each t in Q, $H_F(t)$ is a non- empty, compact, convex subset of $\mathbb{C}$. Further the set valued map H_F is lower Hausdorff semi-continuous on Q.*

Proof. We shall first prove that the values $H_F(t)$ are non-empty. Select any t in Q and consider

$$H_F^{\eta}(t) = \{\beta \in D : F(t) \subset D(\beta, R_1(F) + \eta)\},$$

for $\eta > 0$. The set $H_F^{\eta}(t)$ is non-empty for each $\eta > 0$. Let $\{\beta_n\} \subseteq D$ be a sequence such that $F(t) \subseteq D(\beta_n, R_1(F) + \frac{1}{n})$, for all $n \geq 1$. Then $\{\beta_n\}$ has a convergent subsequence that converges to, say, β. We claim that β is in $H_F(t)$. Suppose not. Then there is a x in $F(t)$ such that $\|\beta - x\| \geq R_1(F) + \eta$, for some $\eta > 0$. Choose positive integer n such that $\frac{1}{n} < \frac{\eta}{2}$ and $\|\beta - \beta_n\| < \frac{\eta}{2}$. Then

$$\|\beta - x\| \leq \|\beta - \beta_n\| + \|\beta_n - x\| \leq \frac{\eta}{2} + R_1(F) + \frac{1}{n} < R_1(F) + \eta,$$

which is a contradiction to our assumption. Thus β is in $H_F(t)$ and the set $H_F(t)$ is non-empty.

That the set $H_F(t)$ is a closed, and hence, is a compact subset of D, follows from the fact that the set $F(t)$ is compact. It is easily verified that $H_F(t)$ is a convex set for each t in Q.

We now show that the set valued map H_F is lower Hausdorff semi-continuous. Fix t_0 in Q and select any β_0 in $H_F(t_0)$ and η, a positive number. Let $\epsilon > 0$

be so chosen that $(\epsilon(2R_1(F)+\epsilon))^{\frac{1}{2}} < \eta$. Since F is upper semi-continuous there exists a neighborhood U_ϵ of t_0 such that

$$F(t) \subset D(\beta_0, R_1(F)+\epsilon), \text{ if } t \in U_\epsilon. \tag{2.7}$$

Select any t in U_ϵ. We will show that $H_F(t) \cap B(\beta_0, \eta) \neq \emptyset$. Since the set U_ϵ is independent of β_0 in $H_F(t_0)$ and t_0 in Q was selected arbitrarily, this would imply H_F is lower Hausdorff semi-continuous on Q.

Pick an element β_1 in $H_F(t)$. Using (2.7), we have

$$F(t) \subset D(\beta_0, R_1(F)+\epsilon) \cap D(\beta_1, R_1(F)).$$

Now, by Lemma 2.1, there is a β in the line segment joining β_0 and β_1, such that $|\beta - \beta_0| \leq \eta$ and $D(\beta, R_1(F)) \supseteq F(t)$. Since β_0 and β_1 are in D, β is also in D. It is now clear that β belongs to $H_F(t)$ and so $\beta \in H_F(t) \cap B(\beta_0, \eta)$. ∎

We conclude this section with still another fact, that is needed in the sequel, about lower Hausdorff semi-continuous maps.

FACT 2.1. *Let f be in $C(Q)$ and δ, a positive number. Assume T is a lower Hausdorff semi-continuous set valued map from Q into the collection of closed, convex non-empty subsets of $\mathbb{C}$ such that $C_f(t) = D(f(t), \delta) \cap T(t)$ is non-empty for each t in Q. Then the set valued map C_f is lower Hausdorff semi-continuous on Q.*

Proof. Fix t in Q and select any w in $C_f(t)$. Since T is lower Hausdorff semi-continuous on Q and f is continuous on Q, given $\eta > 0$, we can select a neighborhood N of t (independent of w) such that

$$|f(s) - f(t)| < \frac{\eta}{2} \text{ and } T(s) \cap D(w, \eta/2) \neq \emptyset$$

for any s in N. Select any z in this intersection. Then

$$|w - z| \leq \frac{\eta}{2}. \tag{2.8}$$

Let s be in N and select z in $T(s)$ to satisfy (2.8). If z is in $C_f(s)$, we are done. Otherwise, $|z - f(s)| > \delta$. Let x be the point of intersection of the disk $D(f(s), \delta)$ and the line segment joining $f(s)$ and z. Clearly, $|x - f(s)| = \delta$.

