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# Convex Sets without Diametral Pairs

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*Abstract*: Let $X$ be an infinite dimensional normed linear space. It is not difficult to see that arbitrarily near (in the Hausdorff metric) to the unit ball of $X$ there exists a nonempty closed convex set whose diameter is not attained. We show that such sets are dense in the metric space of all nonempty bounded closed convex subsets of $X$ if and only if either $X$ is not a reflexive Banach space or $X$ is a reflexive Banach space in which every weakly closed set contained in the unit sphere $S_X$ has empty relative interior in $S_X$.

*Key words*: Diametral pair, bounded closed convex set, Hausdorff metric.

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Let $X$ be a real normed linear space and $\mathcal{BCC}(X)$ denote the metric space (in the Hausdorff metric) of all nonempty bounded closed convex subsets of $X$. Two elements of a set $C \in \mathcal{BCC}(X)$, whose distance realizes the diameter of $C$, are called a *diametral pair* for $C$. Obviously, if $X$ has a finite dimension, then each element of $\mathcal{BCC}(X)$ admits a diametral pair. For an infinite dimensional $X$, it is not difficult to see that the sets admitting a diametral pair are dense in $\mathcal{BCC}(X)$ (Observation 2.2), and, on the other hand, the unit ball of $X$ can be approximated by elements of $\mathcal{BCC}(X)$ having no diametral pair (Proposition 2.1). (This latter proposition provides also an answer to a question from [3] – see Remark 2.10.)

The aim of the present paper is to investigate the normed linear spaces $X$ in which the sets without diametral pairs are dense in $\mathcal{BCC}(X)$. It turns out that not every infinite dimensional $X$ has this property. In our main result, Theorem 2.8, we show that denseness of the sets without diametral pairs is equivalent to a certain geometric property (W) (see Definition 2.6) which is satisfied if and only if either $X$ is not a reflexive Banach space or $X$ is a reflexive Banach space whose unit sphere $S_X$ contains no weakly closed sets with a nonempty relative interior in $S_X$.

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## 1. Notations and preliminaries

Throughout the present paper, $X$ will denote a real normed linear space with the closed unit ball $B_X$, the open unit ball $B_X^0 = \operatorname{int} B_X$ and the unit sphere $S_X = \partial B_X$. The closed and open ball of radius $r > 0$, centered at $x \in X$, are the sets $B(x,r) = x + rB_X$ and $B_X^0(x,r) = x + rB_X^0$.

Given a nonempty set $A \subset X$, its *diameter* is defined as

$$\delta(A) = \operatorname{diam} A = \sup\{\|x-y\| \,:\, x,y \in A\}.$$

It is easy to see that one always has $\delta(A) = \delta(B)$ whenever $A \subset B \subset \overline{\operatorname{conv}} A$. We say that two points $x, y \in X$ form a *diametral pair* for $A$ if $x, y \in A$ and $\|x-y\| = \delta(A)$. Observe that $x, y$ form a diametral pair for $A$ if and only if $x - y$ is a farthest point from the origin for the set $A - A$.

By $\mathcal{BCC}(X)$ we mean the family of all nonempty, bounded, closed and convex subsets of $X$, equipped with the Hausdorff metric

$$\mathrm{d}_H(A,B) = \max\left\{\sup_{a\in A}\operatorname{dist}(a,B)\,,\ \sup_{b\in B}\operatorname{dist}(b,A)\right\}.$$

Let us start with the following easy and well known properties of convex hulls, which we shall use several times.

Fact 1.1. (a) *If $C_1, \dots, C_m$ are convex sets in $X$, then*

$$\operatorname{conv}\left(\bigcup_{i=1}^m C_i\right) = \left\{\sum_{i=1}^m \lambda_i c_i \,:\, c_i \in C_i\,,\ \lambda_i \ge 0\,,\ \sum_1^m \lambda_i = 1\right\}.$$

(b) $\operatorname{conv}(C \cup F)$ *is closed whenever $C \in \mathcal{BCC}(X)$ and $F \subset X$ is finite.*

We shall also need the following two natural lemmas about diametral pairs.

