EXTRACTA MATHEMATICAE Vol. **23**, Núm. 3, 243 – 253 (2008)

Normal Trajectories in Stationary Spacetimes under the Action of an External Field with Quadratic Asymptotic Behavior*

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Presented by Antonio M. Cegarra Received December 12, 2008

Abstract: The aim of this note is to study the existence of normal trajectories joining two given submanifolds under the action of an external field in a standard stationary spacetime. Here, it is assumed that both the growth of the potential and that one of the coefficients of the metric are critical in a suitable sense.

Key words: Standard stationary spacetime, normal trajectory, external field, quadratic growth, variational approach.

AMS *Subject Class.* (2000): 53C50, 53C22, 58E10.

1. Introduction

Taking a Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$, a function $V \in C^1(\mathcal{M}\times[0,T^*],\mathbb{R})$ $(T^* > 0)$, two submanifolds P, Q of $\mathcal{M}$ and an *arrival time* $T \in\ ]0,T^*]$, we use variational tools in order to find accurate conditions ensuring that P and Q can be connected by means of normal trajectories under the action of the potential V in time T, i.e., we seek for solutions of the equation

$$D_s^L \dot{z} + \nabla_L V(z,s) = 0 \quad \text{for all } \ s \in [0,T] \tag{1.1}$$

which satisfy the boundary conditions

$$\begin{cases} z(0) \in P, \ z(T) \in Q, \\ \dot{z}(0) \in T_{z(0)}P^{\perp}, \ \dot{z}(T) \in T_{z(T)}Q^{\perp}, \end{cases} \tag{1.2}$$

*Work supported by M.I.U.R. Research project PRIN07 “Metodi Variazionali e Topologici nello Studio di Fenomeni Nonlineari”.

where D_s^L denotes the covariant derivative along z induced by the Levi–Civita connection of $\langle \cdot, \cdot \rangle_L$ and $\nabla_L V(z, s)$ is the gradient of V with respect to z.

Clearly, this problem generalizes the study of the geodesic connectedness, that is the possibility of joining any two given points in $\mathcal{M}$ by geodesics, but, even in the simplest case, we are able to give only partial answers as the metric is not positive definite (cf. [17] or, for a recent survey, [11] and references therein). As we cannot carry on a general study, we limit our interest to a class of Lorentzian manifolds whose features make them appropriate to be handled from a variational point of view.

Definition 1.1. A spacetime $\mathcal{M}$ (i.e., a Lorentzian manifold with a time–orientation) is called *standard stationary* if it splits as a product $\mathcal{M}_0 \times \mathbb{R}$, where the connected finite dimensional manifold $\mathcal{M}_0$ is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ and the metric $\langle \cdot, \cdot \rangle_L$ is given by

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau - \beta(x) \tau \tau' \tag{1.3}$$

for any $z = (x, t) \in \mathcal{M}$ and $\zeta = (\xi, \tau)$, $\zeta' = (\xi', \tau') \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, where δ and β are respectively a smooth vector field and a smooth strictly positive scalar field on $\mathcal{M}_0$. In particular, $\mathcal{M}$ is named *standard static* if $\delta \equiv 0$.

Let us point out that this assumption is not too restrictive with respect to the general theory dealing with *stationary* spacetimes, i.e. spacetimes admitting a timelike Killing vector field K. In fact, not only locally any stationary spacetime looks like a standard one, but also a global splitting exists in some "good" geometric assumptions. More precisely, recalling that a spacetime is *globally hyperbolic* if it admits a (smooth) spacelike Cauchy hypersurface (i.e., a subset crossed exactly once by any inextendible timelike curve), it is known that a stationary spacetime $\mathcal{M}$ is standard if it is globally hyperbolic and admits at least one complete Killing vector field (see [8, Theorem 2.3]). By the way, in this general case the coefficients in (1.3) are not given *a priori* but depend on the Cauchy hypersurface and the complete Killing vector field which are considered. Even more, in a recent paper it is showed that a necessary and sufficient condition for $\mathcal{M}$ to split globally as a standard *conformastationary* spacetime with respect to a complete timelike conformal vector field K is to be *distinguishing* (see [14, Theorem 1.2]).