Now, w is in $C_f(t)$ and so $|w - f(t)| \leq \delta$. Hence

$$|w - f(s)| \leq |w - f(t)| + |f(s) - f(t)| \leq \delta + \frac{\eta}{4}.$$

Since $|w-z| \leq \frac{\eta}{4}$, we have

$$|z - f(s)| \leq |z - w| + |w - f(s)| < \frac{\eta}{4} + \delta + \frac{\eta}{4} = \delta + \frac{\eta}{2}.$$

Now x lies in the line segment joining $f(s)$ and z and $|x - f(s)| = \delta$, we must have $|x - z| \leq \frac{\eta}{2}$. Therefore,

$$|x - w| \leq |x - z| + |z - w| \leq \eta. \tag{2.9}$$

Observe that $f(s) \in C_f(s) \subset T(s)$ and z is in $T(s)$. Since $T(s)$ is convex, x is in $T(s)$ and $|x - f(s)| = \delta$ and so x is in $C_f(s)$. Since s in N and $\eta > 0$ were chosen arbitrarily, this with (2.9), implies that C_f is lower Hausdorff semi-continuous at t. As t in Q was selected arbitrarily, this completes the proof. ∎

3. Main results

In this section we prove the main results of this paper. We show that if Q is compact, Hausdorff space, then any closed * subalgebra $\mathcal{A}$ of $C(Q)$ is strongly ball proximinal in $C(Q)$. Further, we prove that the metric projection from $C(Q)$ onto $\mathcal{A}_1$ is Hausdorff metric continuous.

We begin with some results that are needed in the sequel.

THEOREM 3.1. *Let Q be a compact, Hausdorff space. For each upper semi-continuous map F from Q into compact subset of $\mathbb{C}$ there exists a best approximation from $C(Q)_1$. That is, there exists an $f_0 \in C(Q)_1$ such that*

$$\varrho(f_0, F) = \inf_{f \in C(Q)_1} \varrho(f, F).$$

Moreover, for each such f_0 we have the equality $\varrho(f_0, F) = R_1(F)$.

Proof. We recall that the set valued map H_F is defined by (2.6). By Lemma 2.2, $H_F(t)$ is a compact, convex and non-empty subset of D for each t in Q and H_F is lower Hausdorff semi-continuous on Q. Now by the Michael selection theorem, there is a continuous selection f_0 of the set valued map H_F. We will show that $\varrho(f_0, F) = \inf_{f \in C(Q)_1} \varrho(f, F) = R_1(F)$ and hence, f_0 is a best approximation to F from $C(Q)_1$.

By (2.5)

$$R_1(F) \leq \inf_{f \in C(Q)_1} \varrho(f, F) \leq \varrho(f_0, F)$$

and by Lemma 2.2, $f_0(t) \in H_F(t)$ for each $t \in Q$. Now (2.6) and (2.1) imply that $\varrho(f_0, F) \le R_1(F)$ and so $\varrho(f_0, F) = R_1(F)$. It now follows from (2.5) that f_0 is a best approximation to F from $C(Q)_1$ and $R_1(F) = \inf_{f \in C(Q)_1} \varrho(f, F)$. ∎

Remark 3.1. If F from Q into subsets of $\mathbb{C}$ is upper semi-continuous map then it is clear from the above theorem that the distance of F from $C(Q)_1$ is $R_1(F)$ and f is a best approximation to F from $C(Q)_1$ if and only if f is a continuous selection of the set valued map H_F.

