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On Operators that Preserve the Radon-Nikodým Property *

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Abstract: We consider a certain class $\mathcal{RN}_+$ of operators that preserve the Radon-Nikodým property. Conjugate operators in $\mathcal{RN}_+$ can be characterized as those operators T such that the kernel $N(T^*+K^*)$ has the Radon-Nikodým property for every compact operator K. A construction by J. Bourgain involving infinite convolution products of measures in the Cantor group provides examples of operators $T\colon L_1 \to L_1$ in the class $\mathcal{RN}_+$. As an application, we show the existence of Banach spaces which are $\mathcal{L}_1$-spaces, have the Radon-Nikodým property and contain infinite-dimensional reflexive subspaces.

Key words: Banach space, Radon-Nikodým property, Asplund space, $\mathcal{L}_1$-space, compact perturbation.

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1. Introduction

In the study of isomorphic properties of Banach spaces, it is sometimes useful to consider the operators that preserve those properties, other than the isomorphisms. For instance, in the factorization of Davis, Figiel, Johnson and Pełczyński [5] of an operator $T\colon X \to Y$, we obtain $T = jA$, where $j\colon E \to Y$ is a tauberian operator, and it has been shown [10] that $\overline{T(B_X)}$ is weakly compact (respectively, weakly precompact or separable) if and only if so is B_E; similarly, if $T\colon X \to Y$ is a tauberian operator, then Y is reflexive (respectively, contains no copies of ℓ_1 or is separable) if and only if the same happens to X.

In this line, several authors have studied those operators $T\colon X \to Y$ such that any operator $A\colon L_1 \to X$ is representable whenever so is the product

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TA; the class of these operators was denoted by $\mathcal{RN}_+$ in [8]. For example, Bourgain and Rosenthal showed [4, proof of Theorem 1.1'] that if $T\colon X \to Y$ is a semiembedding and X is separable, then $T \in \mathcal{RN}_+$ (see also [7]); as a consequence, if Y has the Radon-Nikodým property, then so has X. Also, Ghoussoub and Rosenthal showed [7, Theorem 2.6] that if T is a G_δ-embedding and TA is a Dunford-Pettis operator, then so is A, and asked [7, Problem II.3] if all G_δ-embeddings are in $\mathcal{RN}_+$, although later Ghoussoub and Maurey gave a negative answer by constructing examples of G_δ-embeddings from spaces failing the Radon-Nikodým property into ℓ_2 [6]. The class $\mathcal{RN}_+$ appears, too, in a construction by Bourgain: In [2], he shows that if an operator $T\colon L_1 \to L_1$ belongs to $\mathcal{RN}_+$ then the kernel $N(T)$ can be embedded in a $\mathcal{L}_1$-space with the Radon-Nikodým property. He uses this fact and the construction of a family of convolution operators in $\mathcal{RN}_+$ to prove that the class of all $\mathcal{L}_1$-spaces with the Radon-Nikodým property admits no universal element. In fact, if a $\mathcal{L}_1$-space contains an isomorphic copy of all $\mathcal{L}_1$-spaces with the Radon-Nikodým property, then it must also contain a subspace isomorphic to L_1. We refer to [11] for additional information on operators in $\mathcal{RN}_+$.

In this paper we describe some recent results on operators in $\mathcal{RN}_+$, including the main result in [8] which states that, given an operator $T\colon X \to Y$, the conjugate T^* belongs to $\mathcal{RN}_+$ if an only if for every compact operator $K\colon X \to Y$, the kernel $N(T^* + K^*)$ has the Radon-Nikodým property; or equivalently, for every compact operator $K\colon X \to Y$, the quotient space $Y/\overline{R(T+K)}$ is an Asplund space. Moreover, we apply Bourgain's construction of a family of convolution operators in $\mathcal{RN}_+$ to show the existence of $\mathcal{L}_1$-spaces with the Radon-Nikodým property that contain copies of ℓ_2.

2. A perturbative characterization for conjugate operators in $\mathcal{RN}_+$

In this section we describe the results that lead us to obtain a perturbative characterization for operators $T\colon X \to Y$ such that $T^* \in \mathcal{RN}_+$.