Lemma 1.2. *Assume that a bounded set $A \subset X$ and a point $z \in X$ are such that $\delta(A) < \delta(D)$ where $D = \overline{\operatorname{conv}}(A \cup \{z\})$. Then every diametral pair for $D$ consists of $z$ and a point of $\overline{\operatorname{conv}} A$.*

*Proof.* We may assume that $A$ is closed and convex, and $z = 0$. In this case, $D = \operatorname{conv}(A \cup \{0\})$ by Fact 1.1(b). Let $x, y$ form a diametral pair for $D$. By Fact 1.1(a), we can write $x = \lambda a,\ y = \mu b$ where $\lambda, \mu \in [0,1]$ and $a, b \in A$. Let, for example, $\lambda \ge \mu$. Then

$$\begin{aligned}\delta(D) = \|\lambda a - \mu b\| &\le \|\lambda a - \mu a\| + \|\mu a - \mu b\| \\ &\le (\lambda-\mu)\delta(D) + \mu\delta(A) \le \lambda\delta(D) \le \delta(D)\,.\end{aligned}$$

Thus all inequalities are, in fact, equalities. But this is possible only if $\lambda = 1$ and $\mu = 0$, which implies $x = a \in A = \overline{\mathrm{conv}}A$ and $y = 0 = z$. ∎

LEMMA 1.3. *Assume that bounded sets $A, B \subset X$ and a point $z \in X$ are such that $\delta(B \cup \{z\}) < \delta(D)$ where $D = \overline{\mathrm{conv}}(A \cup B \cup \{z\})$. If $x \in D$ is such that $\|x - z\| = \delta(D)$, then $x \in \overline{\mathrm{conv}}A$.*

*Proof.* We may assume that the sets $A, B$ are closed and convex since this leaves unchanged the set $D$ and the involved diameters. Since $x \in D$, we can write (by Fact 1.1(a))

$$x = \lambda_n a_n + \mu_n b_n + \nu_n z + p_n\,,$$

where $\lambda_n, \mu_n, \nu_n \ge 0$, $\lambda_n + \mu_n + \nu_n = 1$, $a_n \in A$, $b_n \in B$, and $p_n \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\delta(D) = \|x - z\| &\le \lambda_n\|a_n - z\| + \mu_n\|b_n - z\| + \|p_n\| \\ &\le \lambda_n\delta(D) + \mu_n\delta(B \cup \{z\}) + \|p_n\|\,.\end{aligned}$$

Consequently, $\mu_n\big[\delta(D) - \delta(B \cup \{z\})\big] + \nu_n\delta(D) \le \|p_n\|$. Since $\|p_n\| \to 0$, we must have $\mu_n \to 0$ and $\nu_n \to 0$, which implies that $\lambda_n \to 1$. It follows that $x = \lim \lambda_n a_n = \lim a_n \in A$. ∎

The last lemma of this section is a variant of [1, Lemma 3.5.1].

LEMMA 1.4. *Let $A, B \subset X$ be bounded convex sets and $\varphi\colon X \to \mathbb{R}$ a continuous convex function. If $x \in \overline{\mathrm{conv}}(A \cup B)$ is such that*

$$\varphi(x) > \sup \varphi(A)\,, \qquad \varphi(x) \ge \sup \varphi(B)\,,$$

*then $x \in \overline{B}$.*

*Proof.* Put $m = \sup \varphi(A)$. Let $\{x_n\} \subset \mathrm{conv}(A \cup B)$ be a sequence tending to $x$. By Fact 1.1(a), we can write $x_n = \lambda_n a_n + (1 - \lambda_n)b_n$, where $a_n \in A$, $b_n \in B$ and $\lambda_n \in [0, 1]$. Since $\varphi$ is convex,

$$\varphi(x_n) \;\le\; \lambda_n\varphi(a_n) + (1 - \lambda_n)\varphi(b_n) \;\le\; \lambda_n m + (1 - \lambda_n)\varphi(x)\,.$$

It follows that $\lambda_n(\varphi(x_n) - m) \;\le\; (1 - \lambda_n)(\varphi(x) - \varphi(x_n))$. Thus, for any sufficiently large $n$,

$$\lambda_n \le \frac{\varphi(x) - \varphi(x_n)}{\varphi(x_n) - m}\,,$$

which implies that $\lambda_n \to 0$. Now, $x \in \overline{B}$ since $x - b_n = (x - x_n) + \lambda_n(a_n - b_n) \to 0$. ∎