In what follows, we adhere to the notation given below. Throughout X and Y would denote compact Hausdorff spaces and $\phi : Y \to X$ be a continuous surjection. Define $T_\phi : C(X) \to C(Y)$ by $T_\phi(f) = f \circ \phi$ for $f \in C(X)$. Let $Z = T_\phi(C(X))$. Note that

$$\|f\| = \sup_{x \in X} |f(x)| = \sup_{y \in Y} |f(\phi(y))| = \sup_{y \in Y} |g(y)| = \|g\|$$

and hence f is in $C(X)_1$ if and only if g is in Z_1.

For h in $C(Y)$, define a set valued map $F = F_h$ on X by

$$F(x) = \{h(y) : y \in \phi^{-1}(x)\}, \text{ for } x \in X. \tag{3.1}$$

It is easily verified that $F(x)$ is a compact subset of $\mathbb{C}$ for each x in X. If g is in Z then $g = T_\phi(f)$, for some f in $C(X)$. Note that g is constant on $\phi^{-1}(x) = \{y \in Y; \phi(y) = x\}$ for every $x \in X$. Now

$$\begin{aligned}\|h - g\| &= \sup_{y \in Y} |h(y) - g(y)| \\ &= \sup_{x \in X} \sup_{y \in \phi^{-1}(x)} |h(y) - g(y)| \\ &= \sup_{x \in X} \sup_{s \in F(x)} |s - f(x)| \\ &= \varrho(f, F).\end{aligned} \tag{3.2}$$

Let

$$S_1(h) = \sup_{x \in X} \inf_{z \in D} \sup_{y \in \phi^{-1}(x)} |h(y) - z|. \tag{3.3}$$

Then clearly we have

$$S_1(h) = \sup_{x \in X} \inf_{z \in D} \sup_{s \in F(x)} |s - z| = \sup_{x \in X} r_1(x, F) = R_1(F). \tag{3.4}$$

It is easy to see that for each $h \in C(Y)$ and $g \in Z_1$,

$$\|h - g\| \geq S_1(h) \text{ and so } S_1(h) \leq d(h, Z_1). \tag{3.5}$$

We now have

THEOREM 3.2. *For each $h \in C(Y)$, there exists $g_0 \in Z_1$ such that*

$$\|h - g_0\| = d(h, Z_1) = S_1(h).$$

That is, Z is ball proximinal in $C(Y)$.

Proof. Since (3.5) holds, it is enough to show the existence of a $g_0 \in Z_1$ such that

$$\|h - g_0\| = S_1(h).$$

Let $F = F_h$ be given by (3.1). By Theorem 3.1 there exists $f_0 \in C(X)_1$ such that

$$\varrho(f_0, F) = R_1(F).$$

Let $g_0 = T_\phi(f_0)$. Then g_0 is in Z_1. Now using (3.2) and (3.4) we have,

$$\|h - g_0\| = \varrho(f_0, F) = R_1(F) = S_1(h) \tag{3.6}$$

and g_0 is a nearest element to h from Z_1. Hence Z is ball proximinal in $C(Y)$. ∎

Remark 3.2. We observe from the above Theorem 3.2 that $g = f \circ \phi$ in Z_1 is a nearest element to $C(Y)$ if and only if f is a nearest element to F from $C(X)_1$, where F is given by (3.1). Recall that by Remark 3.1, f is a best approximation to F from $C(X)_1$ if and only if $\rho(f, F) = R_1(F)$. Thus $g = f \circ \phi$ in Z_1 is a nearest element to h in $C(Y)$ if and only if $\|h - g\| = S_1(h)$ or equivalently $\varrho(f, F) = R_1(F)$.

We now proceed to show that Z is strongly ball proximinal in $C(Y)$ and the metric projection from $C(Y)$ onto Z_1 is lower Hausdorff semi-continuous.

