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Kleisli and Eilenberg-Moore Constructions as Parts of Biadjoint Situations

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Abstract: We consider monads over varying categories, and by defining the morphisms of Kleisli and of Eilenberg-Moore from a monad to another and the appropriate transformations (2-cells) between morphisms of Kleisli and between morphisms of Eilenberg-Moore, we obtain two 2-categories $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$. Then we prove that $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$ are, respectively, 2-isomorphic to the conjugate of $\mathbf{Kl}$ and to the transpose of $\mathbf{EM}$, for two suitably defined 2-categories $\mathbf{Kl}$ and $\mathbf{EM}$, related, respectively, to the constructions of Kleisli and of Eilenberg-Moore. Next, by considering those morphisms and transformations of monads that are simultaneously of Kleisli and of Eilenberg-Moore, we obtain a 2-category $\mathbf{Mnd}_{\mathrm{alg}}$, of monads, algebraic morphisms, and algebraic transformations, and, to confirm its naturalness, we, on the one hand, prove that its underlying category can be obtained by applying the Ehresmann-Grothendieck construction to a suitable contravariant functor, and, on the other, we provide an explicit 2-embedding of a certain 2-category, $\mathbf{Sig}_{\mathfrak{pd}}$, of many-sorted signatures (hence also of another 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications), arising from the field of many-sorted universal algebra, into a 2-category of the type $\mathbf{Mnd}_{\mathrm{alg}}$. Moreover, we investigate for the adjunctions between varying categories the counterparts of the concepts previously defined for the monads, obtaining several 2-categories of adjunctions, as well as several 2-functors from them to the corresponding 2-categories of monads, and all in such a way that the classical Kleisli and Eilenberg-Moore constructions are left and right biadjoints, respectively, for these 2-functors. Finally, we define a 2-category $\mathbf{Ad}_{\mathrm{alg}}$, of adjunctions, algebraic squares, and algebraic transformations, and prove that there exists a canonical 2-functor $\mathrm{Md}_{\mathrm{alg}}$ from $\mathbf{Ad}_{\mathrm{alg}}$ to $\mathbf{Mnd}_{\mathrm{alg}}$.

Key words: Morphism of Kleisli, morphism of Eilenberg-Moore, transformation of Kleisli, transformation of Eilenberg-Moore, adjoint square of Kleisli, adjoint square of Eilenberg-Moore, algebraic square of adjunctions, transformation of algebraic squares, algebraic morphism of monads, algebraic transformation.

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"In memory of our dear friend Fuensanta Andreu Vaillo (1955–2008)"

1. Introduction

As it is well-known P.J. Huber proved in [19, p. 370] that every adjoint situation gave rise to a monad. In this connection, we remark that Mac Lane, in [23, p. 159] says: "...Then Hilton (and others) raised the question as to whether any monad arises from an adjunction. Two independent answers appeared: Kleisli's construction in [1965] of the "free algebra" realization and the decisive construction by Eilenberg-Moore [1965] of the category of algebras for a monad." In this article, taking into account (and generalizing) the slogan due to Mac Lane: "Adjoint functors arise everywhere", stated in the preface to the first edition of [23], we solve affirmatively the problem as to whether the classical constructions of Kleisli and of Eilenberg-Moore arise as natural examples of biadjoint situations. Specifically, we investigate several 2-categories of monads—from now on understood as pairs $(\mathbf{C}, \mathbb{T})$, where $\mathbf{C}$ is a category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathbf{C}$—and of adjunctions, which enable us to prove that the Kleisli (see [20] and [23]) and Eilenberg-Moore (see [9] and [23]) classical constructions are, respectively, left and right biadjoints to certain 2-functors from some convenient 2-categories of adjunctions to some convenient 2-categories of monads.