Recall that, denoting $L_1 \equiv L_1[0,1]$, an operator $A\colon L_1 \to X$ is said to be *representable* if there exists $g \in L_\infty(X)$ such that $Af = \int f(t)g(t)\,dt$ for all $f \in L_1$. A Banach space X has the *Radon-Nikodým property* if every operator $A\colon L_1 \to X$ is representable. Moreover, a Banach space X is *Asplund* if and only if every separable subspace of X has separable dual; equivalently, if and only if the dual space X^* has the Radon-Nikodým property. In [12], Pietsch defines the class of Radon-Nikodým operators $\mathcal{RN}$ as those operators $T\colon X \to$

Y such that the product TA is representable for any operator $A\colon L_1 \to X$; this class is an operator ideal [12, Theorem 24.2.7]. Pietsch also defines the class of decomposing operators $\mathcal{Q}$ [12, Section 24.4], that can be characterized as those $T\colon X \to Y$ for which $T^* \in \mathcal{RN}$; again, this class is an operator ideal.

Following the notation in [1], we can define the additional classes $\mathcal{RN}_+$ and $\mathcal{Q}_-$. As we said before, $\mathcal{RN}_+$ will be the class of those operators $T\colon X \to Y$ for which any operator $A\colon L_1 \to X$ is representable whenever so is the product TA; equivalently, $A\colon Z \to X$ is in $\mathcal{RN}$ whenever so is the product TA. The class $\mathcal{Q}_-$ will consist of those operators $T\colon X \to Y$ such that any operator $B\colon Y \to Z$ belongs to $\mathcal{Q}$ if $BT \in \mathcal{Q}$.

Our main result in this section states that an operator $T\colon X \to Y$ belongs to $\mathcal{Q}_-$ if and only if every perturbation $T + K$ with $K\colon X \to Y$ compact has an Asplund cokernel $Y/\,\overline{R(T+K)}$. We derive this result from the following, slightly stronger one:

Theorem 1. ([8, Theorem 15]) *Let $T\colon X \to Y$ be an operator such that $T^* \notin \mathcal{RN}_+$ and let $\lambda > 1$. Then there exist $\alpha > 0$ and a pair of biorthogonal sequences $(b_n)_{n\in\mathbb{N}} \subseteq Y$ and $(d_n)_{n\in\mathbb{N}} \subseteq Y^*$ such that, for every $n \in \mathbb{N}$,*

$$1/3 < \|b_n\| < 12/\alpha\,, \qquad \alpha/2 < \|d_n\| < 3\,, \qquad \|T^* d_n\| < \alpha/2^n$$

and $(d_n)_{n\in\mathbb{N}}$ is a λ-basic sequence whose span fails the Radon-Nikodým propery.

The idea behind this result is to build a bounded separated tree in Y^* whose difference will be the sequence $(d_n)_{n\in\mathbb{N}}$, which ensures that

$$\overline{\operatorname{span}}\,\{d_n \,:\, n \in \mathbb{N}\}$$

fails the Radon-Nikodým property, while keeping their images through T^* adequately controlled. We refer to [8] for the details.

We can now state the perturbative characterization for conjugate operators in $\mathcal{RN}_+$.

Theorem 2. ([8, Corollary 17]) *Let $T\colon X \to Y$ be an operator. Then the following are equivalent:*

(i) *$T^* \in \mathcal{RN}_+$;*
(ii) *$T \in \mathcal{Q}_-$;*
(iii) *$Y/\,\overline{R(T+K)}$ is Asplund for every compact operator K.*

3. Examples of $\mathcal{L}_1$-spaces

As we mentioned in the introduction, the class $\mathcal{RN}_+$ also appears in a construction by Bourgain. In [2], he proves that the class of separable $\mathcal{L}_1$-spaces not containing a copy of L_1 has no universal element. One of the key results in his proof is the following:

Theorem 3. ([3, Theorem 5.4 (b)]) *Let $S\colon L_1 \to L_1$ be an operator such that $S \in \mathcal{RN}_+$. Then $N(S)$ embeds isomorphically into a $\mathcal{L}_1$-space with the Radon-Nikodým property.*

This theorem is then applied to a family of operators which are found to be in $\mathcal{RN}_+(L_1, L_1)$ to obtain a variety of $\mathcal{L}_1$-spaces with the Radon-Nikodým property. These operators are interesting because they show a wide collection of highly non-trivial examples in $\mathcal{RN}_+(L_1, L_1)$.