THEOREM 3.3. *The space Z is strongly ball proximinal in $C(Y)$.*

Proof. Let h be in $C(Y)$ and $\epsilon > 0$ be given. Select $\delta > 0$ such that $3\delta(R_1(F) + 3\delta)^{\frac{1}{2}} < \epsilon$, where $F = F_h$ is given by (3.1). Let g be in Z_1 satisfy $\|h - g\| < d(h, Z_1) + \delta = S_1(h) + \delta$. We will show that there is a g_0 such that

$\|h - g_0\| = S_1(h)$ and $\|g - g_0\| < \epsilon$. By the above Remark 3.2, this would imply Z_1 is strongly ball proximinal. Let f in $C(X)_1$ satisfy $T_\phi(f) = g$. Then by (3.2) and (3.4),

$$\varrho(f, F) = \|h - g\| < S_1(h) + \delta = R_1(F) + \delta.$$

Hence for any z in $F(x)$ and $x \in X$, we have

$$\sup_{x \in X} |f(x) - z| < R_1(F) + \delta,$$

which in turn implies

$$D(f(x), R_1(F) + \delta) \supseteq F(x),$$

for each x in X. Fix x in X and select any α in $H_F(x)$. Now by Lemma 2.1, there is a s_x in complex plane such that $|s_x - f(x)| < \epsilon$, s_x lies in the line segment joining α and $f(x)$ and $D(s_x, R_1(F)) \supseteq F(x)$. Since both $f(x)$ and α lie in D, so does s_x and hence s_x is in $H_F(x)$. Thus $d(f(x), H_F(x)) < \epsilon$. If we set

$$C_f(x) = D(f(x), \epsilon/2) \cap H_F(x), \quad \text{for } x \in X,$$

then $C_f(x)$ is non-empty for each x in X and the set valued map C_f is lower Hausdorff semi-continuous by Lemma 2.2 and Fact 2.3. By Michael's selection theorem, C_f has a continuous selection, say, f_0. By Remark 3.2, f_0 is a best approximation to F from $C(X)_1$ and $\varrho(f_0, F) = R_1(F)$. Further $\|f - f_0\| < \epsilon$. Now let $g_0 = T_\phi(f_0)$. Then g_0 is in Z_1 and by (3.4),

$$\|h - g_0\| = \varrho(f_0, F) = R_1(F) = S_1(h)$$

and g_0 is a nearest element to h from Z_1. Also,

$$\|g - g_0\| = \|f - f_0\| < \epsilon$$

and this completes the proof. ∎

Theorem 3.4. *For any surjection map $\phi : Y \to X$ where X and Y are compact Hausdorff spaces and $Z = T_\phi(C(X))$, the metric projection from $C(Y)$ onto the closed unit ball of Z is lower Hausdorff semi-continuous and hence it has a continuous selection.*

Proof. Given $0 < \epsilon < 1$, let h_1, h_2 be elements in $C(Y)$ with $\|h_1 - h_2\| < \epsilon$. Then clearly,

$$d_H(F(x), G(x)) \le \epsilon, \text{ for all } x \in X, \tag{3.7}$$

where

$$F(x) = \{s \in D : s = h_1(y), y \in \phi^{-1}(x)\}$$

and

$$G(x) = \{t \in D : t = h_2(y), y \in \phi^{-1}(x)\}.$$

From the definition of $r_1(x, F)$, for any $\eta > 0$ there exists β in D such that

$$F(x) \subseteq D\Big(\beta, r_1(x, F) + \frac{\eta}{2}\Big).$$

Now (3.7) implies that

$$D\Big(\beta, r_1(x, F) + \epsilon + \frac{\eta}{2}\Big) \supseteq G(x).$$

Since $\eta > 0$ is arbitrarily chosen, this implies that $r_1(x, G) \le r_1(x, F) + \epsilon$. Interchange h_1 and h_2, we conclude, $r_1(x, F) \le r_1(x, G) + \epsilon$. Hence

$$|r_1(x, F) - r_1(x, G)| \leqq \epsilon \text{ for all } x \in X. \tag{3.8}$$

Clearly, (3.8) implies that

$$|R_1(F) - R_1(G)| \leqq \epsilon < 1 \text{ if } \|h_1 - h_2\| < \epsilon. \tag{3.9}$$

