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On Orthocentric Systems in Strictly Convex Normed Planes

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Abstract: It has been shown that the three-circles theorem, which is also known as Ţiţeica's or Johnson's theorem, can be extended to strictly convex normed planes, with various applications. From this it follows that the notions of orthocenters and orthocentric systems in the Euclidean plane have natural analogues in strictly convex normed planes. In the present paper (which can be regarded as continuation of [5] and [14]) we derive several new characterizations of the Euclidean plane by studying geometric properties of orthocentric systems in strictly convex normed planes. All these results yield also geometric characterizations of inner product spaces among all real Banach spaces of dimension ≥ 2 having strictly convex unit balls.

Key words: Birkhoff orthogonality, Busemann angular bisector, $\mathcal{C}$-orthocenter, inner product space, isosceles orthogonality, Minkowski plane, normed linear space, orthocenter, orthocentric system, three-circles theorem.

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1. Introduction

By $(M_2, \mathcal{C})$ we denote an arbitrary *normed* or *Minkowski plane* (i.e., a real two-dimensional normed linear space) with *unit circle* $\mathcal{C}$, *origin* O, and *norm* $\|\cdot\|$. We restrict our discussion to *strictly convex* Minkowski planes, whose unit circles are strictly convex curves with midpoint O, thus not containing a non-degenerate segment. Basic references to the geometry of Minkowski planes and spaces are [18], [16], [17], and the monograph [19].

For any point $x \in (M_2, \mathcal{C})$ and any number $\lambda > 0$, the set $\mathcal{C}(x, \lambda) := x + \lambda\mathcal{C}$ is said to be the *circle* centered at x and having radius λ. For $x \neq y$, we denote by $\langle x, y\rangle$ the *line* passing through x and y, by $[x, y]$ the *segment* between x and y, and by $[x, y\rangle$ the *ray* with starting point x passing through y. The

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distance from a point x to a non-empty set A is denoted by $d(x, A)$, i.e., $d(x, A) = \inf\{\|z - x\| : z \in A\}$.

It has been shown by E. Asplund and B. Grünbaum in [5] that the following theorem, which is the extension of the classical three-circles theorem in the Euclidean plane, also holds in strictly convex, smooth Minkowski planes. (The three-circles theorem is also called Ţiţeica's or Johnson's theorem; see the survey [13] and the monograph [12, p. 75]. For the related concept of orthocentricity, even in Euclidean n-space with $n \geq 2$, we refer to the survey contained in [9] and to [7].)

Theorem 1.1. *If three circles $\mathcal{C}(x_1, \lambda)$, $\mathcal{C}(x_2, \lambda)$, and $\mathcal{C}(x_3, \lambda)$ pass through a common point p_4 and intersect pairwise in the points p_1, p_2, and p_3, then there exists a circle $\mathcal{C}(x_4, \lambda)$ such that $\{p_1, p_2, p_3\} \subseteq \mathcal{C}(x_4, \lambda)$.*

Actually, the smoothness condition can be relaxed, i.e., Theorem 1.1 holds in strictly convex Minkowski planes (cf. [19, Theorem 4.14, p. 104] and [14]). This theorem is also basic for extensions of Clifford's chain of theorems to strictly convex normed planes; see [15].

The point p_4 in Theorem 1.1 is called the *$\mathcal{C}$-orthocenter* of the triangle $p_1p_2p_3$, and by Theorem 1.1 it is also evident that p_i is the $\mathcal{C}$-orthocenter of the triangle $p_jp_kp_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Thus it makes sense to call a set of four points, each of which is the $\mathcal{C}$-orthocenter of the triangle formed by the other three points, a *$\mathcal{C}$-orthocentric system*. See [20] for another concept of orthocenters which is related to Birkhoff orthogonality.

The proof of the above theorem for strictly convex Minkowski planes is based on the following properties of strictly convex Minkowski planes (cf. [5] and [18, p. 106]), which will be used throughout the paper:

Lemma 1.2. *Let $(M_2, \mathcal{C})$ be a strictly convex Minkowski plane. If $x_1 \neq x_2$ and $\{y_1, y_2\} \subseteq \mathcal{C}(x_1, \lambda) \cap \mathcal{C}(x_2, \lambda)$, then $x_1 + x_2 = y_1 + y_2$.*

Lemma 1.3. *Any three non-collinear points in a strictly convex Minkowski plane are contained in at most one circle.*

The following facts are well known in Euclidean geometry:

1. The three altitudes of a triangle intersect in a point called orthocenter of that triangle.

2. The altitude to the base of an isosceles triangle bisects the corresponding vertex angle.

3. If one of the altitudes of a triangle is also an angular bisector, then the triangle is isosceles.

4. The altitude to the base of an isosceles triangle bisects its base.

5. If one of the altitudes of a triangle is also a median (i.e., a segment from a vertex to the midpoint of the opposite side), then the triangle is isosceles.

