EXTRACTA MATHEMATICAE Vol. **24**, Núm. 3, 251 – 258 (2009)

Linear Mapping Preserving the Kernel or the Range of Operators *

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Presented by Mostafa Mbekhta Received April 6, 2009

Abstract: Let X and Y be two infinite dimensional real or complex Banach spaces. In this note we determine the forms of surjective additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ preserving the kernel's dimension or the range's codimension. As consequence, we establish that $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the kernel (respectively, the range) if and only if there exists an invertible operator $A \in \mathcal{L}(X)$ such that $\phi(T) = AT$ (respectively, $\phi(T) = TA$) for all $T \in \mathcal{L}(X)$.

Key words: Additives preservers, kernel operator, range operator.

AMS *Subject Class.* (2000): 47B48, 47A10, 46H05.

Introduction and statement of main results

Let X be a Banach space, and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, write T^* for its adjoint, $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range. Recall that an operator $T \in \mathcal{L}(X)$ is called *semi-Fredholm* if $\mathrm{R}(T)$ is closed and either $\dim \mathrm{N}(T)$ or $\operatorname{codim} \mathrm{R}(T)$ is finite. The *index* of such operator is defined by

$$\operatorname{ind}(T) = \dim \mathrm{N}(T) - \operatorname{codim} \mathrm{R}(T)\,,$$

and if $\operatorname{ind}(T)$ is finite then T is said to be *Fredholm*.

In [9] it is shown that a surjective linear map $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$, where X is an infinite-dimensional complex Banach space, is unital, i.e., $\phi(I) = I$, and preserves injective operators in both direction if and only if there is an invertible operator $A \in \mathcal{L}(X)$ such that $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{L}(X)$. Moreover, if X is assumed to be a Hilbert space, then it is proved that the surjective unital linear maps ϕ preserving surjective operators take the above

* This note were written while the author was visiting the University of Lille1. He would like to thank all the members of mathematic's U.F.R for their beneficial hospitality.

mentioned form. These results are extended to the case of unital surjective additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ where X and Y are a complex Banach spaces, see [1].

Let X and Y be an infinite-dimensional Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The purpose of this note is to determine the forms of all surjective additive maps, non-necessary unital, $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ preserving the kernel's dimension or the range's codimension. We establish also that $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the kernel (respectively, the range) if and only if there exists an invertible operator $A \in \mathcal{L}(X)$ such that $\phi(T) = AT$ (respectively, $\phi(T) = TA$) for all $T \in \mathcal{L}(X)$.

Theorem 1. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be an additive surjective mapping. The following assertions are equivalent.*

(i) $\dim \mathrm{N}(\phi(T)) = \dim \mathrm{N}(T)$ *for all $T \in \mathcal{L}(X)$;*

(ii) *there is two bijective bounded linear, or conjugate linear, mappings $U : X \to Y$ and $V : Y \to X$ such that $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$.*

Theorem 2. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be an additive surjective mapping such that* $\operatorname{codim} \mathrm{R}(\phi(T)) = \operatorname{codim} \mathrm{R}(T)$ *for all $T \in \mathcal{L}(X)$. Then one of the following assertions holds:*

(i) *There exist a bijective linear or conjugate linear mappings $U : X \to Y$ and $V : Y \to X$ such that $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$.*

(ii) *There exist a bijective linear or conjugate linear mappings $U' : X^* \to Y$ and $V' : Y \to X^*$ such that $\phi(T) = U'T^*V'$ for all $T \in \mathcal{L}(X)$. In this case, X and Y are reflexive.*

Notice that the case (ii) in the above theorem can occur in some special Banach spaces. More precisely, it is shown in [2, 3, 4] that there exists an infinite-dimensional complex reflexive Banach space X such that every bounded operator $T \in \mathcal{L}(X)$ is of the form $T = \lambda I + S$ where $\lambda \in \mathbb{C}$ and S is strictly singular; the essential spectrum of such operator is $\sigma_{\mathrm{e}}(T) = \{\lambda\}$. Consider the linear map $\phi(T) = T^*$ for all $T \in \mathcal{L}(X)$. Then ϕ preserves the range's codimension. In fact, for Fredholm operators T, we have $\mathrm{ind}(T) = 0$ and so $\operatorname{codim} \mathrm{R}(T) = \operatorname{codim} \mathrm{R}(\phi(T))$. If T is not Fredholm, then it is strictly singular and $\sigma_{\mathrm{e}}(T) = \{0\}$. Hence, the continuity of the index implies that T and T^* are not semi-Fredholm, and consequently $\operatorname{codim} \mathrm{R}(T) = \operatorname{codim} \mathrm{R}(\phi(T) = \infty$.