After many papers dealing with the geodesic connectedness in standard stationary spacetimes (see [11] and references therein), in the last years the study of geodesic connectedness has been set out in (general) stationary spacetimes by using an intrinsic approach but, firstly, introducing quite technical assumptions (see [13]) while, more recently, considering only geometric hypotheses such as the existence of a complete spacelike Cauchy hypersurface and a complete Killing vector field, which imply also that the spacetime is standard (see [8]).

More in general, the study of normal trajectories joining two given submanifolds P, Q under the action of a potential V has been developed only in standard stationary spacetimes so, here, we want to improve the known results but giving "good" growth assumptions on the coefficients δ, β of (1.3) and on the potential V. Indeed, even in this setting, our problem is interesting not only from a physical point of view, since these spacetimes represent time–independent gravitational fields (as, for example, the Kerr spacetime) but also from a mathematical one. In fact, from a variational viewpoint it is equivalent to find critical points of the action functional

$$f_V(z) = \frac{1}{2}\int_0^T \langle \dot z, \dot z\rangle_L ds - \int_0^T V(z,s) ds$$

on a suitable set of curves (for more details, see Proposition 2.1 in Section 2).

Clearly, when $V \equiv 0$ and $P = \{p\}$, $Q = \{q\}$ (with $p = (x_p, t_p)$, $q = (x_q, t_q) \in \mathcal{M}$), the given problem reduces to the study of geodesics joining p to q in $\mathcal{M}$ and, as geodesics are invariant by affine reparametrizations, the arrival time T between the fixed events is not relevant (in fact, in many related papers it is just $T = 1$). In this case, in the pioneer paper [12] the authors provide a variational principle in order to overcome the unboundedness of the action functional, so that looking for geodesics reduces to studying critical points of the new functional

$$J(x) = \int_0^T \langle \dot x, \dot x\rangle ds \ + \ \int_0^T \frac{\langle \delta(x), \dot x\rangle^2}{\beta(x)} ds \ - \ K_t^2(x) \ \int_0^T \frac{1}{\beta(x)} ds$$

in $\Omega^T(x_p, x_q)$, suitable set of curves joining x_p to x_q in a time T, where it is

$$K_t(x) = \left(\Delta_t - \int_0^T \frac{\langle \delta(x), \dot x\rangle}{\beta(x)} ds\right) \left(\int_0^T \frac{1}{\beta(x)} ds\right)^{-1}, \qquad (1.4)$$

with $\Delta_t = t_q - t_p$.

A couple of years ago, the main technical growth assumptions on δ and β, introduced in [12], have been weakened by far. Indeed, in [2] it is proved that a standard stationary spacetime is geodesically connected in the following hypotheses:

(H_1) the Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is complete and smooth (at least C^3),

(H_2) there exist $\mu_1, \mu_2 \geq 0$, $k_1, k_2 \in \mathbb{R}$ and a point $\bar{x} \in \mathcal{M}_0$ such that for all $x \in \mathcal{M}_0$ it is

$$0 < \beta(x) \leq \mu_1 d^2(x, \bar{x}) + k_1, \tag{1.5}$$

$$\sqrt{\langle\delta(x), \delta(x)\rangle} \leq \mu_2 d(x, \bar{x}) + k_2,$$

where, $d(\cdot,\cdot)$ denotes the distance canonically associated to the Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathcal{M}_0$.

Let us point out that, in the same hypotheses (H_1) and (H_2), a standard stationary spacetime $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is also globally hyperbolic (see [16, Corollary 3.4]).