Corollary 1.5. *Let $\{C_k\}$ be a sequence of convex sets in $X$ and $\varphi\colon X \to \mathbb{R}$ a continuous convex function. Denote $D_n = \overline{\mathrm{conv}}\bigcup_{k\ge n} C_k$ $(n \ge 1)$. If $D_1$ is bounded and $x \in D_1$ is such that $\varphi(x) > \sup\varphi(C_k)$ $(k \in \mathbb{N})$, then $x \in \bigcap_{n\ge 1} D_n$.*

*Proof.* Let $n \ge 2$. We have $x \in D_1 = \overline{\mathrm{conv}}(A \cup B)$ where

$$A = \mathrm{conv}\left(\bigcup\nolimits_{1\le k<n} C_k\right) \qquad \text{and} \qquad B = \mathrm{conv}\left(\bigcup\nolimits_{k\ge n} C_k\right).$$

Put $m = \max_{1\le k<n}\{\sup\varphi(C_k)\}$ and observe that $m < \varphi(x)$. Moreover, convexity of $\varphi$ easily implies $\sup\varphi(A) = m$ and $\sup\varphi(B) \le \varphi(x)$. By Lemma 1.4, $x \in \overline{B} = D_n$. ∎

## 2. Main results

The proof of the following proposition is somewhat similar to the one in [4].

Proposition 2.1. *Let $X$ be an infinite dimensional normed linear space and $\varepsilon \in (0,1)$. There exists a symmetric $C \in \mathcal{BCC}(X)$ without diametral pairs, such that $(1-\varepsilon)B_X \subset C \subset B_X$.*

*Proof.* Since $X$ is infinite dimensional, there exists a basic sequence $\{e_n\} \subset S_X$ (see, e.g., [2, Theorem 4.1.30 and Proposition 4.3.4] applied to the completion of $X$). Fix an arbitrary sequence $\{t_n\} \subset (0,1)$ such that $t_n \to 1$. We claim that the set

$$C = \overline{\mathrm{conv}}\big[\,(1-\varepsilon)B_X \cup \{\pm t_n e_n\}_{n\ge 1}\,\big]$$

has the desired properties. Note that $\delta(C) = 2$ since $C \subset B_X$ and $C$ contains the pairs $\pm t_n e_n$ $(n \in \mathbb{N})$. To show that $C$ has no diametral pairs, it suffices to prove that $C \subset B_X^0$. If this is not the case, take $x \in C \cap S_X$ and $f \in S_{X^*}$ such that $f(x) = 1$. By Corollary 1.5, applied to the sets

$$(1-\varepsilon)B_X\,,\ \{t_1e_1\}\,,\ \{-t_1e_1\}\,,\ \{t_2e_2\}\,,\ \{-t_2e_2\}\,,\ \dots\,,$$

we must have $x \in \bigcap_{n\geq 1} \overline{\mathrm{conv}}\{\pm t_k e_k\}_{k\geq n}$. But the last set is contained in $\bigcap_{n\geq 1} \overline{\mathrm{span}}\{e_k\}_{k\geq n} = \{0\}$, which is a contradiction since $\|x\| = 1$. ∎

On the other hand, the following easy result holds true.

OBSERVATION 2.2. *For every normed linear space $X$, the elements of $\mathcal{BCC}(X)$ that do admit a diametral pair are dense in $\mathcal{BCC}(X)$.*

*Proof.* Let $C \in \mathcal{BCC}(X)$ have no diametral pairs. We may suppose that $\delta(C) = 1$. Fix an arbitrary $\varepsilon > 0$ and choose $\alpha > 0$ so small that $(1+2\varepsilon)(1-\alpha) > 1+\varepsilon$. Choose $u, v \in C$ so that $\|u-v\| > 1-\alpha$. We may suppose that $u = 0$; hence $1-\alpha < \|v\| < 1$. Denote

$$D_0 = C \cup \{-\varepsilon v\} \cup \{(1+\varepsilon)v\}\,, \qquad D = \mathrm{conv}\, D_0\,.$$