For x in X, we have $H_F(x) = \{\beta \in D : D(\beta, R_1(F)) \supseteq F(x)\}$. If β is in $H_F(x)$, then by (3.7), $D(\beta, R_1(F) + \epsilon)$ contains $G(x)$ and by (3.9), $D(\beta, R_1(G) + 2\epsilon)$ contains $G(x)$. Select any γ_0 in $H_G(x) = \{\beta \in D : D(\beta, R_1(G)) \supseteq G(x)\}$. Then

$$G(x) \subseteq D(\beta, R_1(G) + 2\epsilon) \cap D(\gamma_0, R_1(G)),$$

for all x in X. By Lemma 2.1, there exists γ in D such that

$$D(\gamma, R_1(G)) \supset G(x) \text{ and } |\beta - \gamma| \le \alpha, \tag{3.10}$$

where $\alpha = (\epsilon(2R_1(G) + \epsilon))^{\frac{1}{2}}$. Clearly, γ is in $H_G(x)$ and $\alpha \le (\epsilon(2R + \epsilon))^{\frac{1}{2}}$ where $R = R_1(F) + 1$. Let $\eta(\epsilon) = (\epsilon(2R + \epsilon))^{\frac{1}{2}}$. Then $\eta(\epsilon) > 0$, $\eta(\epsilon)$ decreases to zero as ϵ decreases to zero. Further, using (3.10), we have $|\beta - \gamma| < \eta(\epsilon)$. Since β in $H_F(x)$ and x is in X were arbitrary chosen, we have $H_F(x) \subset$

$H_G(x)+\eta(\epsilon)D$, for all x in X. Interchanging h_1 and h_2 in the above argument, we can conclude that if $\|h_1 - h_2\| < \epsilon$, then $H_G(x) \subset H_F(x) + \eta(\epsilon)D$ and

$$d_H(H_F(x), H_G(x)) < \eta(\epsilon), \quad \text{for all } x \in X. \tag{3.11}$$

We now show that the metric projection from $C(Y)$ onto Z_1 is lower Hausdorff semi-continuous on $C(Y)$. We fix $h_1 \in C(Y)$ and consider any g in $P_{Z_1}(h_1)$ and $\delta_0 > 0$. We will show that there is $\epsilon > 0$ such that if h_2 is in $C(Y)$ and $\|h_1 - h_2\| < \epsilon$, then $P_{Z_1}(h_2) \cap D(g, \delta_0) \neq \emptyset$. We recall that, by Remark 3.2, g is in $P_{Z_1}(h_1)$ if and only if there is a continuous selection f of the set valued map H_F, such that $g = f \circ \phi$. Clearly $f(x)$ is in $H_F(x)$ for all $x \in X$. Using (3.11), we choose $0 < \epsilon < 1$ such that for h_2 in $C(Y)$ satisfying $\|h_1 - h_2\| < \epsilon$, we have

$$d_H(H_F(x), H_G(x)) < \frac{\delta_0}{2} \quad \text{for all } x \in X. \tag{3.12}$$

Note that the choice of ϵ is independent of g in $P_{Z_1}(h_1)$. For x in X, we now set $C_f(x) = H_G(x) \cap D(f(x), \delta)$, where $2\delta = \delta_0$. It follows from (3.12) that $C_f(x) \neq \varnothing$ for all x in X. Further, by Fact 2.1, the set valued map C_f is lower Hausdorff semi-continuous on $C(Y)$.

By the Michael selection theorem, C_f has a continuous selection, say f_1. The map f_1 is a continuous selection of H_G and so by Remark 3.2, $g_1 = f_1 \circ \phi$ is in $P_{Z_1}(h_2)$. Also $\|f - f_1\| \leqq \delta < \delta_0$. This proves the lower Hausdorff semi-continuity of the metric projection map P_{Z_1}, at h_1. Since h_1 in $C(Y)$ was chosen arbitrarily, P_{Z_1} is lower Hausdorff semi-continuous on $C(Y)$. ∎

Let Y be a compact, Hausdorff space and $\mathcal{A}$ be a closed * subalgebra of $C(Y)$ containing the unit, that is the constant function 1. Then it is known that (see [8] and [10]) there is a compact, Hausdorff space X and a continuous surjection ϕ from Y onto X such that $\mathcal{A} = Z = T_\phi(C(X))$, where

$$T_\phi(f) = f \circ \phi, \text{ for } f \in C(X).$$

The following corollary follows from Theorems 3.3 and 3.4.