More in general, if $V \equiv 0$ and $P = S_1 \times \{t_p\}$ and $Q = S_2 \times \{t_q\}$, in the hypothesis

(H_3) S_1 and S_2 are two embedded submanifolds in $\mathcal{M}_0$ such that both of them are closed as subsets of $\mathcal{M}_0$ while at least one of them is compact,

the existence of normal geodesics joining P to Q has been studied both in a stationary spacetime, if β is far away from zero and both δ and β have a sublinear growth (cf. [10]), and in a static one but in growth condition (1.5) (cf. [6]).

On the other hand, if $V \not\equiv 0$ and $S_1 = \{x_p\}$, $S_2 = \{x_q\}$, equation (1.1) has been studied not only in a Riemannian manifold (see [7]) but also both in a static and in a stationary one when potential V is time–independent, i.e.,

$$V(z, s) \equiv V(x, s) \qquad \text{for all } z = (x, t) \in \mathcal{M},\ s \in [0, T^*] \tag{1.6}$$

(see [1], respectively [3]). In all these cases, if V satisfies the assumption

(H_4) there exist $\lambda \geq 0$, $k \in \mathbb{R}$, $\bar{x} \in \mathcal{M}_0$ such that it is

$$V(x, s) \leq \lambda d^2(x, \bar{x}) + k \qquad \text{for all } x \in \mathcal{M}_0,\ s \in [0, T^*],$$

the existence of trajectories, which are solutions of (1.1), is not guaranteed (in fact, some counterexamples can be found, see Remark 1.4). Anyway, such solutions exist surely if the coefficient λ in (H_4) and the arrival time T are related by the further condition

$$\lambda T^2 < \frac{\pi^2}{2}. \tag{1.7}$$

In the more general setting, i.e. if S_1 and S_2 do not reduce to a singleton but (H_3) holds, some results on a Riemannian manifold $\mathcal{M}_0$ have been obtained in [5] up to assume a condition which is a little bit stronger than (H_4):

(H_4^*) there exist $\lambda \geq 0$, $k \in \mathbb{R}$ such that

$$V(x,s) \leq \lambda d^2(x,A) + k \qquad \text{for all } x \in \mathcal{M}_0,\ s \in [0,T^*],$$

choosing $A = S_1$ if S_2 is compact, or $A = S_2$ if S_1 is compact, where $d(x,A) = \inf_{y \in A} d(x,y)$.

Here, we consider problem (1.1) – (1.2) on a standard stationary spacetime when the potential V is non–trivial and independent of the *universal time* t (although it may even depend on the parameter s of the curve).

As previously remarked, a variational formulation entirely based on the Riemannian part of the spacetime can be stated; more precisely, the given problem reduces to find critical points of the functional

$$J_V(x) = \frac{1}{2}J(x) - \int_0^T V(x,s)ds \tag{1.8}$$

on $\Omega^T(S_1,S_2)$, suitable set of curves joining S_1 to S_2 in a time T (for the exact definition, see Section 2),

Thus, the main theorem can be stated as follows.

Theorem 1.2. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a standard stationary spacetime which satisfies hypotheses (H_1), (H_2) and let $V \in C^1(\mathcal{M} \times [0,T^*],\mathbb{R})$, $T^* > 0$, be a potential satisfying condition* (1.6)*. Moreover, let $P = S_1 \times \{t_p\}$ and $Q = S_2 \times \{t_q\}$ be two submanifolds of $\mathcal{M}$ with t_p, $t_q \in \mathbb{R}$ and S_1, S_2 submanifolds of $\mathcal{M}_0$ which satisfy (H_3). Then, if the potential V and the arrival time $T \in\ ]0,T^*]$ are such that (H_4^*) and* (1.7) *hold, P and Q can be joined by at least one normal trajectory which solves* (1.1) *and* (1.2).

Remark 1.3. If, in addition to the assumptions of Theorem 1.2, S_1 and S_2 are both contractible in $\mathcal{M}_0$, then a direct application of Ljusternik–Schnirelman Theory implies some multiplicity results either if $\mathcal{M}_0$ is non–contractible in itself (cf., e.g., [6, Proposition 3.6] and the related references) or if it is not (see [9, Theorem 3.7].