Then $D \in \mathcal{BCC}(X)$ (Fact 1.1(b)) and $\mathrm{d}_H(C,D) = \mathrm{d}_H(C,D_0) \leq \varepsilon$. Since $\|(1+\varepsilon)v + \varepsilon v\| > (1+2\varepsilon)(1-\alpha)$ and, for each $x \in C$,

$$\|x+\varepsilon v\| \leq \|x\| + \varepsilon < 1+\varepsilon\,,$$
$$\|x-(1+\varepsilon)v\| \leq \|x-v\| + \varepsilon < 1+\varepsilon\,,$$

the points $-\varepsilon v$ and $(1+\varepsilon)v$ form a diametral pair for $D_0$, and hence also for $D$. ∎

A natural question arises: In which normed spaces $X$ the sets without diametral pairs are dense in $\mathcal{BCC}(X)$? The rest of the present paper is devoted to a complete answer to this question.

DEFINITION 2.3. For a sequence $\{x_n\} \subset X$, we denote

$$\Lambda(\{x_n\}) = \bigcap_{n\geq 1} \overline{\mathrm{conv}}\{x_k\}_{k\geq n}\,.$$

LEMMA 2.4. *Let $\{x_n\}, \{y_n\}$ be sequences in $X$.*

(a) $\Lambda(\{x_n\}) = \Lambda(\{y_n\})$ *whenever* $x_n - y_n \xrightarrow{\text{weak}} 0$.

(b) $\Lambda(\{x_n\}) = \{x\}$ *whenever* $x_n \xrightarrow{\text{weak}} x$.

(c) $\Lambda(\{x_{n_k}\}) \subset \Lambda(\{x_n\})$ *whenever* $\{x_{n_k}\}$ *is a subsequence of* $\{x_n\}$.

*Proof.* (a) If $x \notin \Lambda(\{x_n\})$, then there is an index $n_0$ such that $x \notin \overline{\mathrm{conv}}\{x_k\}_{k\geq n_0}$. By the Hahn-Banach separation theorem, there exists $f \in X^* \setminus \{0\}$ such that $f(x) > \sup_{k\geq n_0} f(x_k) =: \sigma$. Fix an arbitrary $\alpha \in (\sigma, f(x))$. There exists an index $n_1 \geq n_0$ such that $|f(x_k - y_k)| < \alpha - \sigma$ whenever $k \geq n_1$. Then we have $f(y_k) = f(x_k) + f(y_k - x_k) < \sigma + (\alpha - \sigma) = \alpha$ whenever $k \geq n_1$. Consequently, $f(x) > \alpha \geq \sup f(\overline{\mathrm{conv}}\{y_k\}_{k\geq n_1})$, which implies that $x \notin \Lambda(\{y_n\})$. We have proved that $\Lambda(\{y_n\}) \subset \Lambda(\{x_n\})$; the other inclusion follows by symmetry.

The proof of (b) follows from (a) with $y_n = x$ ($n \in \mathbb{N}$), and (c) is obvious. ∎

LEMMA 2.5. *A normed linear space $X$ is not a reflexive Banach space if and only if there exists a sequence $\{x_n\} \subset B_X^0$ with $\Lambda(\{x_n\}) = \emptyset$.*

*Proof.* If $X$ is not a reflexive Banach space, there exists a decreasing sequence $\{C_n\} \subset \mathcal{BCC}(X)$ with empty intersection. (Indeed, this is a well known characterization of non-reflexivity for Banach spaces, and if $X$ is incomplete, we can put $C_n = \left(\hat{y} + \frac{1}{n} B_{\hat{X}}\right) \cap X$ where $\hat{X}$ is the completion of $X$ and $\hat{y} \in \hat{X} \setminus X$.) We may assume that $C_1 \subset B_X^0$. Now, it suffices to choose arbitrarily $x_n \in C_n$ ($n \in \mathbb{N}$) to get the desired sequence. Conversely, if $X$ is a reflexive Banach space, every sequence $\{x_n\} \subset B_X^0$ has a weakly convergent subsequence and hence $\Lambda(\{x_n\}) \neq \emptyset$ by Lemma 2.4 (c)-(b). ∎

DEFINITION 2.6. We shall say that a normed space $X$ has the *property (W)* if for each $v \in S_X$ and each $\varepsilon > 0$ there exists a sequence $\{v_n\} \subset S_X$ such that

$$\|v_n - v\| < \varepsilon \quad (n \in \mathbb{N}) \qquad \text{and} \qquad \Lambda(\{v_n\}) \subset B_X^0.$$

The following standard continuity properties are true in arbitrary normed linear spaces, except for the fact that, in general, the set $A - A$ need not be closed for $A \in \mathcal{BCC}(X)$ (this is the only reason for the reflexivity assumption).