Corollary 3.1. *Every closed * subalgebra $\mathcal{A}$ of $C(Q)$, containing the unit, is strongly ball proximinal and the metric projection $P_{\mathcal{A}_1}$ is Hausdorff metric continuous.*

It is also known that $C(Q)^{**}$ is a $C(K)$ space, for a compact, Hausdorff space K and $C(Q)$ is a * subalgebra of $C(K)$, containing the unit. Now the corollary below is an immediate consequence of Corollary 3.1 above.

Corollary 3.2. *If Q is compact, Hausdorff, then $C(Q)$ is strongly ball proximinal in its bidual and the metric projection from $C(Q)^{**}$ onto $C(Q)_1$ is Hausdorff metric continuous.*

4. * SUBALGEBRAS WITHOUT UNIT

We now consider the case of closed * subalgebras without unit. Our methods here are motivated by those used in the proof of Theorem 2 in [4]. Let Y be a compact, Hausdorff space and $\mathcal{A}$ be any closed * subalgebra of $C(Y)$. Then it is known that (see [8] and [10]) there is a compact, Hausdorff space X, w in X and a continuous surjection ϕ from Y onto X such that $\mathcal{A} = T_\phi(C_0(X))$, where

$$T_\phi(f) = f \circ \phi, \text{ for } f \in C(X)$$

and

$$C_0(X) = \{f \in C(X) : f(w) = 0\}.$$

Let F be a map from X into compact subsets of $\mathbb{C}$. Then $r_1(x, F)$, for $x \in X$, $R_1(F)$, $\varrho(f, F)$ for f in $C_0(X) \subseteq C(X)$ and set valued map H_F, are defined by equations (2.1) to (2.6), in the beginning of Section 2. Further we set

$$r_0(F) = \sup_{z \in F(w)} |z|$$

and

$$\overline{R}_1(F) = \max\{R_1(F), r_0(F)\}. \tag{4.1}$$

We define the set valued map $\overline{H}_F$ from X into closed convex subset of D by

$$\overline{H}_F(x) = \begin{cases} H_F(x) & if \quad x \neq w \\ 0 & if \quad x = w \end{cases} \tag{4.2}$$

Now we have the following lemma, which replaces Lemma 2.2 in this part of the discussion.

Lemma 4.1. *For each x in X, $\overline{H}_F(x)$ is a non- empty, compact, convex subset of $\mathbb{C}$. Further the set valued map $\overline{H}_F$ is lower Hausdorff semi-continuous on X.*

Proof. Clearly $\overline{H}_F(x)$ is a non-empty, compact, convex subset of $\mathbb{C}$, follows from the corresponding statement for $H_F(x)$. Using Lemma 2.2, it is enough to show that $\overline{H}_F$ is lower Hausdorff semi-continuous at w. Fix η, a positive number. Let $\epsilon > 0$ be so chosen that $(\epsilon(2R_1(F) + \epsilon))^{\frac{1}{2}} < \eta$. Since F is upper semi-continuous there exists a neighborhood U_ϵ of w such that

$$F(x) \subset F(w) + \epsilon D, \text{ if } x \in U_\epsilon \tag{4.3}$$

We now discuss two cases.

Case i) $\overline{R}_1(F) = R_1(F)$. In this case $R_1(F) \geq r_0(F)$. Hence

$$F(w) \subseteq D(0, r_0(F)) \subseteq D(0, R_1(F))$$

and this with (4.3) implies

$$F(x) \subseteq D(0, R_1(F) + \epsilon), \quad \text{if } x \in U_\epsilon.$$

Now proceeding as in the proof of Lemma 2.2, we find β in D such that $|\beta| < \eta$ and β is in $F(x)$.

Case ii) $\overline{R}_1(F) = r_0(F)$. In this case, $r_0(F) \geq R_1(F)$. Thus for any x in U_ϵ and α in $F(x)$, we have

$$F(x) \subseteq D(\alpha, R_1(F)) \subseteq D(\alpha, r_0(F)).$$

Note that $F(w) \subseteq D(0, r_0(F)$ and using (4.3),

$$F(x) \subseteq D(0, r_0(F) + \epsilon), \quad \text{if } x \in U_\epsilon.$$

Now we again proceed as in the proof of Lemma 2.2, we find β in D such that $|\beta| < \eta$ and β is in $F(x)$.