Let us notice that the source and motivation for this work has to be found in a previous investigation on two-dimensional many-sorted general algebra carried out in [6] and [29]. One of the aims of the last-mentioned works was to prove the equivalence between clones (represented by Hall algebra) and finitary many-sorted algebraic theories (represented by Bénabou algebras) using the equivalence between the many-sorted specifications of Hall and Bénabou in a "convenient 2-category of many-sorted specifications" and by means of a pseudo-functor from such a 2-category to the 2-category $\mathbf{Cat}$, of categories. The crucial element of the procedure consists in properly defining the aforementioned "convenient 2-category of many-sorted specifications" and to do so the theory of Fujiwara in [12] and [13] (which was proposed to cover the case of ordinary single-sorted algebras) was extended in [6] and [29] into several directions. Firstly, by defining the concept of *polyderivor*, from a many-sorted signature into another, which assigns to basic sorts, words and to formal operations, families of derived terms, which subsumes the known earlier proposals of derivor, defined in [16], and that of morphism between many-sorted algebraic theories. Secondly, by equipping the category of many-sorted signatures and polyderivors, $\mathbf{Sig}_{\mathfrak{pd}}$, with a 2-category structure, by defining the appropriate transformations between the polyderivors, that generalize the equivalences defined by Fujiwara in [13], and allow richer comparisons between many-sorted signatures than those usually considered in the literature devoted to it. Lastly, by introducing the corresponding 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications, $\mathfrak{pd}$-specification morphisms, and transformations between $\mathfrak{pd}$-specification morphisms. Moreover, in [6] and [29] we defined a 2-category $\mathbf{Alg}_{\mathfrak{pd}}$ which has as objects (0-cells) the pairs $(\mathbf{\Sigma}, \mathbf{A})$, with $\mathbf{\Sigma}$ a many-sorted signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, as morphisms (1-cells) from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the pairs $(\mathbf{d}, h)$, with $\mathbf{d}$ a polyderivor from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and h a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$, where $\mathbf{d}^*_{\mathfrak{pd}}$ is a functor from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$ (see [6] or [29] for its definition), and as 2-cells from $(\mathbf{d}, f)$ to $(\mathbf{e}, g)$, where $(\mathbf{d}, f)$ and $(\mathbf{e}, g)$ are morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the 2-cells $\xi\colon \mathbf{\Sigma} \rightsquigarrow \mathbf{\Lambda}$ in $\mathbf{Sig}_{\mathfrak{pd}}$ such that $\xi^{\mathbf{B}} \circ f = g$. The principal significance of the 2-categories $\mathbf{Sig}_{\mathfrak{pd}}$, $\mathbf{Spf}_{\mathfrak{pd}}$, and $\mathbf{Alg}_{\mathfrak{pd}}$ in the present context, which, we add, have already been used for other purposes in [6] and [29], is that they did not fall under the standard frame of formal monad theory as stated by Street in [30] and therefore a more general theory must be sought to account for the aforementioned, and similar, 2-categories.

To show the way in which the aforementioned subjects are developed, next we proceed to, briefly, describe the contents of the subsequent sections of this article.