FACT 2.7. *Let $X$ be a reflexive Banach space. It is elementary to prove that:*

- *the function $A \mapsto \varrho(A) := \sup_{x\in A} \|x\|$ is continuous on $\mathcal{BCC}(X)$;*
- *$A \mapsto A - A$ is a continuous mapping of $\mathcal{BCC}(X)$ into $\mathcal{BCC}(X)$;*
- *$(t, A) \mapsto tA$ is a continuous mapping of $\mathbb{R} \times \mathcal{BCC}(X)$ into $\mathcal{BCC}(X)$.*

THEOREM 2.8. *For a normed linear space $X$, the following assertions are equivalent.*

(i) *The elements of $\mathcal{BCC}(X)$ with no diametral pairs are dense in $\mathcal{BCC}(X)$.*

(ii) *$X$ has the property (W) (see Definition 2.6).*

(iii) *Either $X$ is not a reflexive Banach space, or $X$ is a reflexive Banach space such that every weakly closed set contained in $S_X$ has empty relative interior in $S_X$.*

*Proof.* (i) $\Rightarrow$ (iii) Suppose (iii) is false, i.e., $X$ is a reflexive Banach space and $S_X$ contains a weakly compact set $K$ with a nonempty relative interior in $S_X$. Let $v \in K$ and $r > 0$ be such that $B(v, r) \cap S_X \subset K$. Define $C = [-v, v]$ and fix an arbitrary $\varepsilon \in (0, \frac{r}{3})$. Since the mapping $\Phi(D) = \frac{D-D}{\varrho(D)}$ (where $\varrho$ is as in Fact 2.7) is continuous on the set $\{D \in \mathcal{BCC}(X) : \varrho(D) > 0\}$, and $\Phi(C) = C$, there exists $\delta \in (0, 1)$ such that $\mathrm{d}_H(\Phi(D), C) < \varepsilon$ whenever $\mathrm{d}_H(D, C) < \delta$.

Let $D \in \mathcal{BCC}(X)$ satisfy $\mathrm{d}_H(D, C) < \delta$. Denote $\hat{D} = \Phi(D)$. Observe that $\varrho(\hat{D}) = 1$ and the following equivalences hold: $D$ has a diametral pair iff $D - D$ has a farthest point from 0 iff $\hat{D}$ has a farthest point from 0 iff $\hat{D} \cap S_X \neq \emptyset$. Let us prove that $\hat{D}$ intersects $S_X$.

Since $\varrho(\hat{D}) = 1$, there exists a sequence $\{x_n\} \subset \hat{D}$ such that $\|x_n\| \to 1$. Since $\mathrm{d}_H(\hat{D}, C) < \varepsilon$, there exists a sequence $\{t_n\} \subset [-1, 1]$ such that $\|x_n - t_n v\| < \varepsilon$ $(n \in \mathbb{N})$. Since $\hat{D}$ is symmetric, we can suppose that $t_n \in [0, 1]$ for each $n$. Since $X$ is a reflexive Banach space, we can suppose (by passing to a subsequence) that $\{x_n\}$ weakly converges to some $x \in \hat{D}$. Denote $\hat{x}_n = \frac{x_n}{\|x_n\|}$ and observe that $\|x_n - \hat{x}_n\| \to 0$ and

$$1 - t_n = \|x_n\| - \|t_n v\| \leq \|x_n - t_n v\| < \varepsilon \,.$$

Consequently, $\{\hat{x}_n\}$ weakly converges to $x$ and, for any sufficiently large $n$,

$$\begin{aligned} \|\hat{x}_n - v\| &\leq \|\hat{x}_n - x_n\| + \|x_n - t_n v\| + \|t_n v - v\| \\ &< \varepsilon + \varepsilon + (1 - t_n) < 3\varepsilon < r \,. \end{aligned}$$

It follows that $\hat{x}_n \in K$ for each sufficiently large $n$, and hence $x \in K \subset S_X$. This proves that $D$ has a diametral pair whenever $\mathrm{d}_H(D, C) < \delta$; hence (i) is false.