Remark 1.4. Even if both S_1 and S_2 are singleton, counterexamples can be construct both if hypothesis (H_4) holds, but (1.7) fails (see, e.g., [7, Example 3.6]) and if (H_2) fails (see [4, Section 7] for β that grows more than quadratically; see also [2, Example 2.7] when δ grows more than linearly).

2. Variational setting and abstract tools

Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ be a stationary spacetime with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and $\langle\cdot,\cdot\rangle_L$ as in (1.3), where $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold such that (H_1) holds. Moreover, let S_1 and S_2 be two submanifolds of $\mathcal{M}_0$ and fix t_p, $t_q \in \mathbb{R}$, so that it is $P = S_1 \times \{t_p\}$ and $Q = S_2 \times \{t_q\}$. Now, fixed $T > 0$, for simplicity assume $I = [0, T]$.

Here, our aim is working by means of variational tools. Thus, let us recall some basic definitions (for more details, cf. [11] and references therein).

It is known that there exists a closed embedding of any complete Riemannian manifold in $\mathbb{R}^N$ (cf. [15]). Hence, we can assume that $\mathcal{M}_0$ is a closed submanifold of $\mathbb{R}^N$, $\langle\cdot,\cdot\rangle$ is the restriction to $\mathcal{M}_0$ of the Euclidean metric of $\mathbb{R}^N$ and $d(\cdot,\cdot)$ is the corresponding distance, i.e.,

$$d(x_1, x_2) = \inf\left\{\int_a^b \sqrt{\langle\dot\gamma,\dot\gamma\rangle}ds :\ \gamma \in A_{x_1,x_2}\right\} \qquad \text{if } x_1,\ x_2 \in \mathcal{M}_0,$$

where $\gamma \in A_{x_1,x_2}$ if $\gamma : [a,b] \to \mathcal{M}_0$ is a piecewise smooth curve joining x_1 to x_2.

Thus, the manifold $H^1(I, \mathcal{M}_0)$ can be identified with the set of the absolutely continuous curves $x : I \to \mathbb{R}^N$ with square summable derivative such that $x(I) \subset \mathcal{M}_0$. Furthermore, since $\mathcal{M}_0$ is a complete Riemannian manifold with respect to $\langle\cdot,\cdot\rangle$, $H^1(I, \mathcal{M}_0)$ equipped with its standard Riemannian structure is a complete Riemannian manifold, too.

Let Z be the smooth manifold of all the $H^1(I, \mathcal{M})$–curves joining P to Q, while $\Omega^T(S_1, S_2)$ denotes the smooth submanifold of $H^1(I, \mathcal{M}_0)$ which contains all the curves joining S_1 to S_2 in a time T with

$$T_x\Omega^T(S_1,S_2) = \{\xi \in T_xH^1(I,\mathcal{M}_0) : \xi(0) \in T_{x(0)}S_1,\ \xi(T) \in T_{x(T)}S_2\}$$

for all $x \in \Omega^T(S_1, S_2)$. By the product structure of $\mathcal{M}$, it follows

$$Z \equiv \Omega^T(S_1,S_2) \times W^T(t_p,t_q) \quad \text{and} \quad T_zZ \equiv T_x\Omega^T(S_1,S_2) \times H_0^1$$

for each $z = (x, t) \in Z$, as

$$W^T(t_p,t_q) = \{t \in H^1(I,\mathbb{R}) : t(0) = t_p,\ t(T) = t_q\} = H_0^1 + j^*$$

is a closed affine submanifold of $H^1(I, \mathbb{R})$ with tangent space $T_tW^T(t_p, t_q) = H_0^1$, with

$$j^* : s \in I \mapsto t_0 + s\ \frac{\Delta_t}{T}\ \in \mathbb{R}, \quad H_0^1 = \{t \in H^1(I,\mathbb{R}) : t(0) = t(T) = 0\}.$$

Let us remark that, if S_1 and S_2 are closed, then the submanifold Z of $H^1(I, \mathcal{M})$ can be equipped with the Riemannian structure

$$\langle\zeta,\zeta\rangle_H = \int_0^T \langle D_s\xi, D_s\xi\rangle ds \ + \ \int_0^T \dot{\tau}^2 ds$$

for any $z = (x,t) \in Z$, $\zeta = (\xi,\tau) \in T_z Z$, and the submanifold $\Omega^T(S_1, S_2)$, hence Z, is complete.