As the notion of $\mathcal{C}$-orthocenter can be viewed as a natural extension of that of orthocenter in Euclidean geometry, one may ask whether results related to orthocenters in Euclidean geometry still hold in Minkowski geometry in the sense of $\mathcal{C}$-orthocenters. It is our aim to continue the investigations from [14] in this spirit.

For our discussion we need the definitions of isosceles orthogonality, Birkhoff orthogonality, and Busemann angular bisectors. Let $x, y \in (M_2, \mathcal{C})$. The point x is said to be *isosceles orthogonal to* y if $\|x+y\| = \|x-y\|$, and in this case we write $x \perp_I y$ (cf. [10]). On the other hand, x is said to be *Birkhoff orthogonal to* y if $\|x+ty\| \geq \|x\|$ holds for all $t \in \mathbb{R}$, and for this we write $x \perp_B y$ (cf. [6]). We refer to [10], [11], and [2] for basic properties of isosceles orthogonality and Birkhoff orthogonality, and to [4] and [3] for a detailed study of the relations between them.

For non-collinear rays $[p,a\rangle$ and $[p,b\rangle$, the ray

$$\Big[p,\ \frac{1}{2}\Big(\frac{a-p}{\|a-p\|}+\frac{b-p}{\|b-p\|}\Big)+p\Big\rangle$$

is called the *Busemann angular bisector* of the angle spanned by $[p,a\rangle$ and $[p,b\rangle$, and it is denoted by $A_B\big([p,a\rangle,[p,b\rangle\big)$ (cf. [8]). It is trivial to see that when $\|a-p\| = \|b-p\|$, then

$$A_B\big([p,a\rangle,[p,b\rangle\big) = \Big[p,\frac{1}{2}(a+b)\Big\rangle.$$

2. Some lemmas

The following lemmas are needed for our investigations.

Lemma 2.1. (cf. [1]) *Let* $(M_2, \mathcal{C})$ *be a strictly convex Minkowski plane. Then, for any* $x \in (M_2, \mathcal{C})\backslash\{O\}$ *and any number* $\lambda > 0$, *there exists a point* $y \in \lambda\mathcal{C}$ *(unique except for the sign) such that* $x \perp_I y$.

Lemma 2.2. *Let* $\mathcal{C}(x_1, \lambda)$ *and* $\mathcal{C}(x_2, \lambda)$ *be two circles in a strictly convex Minkowski plane* $(M_2, \mathcal{C})$. *If* $\{w, z\} \subseteq \mathcal{C}(x_1, \lambda)$, $\{w', z'\} \subseteq \mathcal{C}(x_2, \lambda)$, *and* $w - z = w' - z'$, *then*

$$d(x_1, \langle w, z\rangle) = d(x_2, \langle w', z'\rangle).$$

Proof. Without loss of generality, we can suppose that $x_1 = x_2 = O$ and $\lambda = 1$. By the assumed strict convexity, $\|w - z\| = \|w' - z'\| = 2$ implies that $w = -z$ and $w' = -z'$, yielding $d(x_1, \langle w, z\rangle) = d(x_2, \langle w', z'\rangle) = 0$. Now we consider the case when $\|w - z\| < 2$. Again by strict convexity of $(M_2, \mathcal{C})$, any vector with norm < 2 is the sum of two unit vectors in a unique way (cf. [18, Proposition 14, p. 106]). Thus, either $w = w'$ and $z = z'$ or $w = -z'$ and $z = -w'$ hold, and each of these two cases clearly gives that $d(x_1, \langle w, z\rangle) = d(x_2, \langle w', z'\rangle)$. ∎

Lemma 2.3. ([4, (4.12)]) *If for any* $x, y \in (M_2, \mathcal{C})$ *with* $x \perp_I y$ *there exists a number* $0 < t < 1$ *such that* $x \perp_I ty$, *then* $(M_2, \mathcal{C})$ *is Euclidean.*

The following lemma follows immediately from Lemma 2.3.

Lemma 2.4. *If for any* x, $y \in (M_2, \mathcal{C})$ *with* $x \perp_I y$ *there exists a number* $t > 1$ *such that* $x \perp_I ty$, *then* $(M_2, \mathcal{C})$ *is Euclidean.*

Lemma 2.5. (cf. [4, (10.2)]) *A Minkowski plane* $(M_2, \mathcal{C})$ *is Euclidean if and only if the implication*

$$x \perp_B y \Rightarrow x \perp_I y$$

holds for any x, $y \in \mathcal{C}$.

Lemma 2.6. (cf. [4, (10.9)]) *A Minkowski plane* $(M_2, \mathcal{C})$ *is Euclidean if and only if the implication*

$$x \perp_I y \Rightarrow x \perp_B y$$

holds for any x, $y \in \mathcal{C}$.