Thus in either case, the map $\overline{H}_F$ is lower Hausdorff semi-continuous at w. Since $H_F(w) = \{0\}$, this implies the map is lower Hausdorff semi-continuous at w. ∎

We now prove the analog of Theorem 3.1.

THEOREM 4.1. *Let Q be a compact, Hausdorff space, $w \in Q$ and $C_0(Q) = \{f \in C(Q) : f(w) = 0\}$. If F is an upper semi-continuous map from Q into the set of compact subsets of $\mathbb{C}$, then*

$$d(F, C(Q)_1) = \inf\{\varrho(f, F) : f \in C_0(Q)_1\} = \overline{R}_1(F)$$

and there exists an $f_0 \in C_0(Q)_1$ such that

$$\varrho(f_0, F) = \overline{R}_1(F).$$

That is, f_0 is a best approximation to F from $C_0(Q)_1$.

Proof. First observe that

$$r_0(F) \leq \varrho(f, F), \text{ if } f \in C_0(Q)_1.$$

This with (2.5) implies that

$$\overline{R}_1(F) \leq \inf\{\varrho(f, F) : f \in C_0(Q)_1\} = d(F, C_0(Q)_1). \tag{4.4}$$

By Lemma 2.2, $\overline{H}_F$ is lower Hausdorff semi-continuous on Q, where the set valued map $\overline{H}_F$ is defined by (4.2). By the Michael selection theorem, there is a continuous selection f_0 of the set valued map $\overline{H}_F$. Note that $r_0(F) \geq r_1(w, F)$. It is now clear that $\varrho(f_0, F) = \overline{R}_1(F)$ and hence, f_0 is a best approximation to F from $C_0(Q)_1$. ∎

Let X and Y be compact, Hausdorff spaces. For h in $C(Y)$, define $F = F_h$ by (3.1) and $S_1(h)$ by (3.3). Let

$$S_0(h) = \sup_{y \in \phi^{-1}(w)} |h(y)|.$$

Define $\overline{S}_1(h) = \max\{S_1(h), S_0(h)\}$. It is easily seen that if $F = F_h$ then

$$S_0(h) = r_0(F) \text{ and } \overline{S}_1(h) = \overline{R}_1(F). \tag{4.5}$$

Let ϕ be a continuous surjection from Y onto X and define the map T_ϕ from $C_0(X)$ into $C(Y)$ by

$$T_\phi(f) = f \circ \phi, \text{ if } f \in C_0(X).$$

Then T_ϕ is an isometry and let $\mathcal{A} = T_\phi(C_0(X))$. Select any g in $\mathcal{A}$. Then there is a f in $C_0(X)$ such that $g = T_\phi(f)$. For $h \in C(Y)$ and $F = F_h$, we have

$$\begin{aligned} \|h - g\| &= \sup_{y \in Y} |h(y) - g(y)| \\ &= \sup_{x \in X} \sup_{y \in \phi^{-1}(x)} |h(y) - g(y)| \\ &= \sup_{x \in X} \sup_{s \in F(x)} |s - f(x)| \\ &= \varrho(f, F). \end{aligned} \tag{4.6}$$

and if g is in $\mathcal{A}_1$ then

$$\begin{aligned}\|h-g\| &= \sup_{y\in Y}|h(y)-g(y)| \\ &= \sup_{x\in X}\sup_{y\in\phi^{-1}(x)}|h(y)-g(y)| \\ &\geq \max\{\sup_{x\in X}\inf_{z\in D}\sup_{y\in\phi^{-1}(x)}|h(y)-z|, S_0(h)\} \\ &= \overline{S}_1(h).\end{aligned} \tag{4.7}$$

Hence

$$\overline{S}_1(h) \leq d(h,\mathcal{A}_1). \tag{4.8}$$

We now have

THEOREM 4.2. *For each $h\in C(Y)$, there exists $g_0\in\mathcal{A}_1$ such that*

$$\|h-g_0\| = d(h,\mathcal{A}_1) = \overline{S}_1(h).$$

That is, $\mathcal{A}$ is ball proximinal in $C(Y)$.