In the second section we define the morphisms of Kleisli and the morphisms of Eilenberg-Moore from a monad $(\mathbf{C}, \mathbb{T})$ to another $(\mathbf{C}', \mathbb{T}')$, from which we obtain, respectively, the categories $\mathbf{Mnd}_{\mathrm{Kl}}$, of monads and morphisms of Kleisli, and $\mathbf{Mnd}_{\mathrm{EM}}$, of monads and morphisms of Eilenberg-Moore. Then we prove that $\mathbf{Mnd}_{\mathrm{Kl}}$ is isomorphic to the category $\mathbf{Kl}$, of monads and morphisms from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ the pairs (J, H), where J is a functor from $\mathbf{C}$ to $\mathbf{C}'$ and H a functor from $\mathbf{Kl}(\mathbb{T})$, the Kleisli category of $\mathbb{T}$, to $\mathbf{Kl}(\mathbb{T}')$, the Kleisli category of $\mathbb{T}'$, such that $H \circ F_{\mathbb{T}} = F_{\mathbb{T}'} \circ J$, where $F_{\mathbb{T}}$ and $F_{\mathbb{T}'}$ are the canonical functors from $\mathbf{C}$ to $\mathbf{Kl}(\mathbb{T})$ and from $\mathbf{C}'$ to $\mathbf{Kl}(\mathbb{T}')$. As for $\mathbf{Mnd}_{\mathrm{Kl}}$, we prove that $\mathbf{Mnd}_{\mathrm{EM}}$ is isomorphic to the dual of the category $\mathbf{EM}$, of monads and morphisms from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ the pairs (K, H), where K is a functor from $\mathbf{C}'$ to $\mathbf{C}$ and H a functor from $\mathbf{EM}(\mathbb{T}')$, the Eilenberg-Moore category of $\mathbb{T}'$, to $\mathbf{EM}(\mathbb{T})$, the Eilenberg-Moore category of $\mathbb{T}$, such that $G^{\mathbb{T}} \circ H = K \circ G^{\mathbb{T}'}$, where $G^{\mathbb{T}}$ and $G^{\mathbb{T}'}$ are the canonical functors from $\mathbf{EM}(\mathbb{T})$ to $\mathbf{C}$ and from $\mathbf{EM}(\mathbb{T}')$ to $\mathbf{C}'$, respectively. Following this, after equipping the categories $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$ with 2-category structures and defining, on the one hand, the 2-category $\mathbf{Kl}$, with 2-cells from (J, H) to (J', H') the natural transformations from H to H', and, on the other hand, the 2-category **EM**, with 2-cells from (K, H) to (K', H') the natural transformations from H to H', we prove that $\mathbf{Mnd}_{\mathrm{Kl}}$ is 2-isomorphic to $\mathbf{Kl}^{\mathrm{c}}$, the conjugate of **Kl**, and that $\mathbf{Mnd}_{\mathrm{EM}}$ is 2-isomorphic to $\mathbf{EM}^{\mathrm{t}}$, the transpose of **EM** (for the definition of the concepts of "conjugate" and "transpose" see [2, p. 26]). Furthermore, by defining the Street transformations from a Kl-morphism to another, respectively, from an EM-morphism to another, we obtain a sub-2-category of $\mathbf{Mnd}_{\mathrm{Kl}}$, respectively, of $\mathbf{Mnd}_{\mathrm{EM}}$, and characterize its images in **Kl** and **EM**. Finally, by considering those morphisms and transformations of monads that are, in a well-defined sense, simultaneously of Kleisli and of Eilenberg-Moore, we obtain a 2-category $\mathbf{Mnd}_{\mathrm{alg}}$, of monads, algebraic morphisms, and algebraic transformations, and, to confirm its naturalness, we, on the one hand, prove that its underlying category can be obtained by applying the Ehresmann-Grothendieck construction to a suitable contravariant functor, and, on the other, provide an explicit embedding of the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ (thus also of the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$) into a 2-category of the type $\mathbf{Mnd}_{\mathrm{alg}}$. Moreover, we state that $\mathbf{Mnd}_{\mathrm{alg}}$, by its very definition, bears some interesting relations with the 2-categories $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$. Specifically, we show that it is isomorphic to the sub-2-category of $\mathbf{Mnd}_{\mathrm{Kl}}$ for which the underlying functors of the 1-cells have a right adjoint, and to the sub-2-category of $\mathbf{Mnd}_{\mathrm{EM}}$ for which the underlying functors of the 1-cells have a left adjoint.

We must remark here that the 2-cells of the 2-categories $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$ are *essentially* coincident with the 2-cells introduced by Lack and Street in [22]. In the aforementioned article, Lack and Street work out a formal theory of monads, continuing the labor begun by Street in [30], developing the usual elements of the theory of monads in the context of monads in a 2-category $\mathcal{K}$. They give an explicit description of the free completion $\mathbf{EM}(\mathcal{K})$ of a 2-category $\mathcal{K}$ under the Eilenberg-Moore construction, showing that it has the same underlying category as the 2-category $\mathbf{Mnd}(\mathcal{K})$ of monads in $\mathcal{K}$ but different 2-cells. In the case where $\mathcal{K} = \mathbf{Cat}$ we have that the 2-category $\mathbf{Kl}(\mathbf{Cat})$ of Lack and Street, the free completion under Kleisli objects of **Cat**, is the conjugate of our 2-category $\mathbf{Mnd}_{\mathrm{Kl}}$ and that their 2-category $\mathbf{EM}(\mathbf{Cat})$, the free completion under Eilenberg-Moore objects of **Cat**, is the transpose of our 2-category $\mathbf{Mnd}_{\mathrm{EM}}$.