LEMMA 2.7. (cf. [14]) *Let* $\{p_1, p_2, p_3, p_4\}$ *be a* $\mathcal{C}$*-orthocentric system in a strictly convex Minkowski plane* $(M_2, \mathcal{C})$*. If* x_i *is the circumcenter of the triangle* $p_j p_k p_l$*, where* $\{i, j, k, l\} = \{1, 2, 3, 4\}$*, then* $\{x_1, x_2, x_3, x_4\}$ *is also a* $\mathcal{C}$*-orthocentric system and*

$$p_i - p_j = x_j - x_i.$$

LEMMA 2.8. *Let* $(M_2, \mathcal{C})$ *be a strictly convex Minkowski plane. For any* $x, y \in (M_2, \mathcal{C})\backslash\{O\}$ *with* $x \perp_I y$*, let* $p_3 = y$*,* $p_4 = -y$*,* $x_1 = x$*,* $x_2 = -x$*, and* $\lambda = \|x + y\|$*. Then there exist two points* $p_1 \in \mathcal{C}(x_2, \lambda)$ *and* $p_2 \in \mathcal{C}(x_1, \lambda)$ *such that* $\{p_1, p_2, p_3, p_4\}$ *is a* $\mathcal{C}$*-orthocentric system, and that one of the following conditions is satisfied:*

(1) $\|p_3 - p_1\| = \|p_3 - p_2\|$*, and* p_3 *and the line* $\langle p_1, p_2\rangle$ *are separated by the line* l_0*, which is passing through* p_4 *parallel to* $\langle p_1, p_2\rangle$*,*

(2) $p_4 \in \left[p_3, \frac{p_1+p_2}{2}\right]$*,*

(3) p_3 *and the line* $\langle p_1, p_2\rangle$ *are separated by* l_0*, and* $p_4 \in A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big)$*,*

(4) p_3 *and the line* $\langle p_1, p_2\rangle$ *are separated by the line* l_0*, and* $\langle p_1, p_2\rangle$ *is a common supporting line of the circles* $\mathcal{C}(x_2, \lambda)$ *and* $\mathcal{C}(x_1, \lambda)$*.*

Proof. (1) By the assumed strict convexity and the fact that $x \perp_I y$, one can easily verify that the circles $\mathcal{C}(x_1, \lambda)$ and $\mathcal{C}(x_2, \lambda)$ intersect in exactly two points, which are p_3 and p_4. Also, one can easily verify that $2x_2 - y$ lies in the circle $\mathcal{C}(x_2, \lambda)$ and $2x_1 - y$ in $\mathcal{C}(x_1, \lambda)$, and that the point p_4 lies in the segment $[2x_2 - y, 2x_1 - y]$. Denote by H^- the closed half plane bounded by $\langle 2x_2 - y, 2x_1 - y\rangle$ that does not contain p_3, and by $\mathcal{C}(p_4, \lambda)^-$ the intersection of H^- and $\mathcal{C}(p_4, \lambda)$. Then, since the point p_3 and the line $\langle x_2 - 2y, x_1 - 2y\rangle$ are separated by the line $\langle 2x_2 - y, 2x_1 - y\rangle$, the points $x_1 - 2y$ and $x_2 - 2y$ lie in the semicircle $\mathcal{C}(p_4, \lambda)^-$.

Let

$$w = -y - \frac{2}{3}x \quad \text{and} \quad z = -y + \frac{2}{3}x.$$

Then simple calculation shows that $[p_3, x_2 - 2y]$ intersects $[2x_2 - y, 2x_1 - y]$ in w, and $[p_3, x_1 - 2y]$ intersects $[2x_2 - y, 2x_1 - y]$ in z.

Since

$$\|w - p_4\| = \|z - p_4\| = \frac{2}{3}\|x\| \leq \frac{1}{3}(\|x + y\| + \|x - y\|) = \frac{2}{3}\lambda < \lambda,$$

for any $t \in (0, 1)$ there exists a unique point $x(t)$ such that the line $\langle p_3, tw + (1 - t)z\rangle$ intersects the semicircle $\mathcal{C}(p_4, \lambda)^-$ in a point $x(t)$. From the fact