Let $x \in X$ and let f be in the dual space X^* of X, we denote, as usual, by $x \otimes f$ the rank one operator given by $(x \otimes f)z = f(z)x$ for $z \in X$. The spectrum of such operator is $\sigma(x \otimes f) = \{0, f(x)\}$.

As consequence of Theorem 1 and Theorem 2, we derive the following two results.

THEOREM 3. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Then the following assertions are equivalent:*

(i) $\mathrm{N}(\phi(T)) = \mathrm{N}(T)$ *for all* $T \in \mathcal{L}(X)$*;*

(ii) *there is an invertible operator* $A \in \mathcal{L}(X)$ *such that* $\phi(T) = AT$ *for all* $T \in \mathcal{L}(X)$*.*

Proof. Assume that ϕ preserves the kernel, then, obviously, it preserves the kernel's dimension, and by Theorem 1, it takes the form $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$. Let us show that $V = \lambda I$. Suppose, on the contrary, that there exists $x \in X$ such that x and Vx are linearly independent, and let $f \in X^*$ satisfy $f(x) = 1$ and $f(Vx) = 0$. It follows that

$$x \in \mathrm{N}(I - x \otimes f) = \mathrm{N}(U(I - x \otimes f)V) = \mathrm{N}(V - x \otimes fV)\,,$$

and hence $Vx = 0$, a contradiction. Thus $\phi(T) = AT$ for all $T \in \mathcal{L}(X)$, where $A = \lambda U = \phi(I) \in \mathcal{L}(Y)$. This completes the proof. ∎

THEOREM 4. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Then the following assertions are equivalent:*

(i) $\mathrm{R}(\phi(T)) = \mathrm{R}(T)$ *for all* $T \in \mathcal{L}(X)$*;*

(ii) *there exists an invertible operator* $B \in \mathcal{L}(X)$ *such that* $\phi(T) = TB$ *for all* $T \in \mathcal{L}(X)$*.*

Proof. Assume that ϕ preserves the range. Then ϕ preserves the range's dimension. Observe that ϕ can not take the second form in Theorem 1, because otherwise, for $T = x \otimes f$ such that $U'(f)$ and x are linearly independent, we will get

$$\mathrm{Vect}\{x\} = \mathrm{R}(T) = \mathrm{R}(\phi(T)) = \mathrm{Vect}\{U'(f)\}\,,$$

a contradiction. Hence, ϕ takes the form $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$. Now, for an arbitrary $a \in X$ and $g \in X^*$ such that $g(a) \neq 0$, we have

$$\mathrm{R}(a \otimes g) = \mathrm{R}(U(a \otimes g)V) = \mathrm{R}(Ua \otimes g)\,,$$

and so $\{a, Ua\}$ is linearly dependent. This shows that $\phi(T) = TB$ for all $T \in \mathcal{L}(X)$, where $B = \lambda V = \phi(I) \in \mathcal{L}(X)$, as desired. ∎

Before giving the proof of Theorem 1 and Theorem 2, some lemmas are to be established first.

It is well known that the set of semi-Fredholm operators remains invariant under perturbation by finite rank operators.