Perhaps it is appropriate at this point to note that we came upon the concepts of 2-cell between morphisms of Kleisli and of 2-cell between morphisms of Eilenberg-Moore after investigating, as mentioned above, a twofold generalization of the morphisms and transformations of Fujiwara which provide, among others, the aforementioned 2-categories $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ and $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$, and natural 2-embeddings from them into a convenient 2-category $\mathbf{Mnd}_{\mathrm{alg}}$ (see below). It is nonetheless reassuring that what we defined in order to get an abstract rendition of what it happened in some 2-categories of many-sorted algebras and specifications has turned out to have the very important and natural property of being a completion as proved by Lack and Street in [22]. To this we add that Lack and Street in [22] develop a great deal of their formal theory of monads under the proviso that one is willing to work with $\mathbf{EM}(\mathcal{K})$ as the 2-category of monads instead of working with $\mathbf{Mnd}(\mathcal{K})$.

In the third section, turning our attention to adjunctions between varying categories, we obtain from a certain 2-category $\mathbf{Ad}$, of adjunctions, two new 2-categories of adjunctions, $\mathbf{Ad}_{\mathrm{Kl}}$ and $\mathbf{Ad}_{\mathrm{EM}}$, which will allow us to extend to 2-functors the classical, and well-known, construction that assigns to an adjunction a monad, and this in such a way that the Kleisli and Eilenberg-Moore constructions will be left and right biadjoints, respectively, for such 2-functors. Moreover, the morphisms and transformations of Kleisli and of Eilenberg-Moore between monads will be characterized, respectively, as the image of morphisms and transformations of Kleisli and of Eilenberg-Moore between the adjunctions. Finally, we define a 2-category $\mathbf{Ad}_{\mathrm{alg}}$ which has adjunctions as 0-cells, algebraic squares between adjunctions as 1-cells, and algebraic transformations between algebraic squares as 2-cells, and prove that there exists a canonical 2-functor $\mathrm{Md}_{\mathrm{alg}}$ from $\mathbf{Ad}_{\mathrm{alg}}$ to $\mathbf{Mnd}_{\mathrm{alg}}$. The existence of $\mathrm{Md}_{\mathrm{alg}}$ confirms, in particular, that, from an algebraic standpoint, algebraic squares and transformations between algebraic squares play for adjunctions the same role that algebraic morphisms and algebraic transformations between algebraic morphisms play for monads.

In summary, this article gives, among other things, an abstract analysis of the classical Kleisli and Eilenberg-Moore constructions. The two constructions, notably the latter, have proved to be particularly influential within category theory (and theoretical computer science) over the past forty-five years or so; therefore they merit an abstract analysis; and we have obtained results that are not entirely obvious consequences of the work of previous articles (in particular of [22], which is a chronological predecessor, but not a precursor of this article, as explained above). Moreover, we have given a small amount of development of examples that do not seem to have been considered by previous authors.

In this article, unless otherwise stated, we assume that the foundational system underlying category theory is $\mathbf{ZFC} + \exists\, \mathrm{GU}(\boldsymbol{\mathcal{U}})$, i.e., Zermelo-Fraenkel