Now, let $V = V(z,s)$ be a given potential on $\mathcal{M} \times [0, T^*]$ and consider $T < T^*$.

It is quite standard proving the following variational principle (for a hint, see the proof of [10, Proposition 2.1]).

Proposition 2.1. *A curve $z : I \to \mathcal{M}$ is a normal trajectory joining P to Q under the action of the potential V in the time T, i.e. it solves problem* (1.1)–(1.2), *if and only if $z \in Z$ is a critical point of the action functional f_V in the manifold Z.*

In our setting, from Proposition 2.1 and (1.3) it follows that we have to look for critical points of the (strongly unbounded) action functional

$$f_V(z) = \frac{1}{2}\int_0^T (\langle\dot{x},\dot{x}\rangle + 2\langle\delta(x),\dot{x}\rangle\dot{t} - \beta(x)\dot{t}^2)ds - \int_0^T V(z,s)ds \qquad (2.1)$$

for $z = (x,t) \in Z$. But, as in the problem of geodesic connectedness, a way to get over the lack of boundedness of f_V on Z can be to introduce a new functional which depends only on the Riemannian variable x. Clearly, such an approach is allowed if both the metric coefficients and the potential V are independent on the time component t.

Proposition 2.2. *Assume that the potential V satisfies condition* (1.6) *and consider $z^* = (x^*, t^*) \in Z$. The following statements are equivalent:*

(i) *z^* is a critical point of the action functional f_V defined in* (2.1)*;*

(ii) *x^* is a critical point of the functional $J_V : \Omega^T(S_1,S_2) \to \mathbb{R}$ defined in* (1.8) *and $t^* = \Psi(x^*)$, with $\Psi : \Omega^T(S_1,S_2) \to W^T(t_p,t_q)$ such that*

$$\Psi(x)(s) = t_0 + \int_0^s \frac{\langle\delta(x(\sigma)),\dot{x}(\sigma)\rangle}{\beta(x(\sigma))}d\sigma - K_t(x)\int_0^s \frac{1}{\beta(x(\sigma))}d\sigma$$

and $K_t(x)$ defined as in (1.4).

Moreover, for each $x \in \Omega^T(S_1, S_2)$ *and* $(\xi, \tau) \in T_x\Omega^T(S_1, S_2) \times H_0^1$ *it is*

$$f_V(x, \Psi(x)) = J_V(x) \qquad \text{and} \qquad J'_V(x)[\xi] = f'_V(x, \Psi(x))[(\xi, \tau)].$$

So, from now on, assume that the potential V satisfies the hypothesis (1.6). Hence, by Proposition 2.2 our problem is reduced to study the Riemannian functional J_V on $\Omega^T(S_1, S_2)$ and, in order to find at least one critical point, we can use the following classical abstract minimum theorem.

THEOREM 2.3. *Let* Ω *be a complete Riemannian manifold and* $\mathcal{J}$ *a* C^1 *functional on* Ω *which satisfies the Palais–Smale condition, i.e., any* $(x_k)_k \subset \Omega$ *such that*

$$(\mathcal{J}(x_k))_k \text{ is bounded} \quad \text{and} \quad \lim_{k \to +\infty} \mathcal{J}'(x_k) = 0$$

converges in Ω*, up to subsequences. Then, if* $\mathcal{J}$ *is bounded from below, it attains its infimum.*