(iii) $\Rightarrow$ (ii) Let $v \in S_X$ and $0 < \varepsilon < 1$. First, suppose $X$ is not a reflexive Banach space. By Lemma 2.5, there exists a sequence $\{x_n\} \subset \varepsilon B_X^0$ such that

$\Lambda(\{x_n\}) = \emptyset$. Define $y_n = v + x_n$ and $v_n = \frac{y_n}{\|y_n\|}$. Passing to a subsequence if necessary, we can suppose that $\|y_n\| \to \alpha \in \mathbb{R}$. Note that $1-\varepsilon \le \alpha \le 1+\varepsilon$ since $1-\varepsilon < \|y_n\| < 1+\varepsilon$ ($n \in \mathbb{N}$). Then we have

$$\|v_n - v\| \le \|v_n - y_n\| + \|y_n - v\| \le \big|\, 1 - \|y_n\| \,\big| + \|x_n\| < 2\varepsilon$$

and, by Lemma 2.4,

$$\Lambda(\{v_n\}) = \Lambda\Big(\Big\{\frac{y_n}{\alpha}\Big\}\Big) = \frac{1}{\alpha}\Lambda(\{y_n\}) = \frac{1}{\alpha}\big(v + \Lambda(\{x_n\})\big) = \emptyset\,.$$

Now, let $X$ be a reflexive Banach space such that $S_X$ contains no weakly compact set with a nonempty interior (relative to $S_X$). The set $A = S_X \cap B(v,\varepsilon)$ is not weakly compact. Reflexivity of $X$ and the Eberlein-Šmulyan theorem imply that $A$ contains a sequence $\{v_n\}$ that weakly converges to some $y \in X \setminus A$. Now, by Lemma 2.4,

$$\Lambda(\{v_n\}) \;=\; \{y\} \;\subset\; \big(B_X \cap B(v,\varepsilon)\big) \setminus A \;\subset\; B_X^0\,.$$

(ii) $\Rightarrow$ (i) Assume (ii). Then an elementary compactness argument and Lemma 2.4 (c) imply that $X$ cannot be finite dimensional. We have to prove that arbitrarily near to any $C \in \mathcal{BCC}(X)$ there exists $D \in \mathcal{BCC}(X)$ with no diametral pairs. If $C$ is a singleton, we can suppose that $C = \{0\}$ and obtain the assertion easily from Proposition 2.1.

Now, let $C$ have a positive diameter. We may assume that $\delta(C) = 1$, and that the origin 0 and some point $v \in S_X$ form a diametral pair for $C$. Fix an arbitrary $\varepsilon \in (0, \frac{1}{4})$. By (ii), there exists a sequence $\{u_n\} \subset (1+2\varepsilon)S_X$ such that $\|u_n - (1+2\varepsilon)v\| < \frac{\varepsilon}{3}$ and $\Lambda(\{u_n\}) \subset (1+2\varepsilon)B_X^0$. Fix a sequence $\{\sigma_n\} \subset (1-\frac{\varepsilon}{3}, 1)$ such that $\sigma_n \to 1$. Define

$$v_n = \sigma_n u_n - \varepsilon v\,, \qquad D_0 = C \cup \{-\varepsilon v\} \cup \{v_n\}_{n\ge 1}\,, \qquad D = \overline{\mathrm{conv}} D_0\,.$$

Observe that $D \supset C$, $\mathrm{dist}(-\varepsilon v, C) \le \| -\varepsilon v - 0\| = \varepsilon$ and

$$\begin{aligned} \mathrm{dist}(v_n, C) &\le \|v_n - v\| = \|\sigma_n u_n - (1+\varepsilon)v\| \\ &\le \|\sigma_n u_n - u_n\| + \|u_n - (1+2\varepsilon)v\| + \|\varepsilon v\| \\ &< (1-\sigma_n) + \tfrac{\varepsilon}{3} + \varepsilon < \tfrac{5}{3}\,\varepsilon\,. \end{aligned}$$