(-Skolem) set theory with the Axiom of Choice plus the existence of a Grothendieck universe $\boldsymbol{\mathcal{U}}$ fixed once and for all (for an explanation of the concept of *Grothendieck universe* see, e.g., [23, p. 22]). Therefore every set we consider in this article will be either a $\boldsymbol{\mathcal{U}}$-small set, i.e., an element of $\boldsymbol{\mathcal{U}}$, or a $\boldsymbol{\mathcal{U}}$-large set, i.e., a subset of $\boldsymbol{\mathcal{U}}$, or a set which is neither $\boldsymbol{\mathcal{U}}$-small nor $\boldsymbol{\mathcal{U}}$-large. Besides, we let **Set** stand for the category with objects the $\boldsymbol{\mathcal{U}}$-small sets and morphisms the mappings between $\boldsymbol{\mathcal{U}}$-small sets, and, depending on the context, that **Cat** denotes either, the category of the $\boldsymbol{\mathcal{U}}$-categories (i.e., categories $\mathbf{C}$ such that the set of objects of $\mathbf{C}$ is a subset of $\boldsymbol{\mathcal{U}}$, and the hom-sets of $\mathbf{C}$ elements of $\boldsymbol{\mathcal{U}}$), and functors between $\boldsymbol{\mathcal{U}}$-categories, or the 2-category of the $\boldsymbol{\mathcal{U}}$-categories, functors between $\boldsymbol{\mathcal{U}}$-categories, and natural transformations between functors.

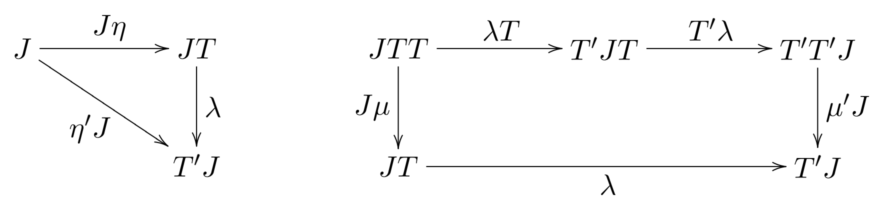
In all that follows we use standard concepts and constructions from category theory, see e.g., [1], [2], [8], [11], [17], [23], [24], and [31].

2. Monads, morphisms, and transformations

In this section we define two types of morphisms from a monad to another, called morphisms of Kleisli and of Eilenberg-Moore, respectively. Then, for each type of morphism, we define its corresponding notion of transformation, which is more general than that of 2-cell between morphisms of monads defined by Street in [30]—providing some examples of this fact—and which are *essentially* coincident with those defined by Lack and Street in [22]. From this we obtain the 2-categories $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$ and prove that $\mathbf{Mnd}_{\mathrm{Kl}}$ is 2-isomorphic to the conjugate of another 2-category, **Kl**, related to the Kleisli construction, and that $\mathbf{Mnd}_{\mathrm{EM}}$ is 2-isomorphic to the transpose of another 2-category, **EM**, related to the Eilenberg-Moore construction. Finally, by considering those morphisms and transformations that are simultaneously of Kleisli and of Eilenberg-Moore we obtain the algebraic morphisms and transformations, and we provide an explicit 2-embedding of a certain 2-category, $\mathbf{Sig}_{\mathfrak{pd}}$, of many-sorted signatures (thus also of another 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications), arising from the field of many-sorted universal algebra, into a 2-category of the type $\mathbf{Mnd}_{\mathrm{alg}}$.

We next turn to defining the morphisms of Kleisli between monads, the identity at a monad, and the composition of two composable morphisms of Kleisli. But before doing that we recall, once more, that for us a monad is a pair $(\mathbf{C}, \mathbb{T})$, where $\mathbf{C}$ is a category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathbf{C}$.

Definition 2.1. Let $(\mathbf{C}, \mathbb{T})$ and $(\mathbf{C}', \mathbb{T}')$ be two monads. A *morphism of Kleisli* or, for brevity, a *Kl-morphism*, from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ is a pair (J, λ), where J is a functor from $\mathbf{C}$ to $\mathbf{C}'$ and λ a natural transformation from JT to $T'J$ such that the following diagrams commute