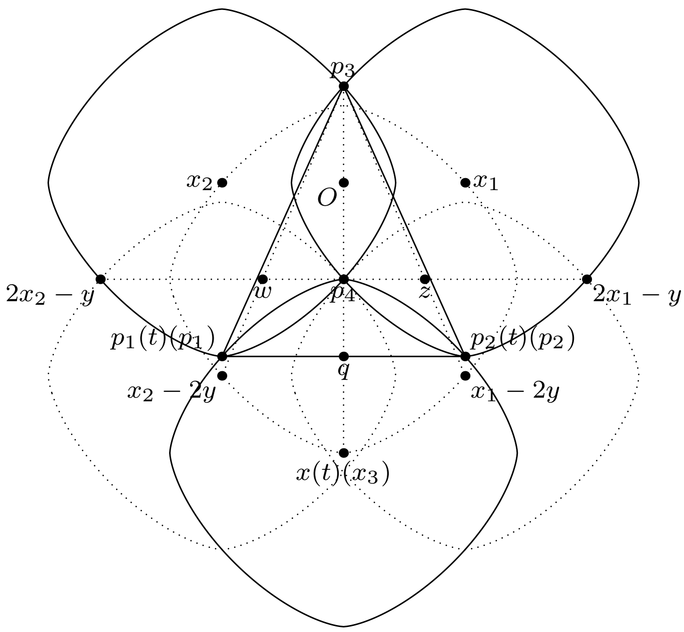

Figure 1: The proof of Lemma 2.8.

that the segment $[x_1, x_2 - 2y]$ intersects $[x_2, x_1 - 2y]$ in p_4 it follows that $\|x(t) - x_1\| < 2\lambda$ and $\|x(t) - x_2\| < 2\lambda$ hold for any $t \in (0,1)$. Thus there exist two points $p_1(t)$ and $p_2(t)$ such that $\mathcal{C}(x(t), \lambda)$ intersects $\mathcal{C}(x_2, \lambda)$ exactly in $p_1(t)$ and p_4, and $\mathcal{C}(x(t), \lambda)$ intersects $\mathcal{C}(x_1, \lambda)$ exactly in $p_2(t)$ and p_4. Then, for any $t \in (0,1)$, $\{p_1(t), p_2(t), p_3, p_4\}$ is a $\mathcal{C}$-orthocentric system; see Figure 1.

Moreover, for any $t \in (0,1)$ we have by Lemma 2.7 that $p_2(t) - p_1(t) = x_1 - x_2$. Then, by Lemma 2.2,

$$d\big(x(t), \langle p_1(t), p_2(t)\rangle\big) = d\big(x_2 - 2y, \langle 2x_2 - y, 2x_1 - y\rangle\big),$$

and therefore

$$d\big(x(t), \langle p_1(t), p_2(t)\rangle\big) < d\big(x(t), \langle 2x_2 - y, 2x_1 - y\rangle\big),$$

which implies that p_3 and the line $\langle p_1(t), p_2(t)\rangle$ are separated by the line $\langle 2x_2 - y, 2x_1 - y\rangle$.

Now we show the existence of the points p_1 and p_2 with the desired properties. It is trivial that the functions $x(t)$, $p_1(t)$, and $p_2(t)$ as well as the function

$$f(t) = \|p_3 - p_2(t)\| - \|p_3 - p_1(t)\|$$

are continuous. So

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(\|p_3 - p_2(t)\| - \|p_3 - p_1(t)\|) = \|p_3 - (2x_1 - y)\| - \|p_3 - p_4\| > 0$$

and

$$\lim_{t\to 1} f(t) = \lim_{t\to 1}(\|p_3 - p_2(t)\| - \|p_3 - p_1(t)\|) = \|p_3 - p_4\| - \|p_3 - (2x_2 - y)\| < 0.$$

Hence there exists a number $t_0 \in (0,1)$ such that $f(t_0) = 0$. Let $x_3 = x(t_0)$, $p_1 = p_1(t_0)$, and $p_2 = p_2(t_0)$. Then p_1 and p_2 are two points having the desired properties.

(2) For any $t \in (0,1)$, let the functions $x(t)$, $p_1(t)$, and $p_2(t)$ be defined as in (1), and $w(t)$, $z(t)$ be the points where the line $\langle 2x_2 - y, 2x_1 - y\rangle$ meets $\langle p_3, p_1(t)\rangle$ and $\langle p_3, p_2(t)\rangle$, respectively. It is clear that when t is sufficiently close to 0, the midpoint of $[w(t), z(t)]$ has to lie strictly between p_4 and $2x_1 - y$, and when t is sufficiently close to 1, the midpoint of $[w(t), z(t)]$ has to lie strictly between p_4 and $2x_2 - y$. Thus there exists a number $t_0 \in (0,1)$ such that $\frac{1}{2}\big(w(t_0) + z(t_0)\big) = p_4$. Then $p_1 = p_1(t_0)$ and $p_2 = p_2(t_0)$ are two points having the desired properties.