Lemma 5. *Let T be a non-zero operator in $\mathcal{L}(X)$. Then the following assertions are equivalent:*

(i) $\mathrm{rg}(T) = 1$;

(ii) *If $S \in \mathcal{L}(X)$, then the map $\lambda \to \dim \mathrm{N}(S + \lambda T)$ is constant on $\mathbb{Q}$ minus at most one point;*

(iii) *If $S \in \mathcal{L}(X)$, then the map $\lambda \to \operatorname{codim} \mathrm{R}(S + \lambda T)$ is constant on $\mathbb{Q}$ minus at most one point.*

Proof. (i) $\Rightarrow$ (ii) : Let $x \in X$ and $f \in X^*$ be such that $T = x \otimes f$, and $S \in \mathcal{L}(X)$. Suppose that there exists a scalar μ such that $\mathrm{N}(S+\mu T)\backslash \mathrm{N}(S) \neq \emptyset$. Then we get easily that $x = Sa$ for some $a \in X$, and so $S+\lambda T = S(I+\lambda a\otimes f)$ for all λ. Therefore, if $\dim \mathrm{N}(S+\lambda T) \neq \dim \mathrm{N}(S)$, $I+\lambda a \otimes f$ is not invertible, and consequently $\lambda f(a) = -1$. This shows that the map $\lambda \to \dim \mathrm{N}(S + \lambda T)$ is constant on $\mathbb{Q} \setminus \{- f(a)^{-1}\}$.

Now, if $\mathrm{N}(S+\lambda T) \subseteq \mathrm{N}(S)$ for all λ, then $\mathrm{N}(S+\lambda T) \subseteq \mathrm{N}(S)\cap\mathrm{N}(f)$ for $\lambda \neq 0$. But, since $\mathrm{N}(S)\cap\mathrm{N}(f) \subseteq \mathrm{N}(S+\lambda T)$, we get that $\mathrm{N}(S+\lambda T) = \mathrm{N}(S)\cap\mathrm{N}(f)$ for all $\lambda \neq 0$, as desired.

(i) $\Rightarrow$ (iii) : Let $S \in \mathcal{L}(X)$. Without loss of generality we can suppose the existence of some $\mu \in \mathbb{Q}$ for which $\operatorname{codim} \mathrm{R}(S + \mu T)$ is finite, and it follows in this case that $S + \mu T$ is semi-Fredholm. Hence $S + \lambda T$ is semi-Fredholm for all λ. Consequently, $S^* + \lambda T^*$ is semi-Fredholm and so $\operatorname{codim} \mathrm{R}(S + \lambda T) = \dim \mathrm{N}(S^* + \lambda T^*)$ for all λ. Finally, since T^* is rank one, the first implication implies that the map $\lambda \to \dim \mathrm{N}(S^* + \lambda T^*) = \operatorname{codim} \mathrm{R}(S + \lambda T)$ is constant on $\mathbb{Q}$ minus at most one point.

$\big[$(ii) or (iii)$\big] \Rightarrow$ (i) : Let δ denote the kernel's dimension or the range's codimension. Assume, on the contrary, that $\mathrm{R}(T)$ contains two linearly independent vectors $u = Tx$ and $v = Ty$, and let N be a closed subspace such that $X = \mathrm{Vect}\{u, v\} \oplus N$. Then it follows easily that $X = \mathrm{Vect}\{x, y\} \oplus M$ where $M = T^{-1}N$. Now, let S be a bounded operator satisfying $Sx = Tx$,

$Sy = -Ty$ and $S : M \mapsto N$ is invertible. Then S is invertible, and

$$\delta(S - \lambda T) = \delta\big(I - \lambda S^{-1}T\big) = 0 \qquad \text{for } \lambda^{-1} \notin \sigma\big(S^{-1}T\big)\,.$$

Hence, $\delta(S-\lambda T) = 0$ for all λ in $\mathbb{Q}$ minus at most one point. This contradicts the fact that $S - T$ and $S + T$ are neither injective nor surjective because $(S - T)x = (S + T)y = 0$, $u \notin \mathrm{R}(S - T)$ and $v \notin \mathrm{R}(S + T)$. Thus, T is rank one operator. ∎

Lemma 6. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map preserving the kernel's dimension or the range's codimension. Then ϕ is injective.*

Proof. Let $T \in \mathcal{L}(X)$ be such that $\phi(T) = 0$. Then, by Lemma 5, T is of rank less than one. Assume that $Tx = y \neq 0$ for some $x, y \in X$, and let S be an invertible operator such that $Sx = y$. It follows that $\phi(S - T) = \phi(S)$ is either injective or surjective. But, since $x \in \mathrm{N}(S - T)$ and $y \notin \mathrm{R}(S - T)$, $\dim \mathrm{N}(S - T)$ and $\operatorname{codim} \mathrm{R}(T - S)$ are non-zero, a contradiction. ∎