We write $(J, \lambda)\colon (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}', \mathbb{T}')$ to denote that (J, λ) is a Kl-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$. For every monad $(\mathbf{C}, \mathbb{T})$, the *identity at* $(\mathbf{C}, \mathbb{T})$, denoted by $\mathrm{id}_{(\mathbf{C}, \mathbb{T})}$, is the morphism $(\mathrm{Id}_{\mathbf{C}}, \mathrm{id}_T)$. If (J, λ) is a Kl-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ and (J', λ') a Kl-morphism from $(\mathbf{C}', \mathbb{T}')$ to $(\mathbf{C}'', \mathbb{T}'')$, then the *composition of* (J, λ) *with* (J', λ'), denoted by $(J', \lambda') \circ (J, \lambda)$, is

$$(J', \lambda') \circ (J, \lambda) = (J' \circ J, \lambda' J \circ J' \lambda).$$

We now give an example of the concept of Kl-morphism that comes from the theory of closure spaces (in this respect we recall that every Galois correspondence, which is an adjunction, gives rise in a canonical way to a closure space).

Example 2.2. As it is well-known, see, e.g., [23, p. 139] to every closure space (A, C) there corresponds a monad $(\mathbf{Sub}(A), \mathbb{C})$, where $\mathbf{Sub}(A)$ is the category determined by the ordered set $(\mathrm{Sub}(A), \subseteq)$, with $\mathrm{Sub}(A)$ the power set of A, and $\mathbb{C}$ the monad on $\mathbf{Sub}(A)$ obtained from the closure operator C on A. Moreover, to every continuous mapping (or, synonymously, morphism of closure spaces) j from (A, C) to (B, D) there corresponds a Kl-morphism (J^j, λ^j) from $(\mathbf{Sub}(A), \mathbb{C})$ to $(\mathbf{Sub}(B), \mathbb{D})$, where the functor J^j from $\mathbf{Sub}(A)$ to $\mathbf{Sub}(B)$ is precisely $j[\cdot]$, i.e., the formation of j-direct images, and λ^j the trivial natural transformation from $j[\cdot] \circ C$ to $D \circ j[\cdot]$. From now on, $(j[\cdot], \lambda)$ stands for (J^j, λ^j).

We leave it to the reader to verify the following proposition.

Proposition 2.3. *Monads and* Kl-*morphisms yield a category, hereafter denoted by* $\mathbf{Mnd}_{\mathrm{Kl}}$.

In the following proposition, for a given pair of monads, we prove that there exists a one-to-one correspondence between the set of all pairs of functors that (in the same direction) relate, respectively, the underlying categories of the monads and the categories of Kleisli associated to the underlying monads on the underlying categories of the monads and satisfy, in addition, a suitable condition (specified below) and the set of all Kl-morphisms between the given pair of monads.

But before stating the aforementioned proposition we recall that, for a category $\mathbf{C}$ and a monad $\mathbb{T}$ on $\mathbf{C}$, the *Kleisli category* of $\mathbb{T}$, denoted by $\mathbf{Kl}(\mathbb{T})$, has the same objects that $\mathbf{C}$ and, for every $X, Y \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{Kl}(\mathbf{C})}(X, Y)$ is $\mathrm{Hom}_{\mathbf{C}}(X, T(Y))$. If $P\colon X \longrightarrow T(Y)$ and $Q\colon Y \longrightarrow T(Z)$, then we define the composition of P with Q as $Q \diamond P = \mu_Z \circ T(Q) \circ P$. The identity morphism at an object X is η_X. Moreover, let $F_{\mathbb{T}}$ denote the functor from $\mathbf{C}$ to $\mathbf{Kl}(\mathbb{T})$ which is the identity mapping on the objects and sends a morphism $f\colon X \longrightarrow Y$ to $\eta_Y \circ f\colon X \longrightarrow T(Y)$, and $G_{\mathbb{T}}$ the functor from $\mathbf{Kl}(\mathbb{T})$ to $\mathbf{C}$ which sends an object X to $T(X)$ and a morphism $P\colon X \longrightarrow T(Y)$ to $\mu_Y \circ T(P)$.