(3) For any $t \in (0,1)$, let the functions $x(t)$, $p_1(t)$, and $p_2(t)$ be defined as in (1). It is clear that when t moves from 0 to 1, the ray $A_B\big([p_3, p_1(t)\rangle, [p_3, p_2(t)\rangle\big)$ turns continuously from $A_B\big([p_3, 2x_1 - y\rangle, [p_3, p_4\rangle\big)$ to $A_B\big([p_3, 2x_2 - y\rangle, [p_3, p_4\rangle\big)$. Thus there exists a number $t_0 \in (0,1)$ such that $A_B\big([p_3, p_1(t_0)\rangle, [p_3, p_2(t_0)\rangle\big) = [p_3, p_4\rangle$. Let $p_1 = p_1(t_0)$ and $p_2 = p_2(t_0)$. Then p_1 and p_2 are two points having the desired property.

(4) Let $y' \in \mathcal{C}$ be a point such that $y' \perp_B x$ and $\{x_2 + \lambda y', x_1 + \lambda y'\} \subseteq H^-$. Then $\langle x_2 + \lambda y', x_1 + \lambda y'\rangle$ is a common supporting line of the circles $\mathcal{C}(x_2, \lambda)$ and $\mathcal{C}(x_1, \lambda)$.

Let $p_1 = x_2 + \lambda y'$, $p_2 = x_1 + \lambda y'$, and $x_3 = p_1 + p_4 - x_2$. Then one can easily verify that $\{p_1, p_2, p_4\} \subseteq \mathcal{C}(x_3, \lambda)$, and therefore p_1 and p_2 are the two points with the desired properties.

3. Main Results

Now we will present our main results which are new characterizations of the Euclidean plane among all strictly convex normed planes via properties of $\mathcal{C}$-orthocentric systems.

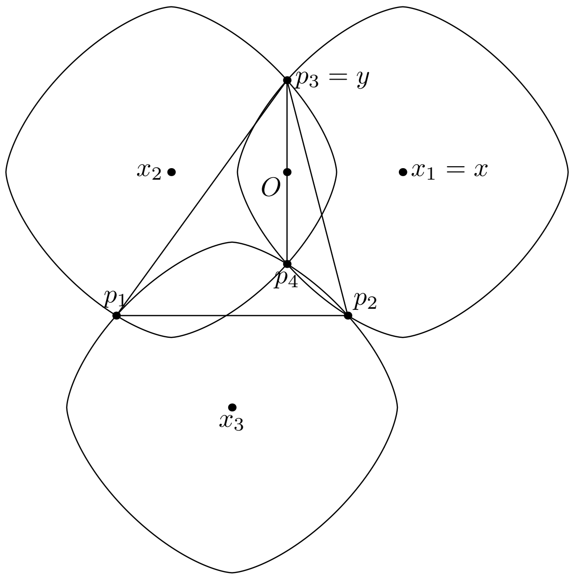

Figure 2: The proof of Theorem 3.1.

THEOREM 3.1. *A strictly convex Minkowski plane is Euclidean if and only if for any $\mathcal{C}$-orthocentric system $\{p_1, p_2, p_3, p_4\}$ the relation*

$$p_i - p_j \perp_B (p_k - p_l)$$

holds, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. If $(M_2, \mathcal{C})$ is Euclidean then, for any $\mathcal{C}$-orthocentric system $\{p_1, p_2, p_3, p_4\}$, p_i is the orthocenter of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Thus

$$p_i - p_j \perp_B (p_k - p_l).$$

Conversely, for any x, $y \in \mathcal{C}$ with $x \perp_I y$ let

$$p_3 = y, \quad p_4 = -y, \quad x_1 = x, \quad \text{and} \quad x_2 = -x.$$

By Lemma 2.8, there exist two points p_1 and p_2 such that $\{p_1, p_2, p_3, p_4\}$ is a $\mathcal{C}$-orthocentric system; see Figure 2. By Lemma 2.7, $p_2 - p_1 = x_1 - x_2 = 2x$. On the other hand, by the assumption of the theorem we have

$$(p_2 - p_1) \perp_B (p_3 - p_4)$$

or, equivalently, $x \perp_B y$. By Lemma 2.6, $(M_2, \mathcal{C})$ is Euclidean.

THEOREM 3.2. *A strictly convex Minkowski plane* $(M_2, \mathcal{C})$ *is Euclidean if and only if for any* $\mathcal{C}$*-orthocentric system* $\{p_1, p_2, p_3, p_4\}$ *with* $\|p_3 - p_1\| = \|p_3 - p_2\|$ *it holds that* $p_4 \in \langle p_3, \frac{p_1+p_2}{2} \rangle$.

Proof. We only have to prove sufficiency. By Lemma 2.4, we just need to show that for any x, $y \in (M_2, \mathcal{C})$ with $x \perp_I y$ there exists a number $t > 1$ such that $x \perp_I ty$, and it is trivial in the case where at least one of x and y is O.