Let τ be a ring automorphism of $\mathbb{K}$. An additive map $A : X \to Y$ will be called τ-quasilinear if $A(\lambda x) = \tau(\lambda)Ax$ holds for all numbers $\lambda \in \mathbb{C}$ and $x \in X$. Notice that in the real case all the quasilinear maps are linear because the identity is the only ring automorphism of $\mathbb{R}$, while in the complex case the ring continuous automorphisms are the identity and the complex conjugation.

From Lemmas 5 and 6 it follows that ϕ preserves in both direction the set of operators of rank one, and consequently it takes one of the following forms:

$$\phi(x \otimes f) = Gx \otimes Hf \qquad \text{for all } x \in X \text{ and } f \in X^*, \tag{1}$$

or

$$\phi(x \otimes f) = Kf \otimes Lx \qquad \text{for all } x \in X \text{ and } f \in X^*, \tag{2}$$

where $G : X \to Y$, $H : X^* \to Y^*$, $K : X^* \to Y$ and $L : X \to Y^*$ are τ-quasilinear bijective maps, and $\tau : \mathbb{K} \to \mathbb{K}$ is a ring automorphism, see [8].

Lemma 7. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map preserving the kernel's dimension or the range's codimension. Then $\phi(I)$ is invertible.*

Proof. Let $S = \phi(I)$. Suppose that ϕ preserves the kernel's dimension, then in particular S is injective. To show that S is surjective, let y be a non-zero vector in Y. By (1) and (2) we obtain the existence of $x \in X$, $f \in X^*$ and $g \in Y^*$ such that $f(x) = 1$ and $\phi(x \otimes f) = y \otimes g$. Since

$$\dim \mathrm{N}(S - y \otimes g) = \dim \mathrm{N}(I - x \otimes f) = 1\,,$$

$S - y \otimes g$ is not injective, and hence $y \in \mathrm{R}(S)$ because S is injective.

Now assume that ϕ preserves the range's codimension, then S is surjective. Suppose that $\mathrm{N}(S)$ contains a non-zero vector y, and using (1) or (2) one can find $x \in X$, $f \in X^*$ and $y \in Y$ such that $f(x) = 1$, $g(y) \neq 0$ and $\phi(x \otimes f) = y \otimes g$. Hence, $\operatorname{codim}(S - y \otimes g) = \operatorname{codim}(I - x \otimes f) = 1$. But, since

$$S = (S - y \otimes g)\big(I - g(y)^{-1} y \otimes g\big)\,,$$

$S - y \otimes g$ is surjective, a contradiction. ∎

LEMMA 8. *Let S, T be two bounded invertible operators on X, and denote by δ and δ^* the dimension of the kernel or the codimension of the range. If $\delta(T + F) = \delta^*(S + F)$ for all rank one operator F, then $S = T$.*

Proof. Let $x \in X$, and consider an arbitrary $f \in X^*$ such that $f\big(T^{-1}x\big) = 1$. It follows that $\delta\big(I - T^{-1}x \otimes f\big) = 1$, and

$$\begin{aligned} \delta^*\big(I - S^{-1}x \otimes f\big) &= \delta^*(S - x \otimes f) = \delta(T - x \otimes f) \\ &= \delta\big(I - T^{-1}x \otimes f\big) = 1\,. \end{aligned}$$

Therefore, $I - S^{-1}x \otimes f$ is not invertible, and so $f\big(S^{-1}x\big) = 1$. This shows that $T^{-1}x = S^{-1}x$ for all x. Consequently, $T = S$. ∎

Proof of Theorem 1 *and Theorem* 2. Let δ denote the kernel's dimension or the range's codimension, and suppose that ϕ preserves δ. Then, by Lemma 7, $\phi(I)$ is invertible, and the unital map $\tilde{\phi} = \phi(I)^{-1}\phi$ preserves δ.