Consequently, $d_H(D,C) \le \frac{5}{3}\varepsilon$. Let us compute $\delta(D)$. For $x \in C$, $n,k \in \mathbb{N}$, $n \ne k$, we have

$$\|x+\varepsilon v\| \le \|x\| + \|\varepsilon v\| \le 1+\varepsilon\,,$$
$$\|x-v_n\| \le \|x-v\| + \|v-v_n\| < 1+\tfrac{5}{3}\varepsilon\,,$$
$$\|v_n-v_k\| \le \|v_n-v\| + \|v-v_k\| < \tfrac{5}{3}\varepsilon + \tfrac{5}{3}\varepsilon < 1\,,$$
$$\|v_n+\varepsilon v\| = \sigma_n\|u_n\| = \sigma_n(1+2\varepsilon) \ \nearrow\ 1+2\varepsilon \qquad (\text{as } n\to\infty)\,.$$

Thus $\delta(D) = \delta(D_0) = 1+2\varepsilon$. It remains to show that $D$ has no diametral pairs.

Suppose that $x, y \in C$ are such that $\|x-y\| = \delta(C)$. By Lemma 1.2, applied to $z = -\varepsilon v$ and $A = C \cup \{v_n\}_{n\ge1}$, one of the two points, say $y$, equals $-\varepsilon v$. By Lemma 1.3, applied to $A = \{v_n\}_{n\ge0}$, $B = C$ and $z = -\varepsilon v$, we must have $x \in \overline{\mathrm{conv}}\{v_n\}_{n\ge1}$. Hence

$$x-y \in \overline{\mathrm{conv}}\{v_n+\varepsilon v\}_{n\ge1} = \overline{\mathrm{conv}}\{\sigma_n u_n\}_{n\ge1}\,,$$

$\|x-y\| = 1+2\varepsilon$ and $\|\sigma_n u_n\| = \sigma_n(1+2\varepsilon) < 1+2\varepsilon$ ($n \in \mathbb{N}$). By Corollary 1.5 (with $\varphi = \|\cdot\|$ and $C_n$ being the singleton $\{\sigma_n u_n\}$ for each $n$) and Lemma 2.4, we have

$$x-y \in \Lambda(\{\sigma_n u_n\}) = \Lambda(\{u_n\}) \subset (1+2\varepsilon)B_X^0$$

which is a contradiction that completes the proof. ∎

Let us conclude with two simple remarks. Recall that $X$ is said to have the *Kadets-Klee property* if the topological spaces $(S_X,\text{weak})$ and $(S_X,\text{norm})$ have the same convergent sequences.

*Remarks* 2.9. (a) Every finite dimensional normed linear space fails the property (W).

(b) Every normed linear space that is not a reflexive Banach space has the property (W).

(c) Every infinite dimensional reflexive Banach space with the Kadets-Klee property has the property (W).

(To see this, fix $v \in S_X$ and $\varepsilon > 0$. Since $S_X \cap B(v,\varepsilon)$ is not norm compact, it contains a sequence having no convergent subsequence. This sequence admits a subsequence $\{v_n\}$ that weakly converges to some $y \in B_X \cap B(v,\varepsilon)$. By the Kadets-Klee property, we must have $y \in B_X^0$. By Lemma 2.4 (b), $\Lambda(\{v_n\}) = \{y\} \subset B_X^0$.)

(d) To get an example of an infinite dimensional reflexive Banach space without the property (W), start with an arbitrary infinite dimensional reflexive Banach space $(X, \|\cdot\|)$, fix an $f \in S_{X^*}$, and define an equivalent norm by $|\!|\!|x|\!|\!| = \max\{\|x\|, 2|f(x)|\}$. Then $(X, |\!|\!|\cdot|\!|\!|)$ fails (W) since the corresponding unit sphere contains the closed convex set $B_{(X,\|\cdot\|)} \cap f^{-1}(\frac{1}{2})$ with a nonempty relative interior.

*Remark* 2.10. Note that the symmetric set $C$ constructed in Proposition 2.1 has no farthest points from the origin. It follows that *a normed linear space $X$ is finite dimensional iff every symmetric element of $\mathcal{BCC}(X)$ has a farthest point from the origin iff every element of $\mathcal{BCC}(X)$ is remotal.* This answers a question from [3] (see Remark 2.8 therein) where a similar result was proved under the additional assumption that $X$ is a reflexive Banach space.

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