For any $x, y \in (M_2, \mathcal{C})\backslash\{O\}$ with $x \perp_I y$, let

$$p_3 = y, \quad p_4 = -y, \quad x_1 = x, \quad \text{and} \quad x_2 = -x.$$

By (1) of Lemma 2.8, there exist two points p_1 and p_2 such that $\{p_1, p_2, p_3, p_4\}$ is a $\mathcal{C}$-orthocentric system, $\|p_3 - p_1\| = \|p_3 - p_2\|$, and that p_3 and the line $\langle p_1, p_2 \rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2 \rangle$. Then, by the assumption of the theorem,

$$p_4 \in \left\langle p_3, \frac{p_1 + p_2}{2} \right\rangle.$$

Since p_3 and the line $\langle p_1, p_2 \rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2 \rangle$, we have

$$p_4 \in \left[p_3, \frac{p_1 + p_2}{2} \right].$$

Thus there exists a number $t > 2$ such that

$$p_3 - \frac{p_1 + p_2}{2} = \frac{t}{2}(p_3 - p_4) = ty.$$

On the other hand, by Lemma 2.7 we have

$$\begin{aligned} \|x + ty\| &= \left\| \frac{x_1 - x_2}{2} + \left(p_3 - \frac{p_1 + p_2}{2} \right) \right\| = \|p_3 - p_1\| \\ &= \left\| \frac{p_2 - p_1}{2} - \left(p_3 - \frac{p_1 + p_2}{2} \right) \right\| = \|x - ty\|. \end{aligned} \tag{3.1}$$

Hence there exists a number $t > 2$ such that $x \perp_I ty$, which completes the proof. ∎

THEOREM 3.3. *A strictly convex Minkowski plane* $(M_2, \mathcal{C})$ *is Euclidean if and only if for any* $\mathcal{C}$*-orthocentric system* $\{p_1, p_2, p_3, p_4\}$ *the equality* $\|p_3 - p_1\| = \|p_3 - p_2\|$ *holds whenever* $p_4 \in \langle p_3, \frac{p_1+p_2}{2} \rangle$.

The proof of Theorem 3.3 makes use of (2) of Lemma 2.8 and is very similar to that of Theorem 3.2, and so we omit it.

THEOREM 3.4. *A strictly convex Minkowski plane* $(M_2, \mathcal{C})$ *is Euclidean if and only if for any* $\mathcal{C}$*-orthocentric system* $\{p_1, p_2, p_3, p_4\}$, p_4 *lies on the line containing* $A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big)$ *whenever* $\|p_3 - p_1\| = \|p_3 - p_2\|$.

Proof. We only have to prove sufficiency. By Theorem 3.2 it is sufficient to show that for any $\mathcal{C}$-orthocentric system $\{p_1,\, p_2,\, p_3,\, p_4\}$, $p_4 \in \big\langle p_3, \frac{p_1+p_2}{2}\big\rangle$ whenever $\|p_3 - p_1\| = \|p_3 - p_2\|$.

By the assumption of the theorem, for any $\mathcal{C}$-orthocentric system $\{p_1, p_2, p_3, p_4\}$ with $\|p_3 - p_1\| = \|p_3 - p_2\|$, p_4 lies on the line containing $A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big)$. By the definition of Busemann angular bisectors and the fact that $\|p_3 - p_1\| = \|p_3 - p_2\|$, we have

$$A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big) = \Big[p_3, \frac{p_1 + p_2}{2}\Big\rangle.$$

Thus $\big\langle p_3, \frac{p_1+p_2}{2}\big\rangle$ is the line containing $A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big)$, and therefore

$$p_4 \in \Big\langle p_3, \frac{p_1 + p_2}{2}\Big\rangle.$$

The proof is complete. ∎

THEOREM 3.5. *A strictly convex Minkowski plane* $(M_2, \mathcal{C})$ *is Euclidean if and only if for any* $\mathcal{C}$*-orthocentric system* $\{p_1,\, p_2,\, p_3,\, p_4\}$ *the equality* $\|p_3 - p_1\| = \|p_3 - p_2\|$ *holds whenever* p_4 *lies on the line containing* $A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big)$.

Proof. We only have to prove sufficiency. By Lemma 2.4, we just need to show that for any $x,\, y \in (M_2, \mathcal{C})$ with $x \perp_I y$ there exists a number $t > 1$ such that $x \perp_I ty$.