We first treat the case when $\tilde{\phi}$ takes the form (1), i.e., $\tilde{\phi}(x \otimes f) = Gx \otimes Hf$ for all $x \in X$ and $f \in X^*$. Observe that for every non-zero scalar λ, $x \in X$ and $f \in X^*$, we have $\delta(I - \lambda x \otimes f) = \delta(I - \tau(\lambda)Gx \otimes Hf)$, and so $I - \lambda x \otimes f$ is invertible if and only if $I - \tau(\lambda)Gx \otimes Hf$ is invertible. This shows that $H(f)(Gx) = \tau(f(x)) = \big(\tau \circ f \circ G^{-1}\big)(Gx)$ for all x and f. Hence $H(f) = \tau \circ f \circ G^{-1}$, and consequently $\tilde{\phi}(x \otimes f) = G(x \otimes f)G^{-1}$ for all x and f. Arguing as in [1, 8] we get that τ and G are bounded, and so τ is either an identity or the complex conjugation.

Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{R} \setminus \big(\sigma(T) \cup \sigma\big(\tilde{\phi}(T)\big)\big)$. For an arbitrary rank one operator F we have

$$\begin{aligned} \delta(T - \lambda I + F) &= \delta\big(\tilde{\phi}(T) - \lambda I + GFG^{-1}\big) \\ &= \delta\big(G^{-1}\big(\tilde{\phi}(T) - \lambda I\big)G + F\big)\,. \end{aligned}$$

Hence, according to Lemma 8, $\tilde{\phi}(T) = GTG^{-1}$. Therefore $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$ where $U = \phi(I)G$ and $V = G^{-1}$.

Now assume that $\tilde{\phi}$ is of the second form $\tilde{\phi}(x \otimes f) = Kf \otimes Lx$. By an argument similar to the previous case one can establish that K and L are a bounded linear, or conjugate linear, operators and that $\tilde{\phi}(F) = KF^*K^{-1}$ for every rank one operator F. Moreover, in this case, the spaces X and Y are reflexive, see [1]. Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{R} \setminus \big(\sigma(T) \cup \sigma\big(\tilde{\phi}(T)\big)\big)$. Consider an arbitrary rank one operator F, then it follows that

$$\begin{aligned}\delta(T - \lambda I + F) &= \delta\big(\tilde{\phi}(T) - \lambda I + KF^*K^{-1}\big)\\ &= \delta\big(K^{-1}\big(\tilde{\phi}(T) - \lambda I\big)K + F^*\big)\,.\end{aligned}$$

But, since $K^{-1}\big(\tilde{\phi}(T) - \lambda I\big)K$ is invertible, $K^{-1}\big(\tilde{\phi}(T) - \lambda I\big)K + F^*$ is semi-Fredholm and hence it has a closed range. Then, using Lemma 8 we get that $T = K^{-1}\tilde{\phi}(T)K$. Therefore, $\phi(T) = U'T^*V'$ for all $T \in \mathcal{L}(X)$ where $U' = \phi(I)K$ and $V' = K^{-1}$.

To complete the proof it remains to show that ϕ cannot take the second form when ϕ preserves the kernel's dimension. Assume on the contrary that $\phi(T) = U'T^*V'$ for all $T \in \mathcal{L}(X)$. Since Y is reflexive, there exists a non-invertible injective operator $S \in \mathcal{L}(Y)$, see [9, 1]. As ϕ is surjective, $S = \phi(T)$ where $T \in \mathcal{L}(X)$ is injective. Consider a nonzero vector $y \in Y$, and let $f = U'^{-1}y$ and $x \in X$ be such that $f(x) = 1$. It follows that

$$\dim \mathrm{N}\left(S - U'(Tx \otimes f)^*V'\right) = \dim \mathrm{N}(T - Tx \otimes f) = 1\,.$$

Consequently, $S - U'(Tx \otimes f)^*V'$ is not injective, and since S is injective, we obtain that

$$\mathrm{Vect}\{U'f\} = \mathrm{R}\left(U'(Tx \otimes f)^*V'\right) \subseteq \mathrm{R}(S)\,.$$

Thus, $y \in \mathrm{R}(S)$ and so S is surjective, a contradiction. ∎

Acknowledgements

The author would like to thank professors Mostafa Mbekhta for several discussions concerning the improvement of this paper.

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