The construction relies on the existence of a certain convolution operator $\Lambda\colon L_1(G) \to L_1(G)$. Here G is the Cantor group, only that, instead of the usual representation $\{-1,1\}^{\mathbb{N}}$, we use $G = \{-1,1\}^{\Xi}$, where Ξ is the infinite countably-branching set $\Xi = \bigcup_{n=0}^{\infty} \mathbb{N}^n$ (which is countable), taking $\mathbb{N}^0 = \{\emptyset\}$. By using this notation, we can define a partial ordering in Ξ by $(n_1,\dots,n_k) \preceq (n_1,\dots,n_k,m_1,\dots,m_l)$.

A tree in Ξ will be a subset $T \subseteq \Xi$ such that $\gamma \in T$ whenever $\gamma \preceq \xi \in T$; a branch in Ξ will be a linearly ordered tree. Branches can be either finite or infinite; a tree containing no infinite branches is said to be well-founded.

The key property of this convolution operator $\Lambda\colon L_1(G) \to L_1(G)$ is that $\Lambda(w_S) = w_S$ if S is a finite branch in Ξ, where w_S is the walsh function in $L_1(G)$ associated to S, that is, $w_S(x) = \prod_{\xi\in S} x_\xi$ [3, Proposition 5.11]. Also, if we take $T \subseteq \Xi$ to be a tree, we can consider the conditional expectation operator $\mathbb{E}_T\colon L_1(G) \to L_1(G)$ with respect to the sub-σ-algebra generated by the walsh functions $\{w_S : S \subseteq T \text{ finite}\}$ and define the operator $\Lambda_T = \mathbb{E}_T\Lambda$. It then follows [3, Proposition 5.12] that $I - \Lambda_T \in \mathcal{RN}_+$ for any well-founded tree $T \subseteq \Xi$.

The construction ressembles (in fact, extends) that of Rosenthal regarding convolution by biased coins, given in [13]. The results shown there can be recovered in an almost literal way from those of Bourgain, by choosing an appropriate tree $T \subseteq \Xi$.

The operators Λ_T can be exploited to find new examples of $\mathcal{L}_1$-spaces with the Radon-Nikodým property and a particular structure embedded, applying

Theorem 3, if we manage to devise a tree T such that the kernel $N(I - \Lambda_T)$ contains the required structure. For instance, this can be done to prove the following result.

Theorem 4. ([9]) *There exists a Banach space X which is a $\mathcal{L}_1$-space, has the Radon-Nikodým property and contains a subspace isomorphic to ℓ_2.*

Idea of the proof. Let $T \subseteq \Xi$ be the tree consisting of the root and the level of depth 1, that is,

$$T = \mathbb{N}^0 \cup \mathbb{N}^1 = \{\emptyset(1), (2), (3), \ldots\},$$

which is clearly well-founded, so the kernel $N(I - \Lambda_T)$ embeds isomorphically into a $\mathcal{L}_1$-space with the Radon-Nikodým property. Now, for each $n \in \mathbb{N}$, take the branch $T_n = \{\emptyset, (n)\}$; by the properties of Λ we cited, we have

$$\Lambda_T(w_{T_n}) = \mathbb{E}_T \Lambda(w_{T_n}) = \mathbb{E}_T(w_{T_n}) = w_{T_n}$$

so $w_{T_n} \in N(I - \Lambda_T)$. Finally, the set of Walsh functions $(w_{T_n})_{n\in\mathbb{N}}$ is equivalent to the sequence of Rademacher functions, so it generates a subspace isomorphic to ℓ_2 within $N(I - \Lambda_T)$, giving the desired result. ∎

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