For any $x,\, y \in (M_2, \mathcal{C})\backslash\{O\}$ with $x \perp_I y$, let

$$p_3 = y, \quad p_4 = -y, \quad x_1 = x, \quad \text{and} \quad x_2 = -x.$$

By (3) of Lemma 2.8, there exist two points p_1 and p_2 such that $\{p_1, p_2, p_3, p_4\}$ is a $\mathcal{C}$-orthocentric system, p_3 and the line $\langle p_1, p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2\rangle$, and that

$$p_4 \in A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big).$$

By the assumption of the theorem we have $\|p_3 - p_1\| = \|p_3 - p_2\|$, and then

$$A_B\big([p_3, p_1\rangle, [p_3, p_2\rangle\big) = \Big[p_3, \frac{p_1 + p_2}{2}\Big\rangle.$$

Since p_3 and the line $\langle p_1, p_2\rangle$ are separated by the line passing through p_4 parallel to $\langle p_1, p_2\rangle$, we have

$$p_4 \in \Big[p_3, \frac{p_1 + p_2}{2}\Big].$$

Hence there exists a number $t > 2$ such that

$$p_3 - \frac{p_1 + p_2}{2} = \frac{t}{2}(p_3 - p_4) = ty.$$

In a way analogous to that referring to Theorem 3.2 it can be proved that $x \perp_I ty$, which completes the proof. ∎

Theorem 3.6. *A strictly convex Minkowski plane is Euclidean if and only if for any $\mathcal{C}$-orthocentric system $\{p_1, p_2, p_3, p_4\}$ with $p_3 \neq p_4$ the equality $\|p_3 - p_1\| = \|p_3 - p_2\|$ holds whenever $\langle p_1, p_2\rangle$ is a common supporting line of the circle containing $\{p_1, p_3, p_4\}$ and the circle containing $\{p_2, p_3, p_4\}$.*

Proof. Suppose first that $(M_2, \mathcal{C})$ is Euclidean. For any $\mathcal{C}$-orthocentric system $\{p_1, p_2, p_3, p_4\}$, let $\mathcal{C}(x_i, \lambda)$ be the circumcircle of the triangle $p_j p_k p_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $\langle p_3, p_4\rangle$ is the radical axis of $\mathcal{C}(x_1, \lambda)$ and $\mathcal{C}(x_2, \lambda)$. (Note that the radical axis of two circles is the locus of points of equal circle power with respect to two non-concentric circles, where the power of a point with respect to a circle is equal to the squared distance from the point to the center of the circle minus the squared radius of the circle; cf. [12].) If $\langle p_1, p_2\rangle$ is a common supporting line of $\mathcal{C}(x_1, \lambda)$ and $\mathcal{C}(x_2, \lambda)$, then $\langle p_3, p_4\rangle$ will be the perpendicular bisector of $[p_1, p_2]$, and therefore $\|p_3 - p_1\| = \|p_3 - p_2\|$.

Now we prove sufficiency. By Lemma 2.5 we only have to show that for any $x, z \in \mathcal{C}$, $z \perp_B x \Rightarrow z \perp_I x$. Let ω be a fixed orientation on $(M_2, \mathcal{C})$. Since $z \perp_B x$ if and only if $(-z) \perp_B x$, it will be sufficient to prove that for any $x, z \in \mathcal{C}$ with $\overrightarrow{zx} = \omega$ (i.e., the orientation $\overrightarrow{zx}$ is given by ω), $z \perp_B x$ implies $z \perp_I x$.

By strict convexity of $(M_2, \mathcal{C})$, for any $x \in \mathcal{C}$ there exists a unique point $z \in \mathcal{C}$ such that $z \perp_B x$ and $\overrightarrow{zx} = \omega$. On the other hand, by Lemma 2.1, for any number $t > 0$ there is a unique point $y(t) \in t\mathcal{C}$ such that $x \perp_I y(t)$ and $\overrightarrow{y(t)x} = \omega$. Let

$$x_1 = x, \quad x_2 = -x, \quad p_3(t) = y(t), \quad \text{and} \quad p_4(t) = -y(t).$$

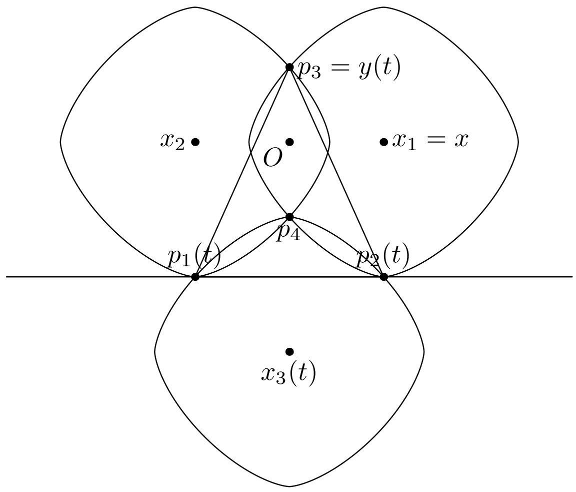

Figure 3: The proof of Theorem 3.6.