3. Proof of Theorem 1.2

As already observed in Section 2, the functional J_V in (1.8) is C^1 on the Riemannian manifold $\Omega^T(S_1, S_2)$ which is complete if S_1 and S_2 are closed submanifolds of $\mathcal{M}_0$. Thus, in order to apply Theorem 2.3, we just need to prove that J_V is bounded from below and satisfies the Palais–Smale condition. Or better, it is enough to prove that J_V is bounded from below and coercive in $\Omega^T(S_1, S_2)$, i.e.,

$$J_V(x_k) \to +\infty \quad \text{if} \quad \|\dot{x}_k\| \to +\infty$$

(here, $\|\cdot\|$ is the L^2–norm). In fact, if J_V is coercive in $\Omega^T(S_1, S_2)$, then a sequence $(x_k)_k$ has to be bounded if $(J_V(x_k))_k$ is bounded, and the Palais–Smale condition is a consequence of the following lemma.

LEMMA 3.1. *If* S_1 *and* S_2 *are two submanifolds of* $\mathcal{M}_0$ *such that* (H_3) *holds, then each sequence* $(x_k)_k$*, bounded in* $\Omega^T(S_1, S_2)$ *and such that* $J'_V(x_k) \to 0$*, converges up to subsequences.*

Proof. It is enough reasoning as in the proof of [11, Lemma 3.4.1] taking into account some comments in the proof of [6, Proposition 4.2]. ∎

Taking any $\epsilon \in \,]0,1[$, it is easy to check that the functional J_V can be written as

$$J_V(x) = \frac{\epsilon}{2} J^\epsilon(x) + (1-\epsilon) J_T^\epsilon(x),$$

where

$$J^\epsilon(x) = \int_0^T \langle \dot{x}, \dot{x} \rangle ds + \int_0^T \frac{\langle \delta(x), \dot{x} \rangle^2}{\beta^\epsilon(x)} ds \\ - \left(\Delta_t^\epsilon - \int_0^T \frac{\langle \delta(x), \dot{x} \rangle}{\beta^\epsilon(x)} ds \right)^2 \left(\int_0^T \frac{1}{\beta^\epsilon(x)} ds \right)^{-1},$$

with $\beta^\epsilon(x) = \epsilon\, \beta(x)$ and $\Delta_t^\epsilon = \frac{\Delta_t}{\epsilon}$, and

$$J_T^\epsilon(x) = \frac{1}{2} \int_0^T \langle \dot{x}, \dot{x} \rangle ds - \int_0^T V^\epsilon(x,s) ds, \qquad \text{with } V^\epsilon(x,s) = \frac{V(x,s)}{1-\epsilon}.$$

Then, the following lemmas can be stated.

Lemma 3.2. *If* (H_2) *and* (H_3) *hold, then for each* $\epsilon \in \,]0,1[$ *the functional* J^ϵ *is bounded from below and coercive in* $\Omega^T(S_1, S_2)$.

Proof. The proof can be obtained by reasoning as in the proofs of [2, Lemma 2.6] and [6, Proposition 4.1] with some minor changes according to assume S_1 or S_2 as compact set. ∎

Lemma 3.3. *If* (H_3), (H_4^*) *and* (1.7) *hold, then, taken* $\epsilon \in \,]0,1[$ *small enough so that*

$$\frac{\lambda}{1-\epsilon} T^2 < \frac{\pi^2}{2},$$

the functional J_T^ϵ *is bounded from below and coercive in* $\Omega^T(S_1, S_2)$.

Proof. See [5, Lemma 3.1]. ∎

Proof of Theorem 1.2. Obviously, in the hypotheses of Theorem 1.2, Lemmas 3.2 and 3.3 imply that the functional J_V is bounded from below and coercive in $\Omega^T(S_1, S_2)$; hence, it satisfies the Palais–Smale condition (see Lemma 3.1) and Theorem 2.3 applies. So, J_V attains its infimum, and, from Proposition 2.2, a solution of the given problem must exist. ∎

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