Proof. Because of (4.7), it is enough to show the existence of a $g_0\in\mathcal{A}_1$ such that

$$\|h-g_0\| = \overline{S}_1(h).$$

Let $F=F_h$ be given by (3.1). By Theorem 4.2 there exists $f_0\in C_0(X)_1$ such that

$$\varrho(f_0,F) = \overline{R}_1(F). \tag{4.9}$$

Let $g_0=T_\phi(f_0)$. Then g_0 is in $\mathcal{A}_1$. Now using (4.5) and (4.6) we have,

$$\|h-g_0\| = \varrho(f_0,F) = \overline{R}_1(F) = \overline{S}_1(h)$$

and g_0 is a nearest element to h from $\mathcal{A}_1$. Hence $\mathcal{A}$ is ball proximinal in $C(Y)$. ∎

Hereafter using similar arguments as in Theorems 3.3 and 3.4 we conclude

THEOREM 4.3. *Let X,Y and $\mathcal{A}$ be as above. Then $\mathcal{A}_1$ is strongly proximinal in $C(Y)$ and the metric projection $P_{\mathcal{A}_1}$ is Hausdorff metric continuous on $C(Y)$.*

If Q is a compact Hausdorff space and J a closed subspace of $C(Q)$. Then J is an M-ideal in $C(Q)$ (see [5]) if and only if there is a closed subset E of Q such that $J = \{f \in C(Q) : f \equiv 0 \text{ on } E\}$. ([8] and [10]) Recall that the second dual of $C(Q)$ is again a $C(K)$ space for a compact, Hausdorff K. Hence J is a closed * subalgebra of $C(K)$ (and C(Q)). The following corollary now follows from Theorem 4.4.

Corollary 4.1. *Let Q be a compact Hausdorff space and J, an M-ideal in $C(Q)$. Then J is strongly ball proximinal in $C(Q)^{**}$ and the metric projection from $C(Q)^{**}$ onto J_1 is Hausdorff metric continuous.*

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References

[1] Blatter, J., "Grothendieck Spaces in Approximation Theory, Memoirs of the A. M. S., 120, American Mathematical Society, Providence, R.I., 1972.

[2] Bandyopadhyay, P., Bor-Luh Lin, Rao, T.S.S.R.K., Ball proximinality in Banach spaces, Preprint 2006.

[3] Godefroy, G., Indumathi, V., Strong proximinality and polyhedral spaces, *Rev. Mat. Complut.*, **14** (1) (2001), 105-125.

[4] Holmes, R.B., Ward, J.D., An approximative property of spaces of continuous functions, *Glasgow Math. J.*, **15** (1974), 48-53.

[5] Harmand, P., Werner, D., Werner, W., "*M*-Ideals in Banach Spaces and Banach Algebras", Lecture Notes in Math., 1574, Springer-Verlag, Berlin, 1993.

[6] Mach, J., Best simultaneous approximation of bounded functions with values in certain Banach spaces, *Math. Ann.*, **240** (2) (1979), 157-164.

[7] Olech, C., Approximation of set-valued functions by continuous functions, *Colloq. Math.*, **19** (1968), 285-293.

[8] Rickart, C.E., "General Theory of Banach Algebras", Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York 1960.

[9] Saidi, Fathi B., On the proximinality of the unit ball of proximinal subspaces in Banach spaces: A counterexample, *Proc. Amer. Math. Soc.*, **133** (9) (2005), 2697-2703.

[10] Semadeni, Z., "Banach Spaces of Continuous Functions, Vol. 1", Monografie Matematyczne, 55, PWN–Polish Scientific Publishers, Warsaw, 1971.

[11] Yost, D.T., Best approximation and intersection of balls in Banach spaces, *Bull. Austral. Math. Soc.*, **20** (2) (1979), 285-300.