Then, by (4) of Lemma 2.8, there exist two points $p_1(t)$ and $p_2(t)$ such that the set $\{p_1(t), p_2(t), p_3(t), p_4(t)\}$ is a $\mathcal{C}$-orthocentric system, $p_3(t)$ and the line $\langle p_1(t), p_2(t)\rangle$ are separated by the line passing through $p_4(t)$ which is parallel to $\langle p_1(t), p_2(t)\rangle$, and that the line $\langle p_1(t), p_2(t)\rangle$ is the common supporting line of $\mathcal{C}(x_1, \|x + y(t)\|)$ and $\mathcal{C}(x_2, \|x + y(t)\|)$; see Figure 3. From Lemma 2.7 it follows that

$$p_1(t) - p_2(t) = x_2 - x_1.$$

Thus $x_2 - p_1(t) \perp_B x$ and $x_1 - p_2(t) \perp_B x$, and therefore

$$\|x + y(t)\| \, z = x_2 - p_1(t) = x_1 - p_2(t).$$

Let $z(t) = \|x + y(t)\| \, z$. Then

$$\|p_3(t) - p_2(t)\| = \|p_3(t) - x_1 + x_1 - p_2(t)\| = \|p_3(t) - x_1 + z(t)\|$$

and

$$\|p_3(t) - p_1(t)\| = \|p_3(t) - x_2 + x_2 - p_1(t)\| = \|p_3(t) - x_2 + z(t)\| .$$

By assumption $\|p_3(t) - p_1(t)\| = \|p_3(t) - p_2(t)\|$, and therefore

$$\|p_3(t) - x_2 + z(t)\| = \|p_3(t) - p_1(t)\| = \|p_3(t) - p_2(t)\| = \|p_3(t) - x_1 + z(t)\| ,$$

i.e.,

$$\big\|\big(y(t)+z(t)\big)+x\big\| = \big\|\big(y(t)+z(t)\big)-x\big\|.$$

It is evident that

$$\lim_{t\to 0} y(t) = O,$$

and therefore

$$\lim_{t\to 0} \|x+y(t)\| = \|x\| = 1.$$

It follows that

$$\lim_{t\to 0}\big(y(t)+z(t)\big) = \lim_{t\to 0}\big(y(t)+\|x+y(t)\|\,z\big) = z,$$

and then

$$\|z+x\| = \left\|\lim_{t\to 0}\big(y(t)+z(t)\big)+x\right\| = \left\|\lim_{t\to 0}\big(y(t)+z(t)\big)-x\right\| = \|z-x\|\,.$$

Hence $z \perp_I x$. ∎

The theorem above says that a Minkowski plane is Euclidean if and only if the implication

$$\langle p_1,p_2\rangle \text{ supports the circles } \mathcal{C}(x_1,\lambda) \text{ and } \mathcal{C}(x_2,\lambda) \implies \|p_3-p_1\| = \|p_3-p_2\|$$

holds for any $\mathcal{C}$-orthocentric system $\{p_1,p_2,p_3,p_4\}$. In the next theorem we show that the reverse implication also characterizes the Euclidean plane.

THEOREM 3.7. *A strictly convex Minkowski plane is Euclidean if and only if for any $\mathcal{C}$-orthocentric system $\{p_1, p_2, p_3, p_4\}$ with $p_3 \neq p_4$, $\langle p_1,p_2\rangle$ is a common supporting line of the circle containing $\{p_1,p_3,p_4\}$ and the circle containing $\{p_2,p_3,p_4\}$ whenever $\|p_3-p_1\| = \|p_3-p_2\|$.*

Proof. For any $x,z \in \mathcal{C}$ with $z \perp_B x$ and any number $t > 0$, we can define $y(t)$, x_1, x_2, $p_3(t)$, and $p_4(t)$ as in the proof of Theorem 3.6. Then, by (1) of Lemma 2.8, there exist two points $p_1(t)$ and $p_2(t)$ such that the set $\big\{p_1(t),\, p_2(t),\, p_3(t),\, p_4(t)\big\}$ is a $\mathcal{C}$-orthocentric system, $p_3(t)$ and the line $\langle p_1(t),p_2(t)\rangle$ are separated by the line passing through $p_4(t)$ parallel to $\langle p_1(t), p_2(t)\rangle$, and $\|p_3(t)-p_1(t)\| = \|p_3(t)-p_2(t)\|$. By assumption, the line $\langle p_1(t), p_2(t)\rangle$ is the common supporting line of $\mathcal{C}(x_1,\|x+y(t)\|)$ and $\mathcal{C}(x_2,\|x+y(t)\|)$. Then, as in the proof of Theorem 3.6, it can be shown that $z \perp_I x$, which completes the proof. ∎

Final remark. Since a real Banach space of dimension ≥ 2 is an inner product space if and only if each of its two-dimensional subspaces is isometric to the Euclidean plane, it is clear that all our theorems can be interpreted (in the spirit of the monograph [4]) as characterizations of inner product spaces among all strictly convex real Banach spaces of dimension ≥ 2.

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