Proposition 2.4. *Let* $(\mathbf{C}, \mathbb{T})$ *and* $(\mathbf{C}', \mathbb{T}')$ *be two monads. Then there exists a one-to-one correspondence between the* Kl-*morphisms* (J, λ) *from* $(\mathbf{C}, \mathbb{T})$ *to* $(\mathbf{C}', \mathbb{T}')$ *and the pairs* (J, H), *where* J *is a functor from* $\mathbf{C}$ *to* $\mathbf{C}'$ *and* H *a functor from* $\mathbf{Kl}(\mathbb{T})$ *to* $\mathbf{Kl}(\mathbb{T}')$, *such that the following equality holds* $H \circ F_{\mathbb{T}} = F_{\mathbb{T}'} \circ J$.

Proof. Let $(J, \lambda)\colon (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}, \mathbb{T}')$ be a Kl-morphism. Then the pair (J, H_λ), where H_λ is the functor from $\mathbf{Kl}(\mathbb{T})$ to $\mathbf{Kl}(\mathbb{T}')$ which assigns to a $\mathbf{C}$-morphism P from Y to $T(X)$ the $\mathbf{C}'$-morphism $\lambda_X \circ J(P)$, is such that $H_\lambda \circ F_{\mathbb{T}} = F_{\mathbb{T}'} \circ J$.

Reciprocally, if (J, H), where J is a functor from $\mathbf{C}$ to $\mathbf{C}'$ and H a functor from $\mathbf{Kl}(\mathbb{T})$ to $\mathbf{Kl}(\mathbb{T}')$, is such that $H \circ F_{\mathbb{T}} = F_{\mathbb{T}'} \circ J$, then let κ be the conjugate natural transformation of the identity natural transformation from $F_{\mathbb{T}'} \circ J$ to $H \circ F_{\mathbb{T}}$ and λ_H the composition of $F_{\mathbb{T}}$ with κ. Then

$$\lambda_H (= \kappa F_{\mathbb{T}})\colon JT = JG_{\mathbb{T}} F_{\mathbb{T}} \Longrightarrow G_{\mathbb{T}'} H F_{\mathbb{T}} = G_{\mathbb{T}'} F_{\mathbb{T}'} J = T' J,$$

and, for every $\mathbf{Kl}(\mathbb{T})$-object X, $(\lambda_H)_X$ is $H(\mathrm{id}^{\mathbf{C}}_{T(X)})$.

To prove that the pair (J, λ_H) is a Kl-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ it is sufficient to take into account the defining diagrams of the Kl-morphisms for the natural transformation λ_H and the triangular equations of the involved adjunctions.

It is easy to check that both correspondences are mutually inverse. ∎

Definition 2.5. The category $\mathbf{Kl}$ has as objects the monads and as morphisms from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ the pairs (J, H), where J is a functor from $\mathbf{C}$ to $\mathbf{C}'$ and H a functor from $\mathbf{Kl}(\mathbb{T})$ to $\mathbf{Kl}(\mathbb{T}')$, such that the following equality holds

$$H \circ F_{\mathbb{T}} = F_{\mathbb{T}'} \circ J.$$

On account of the above definition and from the functorial character of the one-to-one correspondence defined in Proposition 2.4 we have the following proposition.

Proposition 2.6. *The categories* $\mathbf{Mnd}_{\mathrm{Kl}}$ *and* $\mathbf{Kl}$ *are isomorphic.*

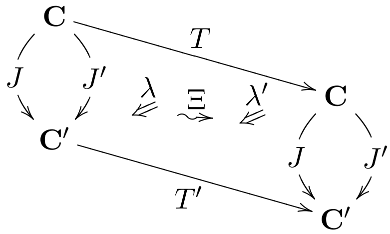
As stated in the definition immediately below, just as, e.g., natural transformations compare functors, the so-called transformations of Kleisli compare Kl-morphisms of monads by means of a certain class of 2-cells (that, for a given pair of Kl-morphisms of monads, are in a one-to-one correspondence with a suitable subset of the set of all natural transformations between the functors on the categories of Kleisli associated to the given pair of Kl-morphisms).

Remark 2.7. It is worth pointing out that Lack and Street in [22, p. 248] from a 2-category $\mathcal{K}$, define another 2-category $\mathbf{Kl}(\mathcal{K})$ which has (as objects the monads, as 1-cells the morphisms of Kleisli, and) as 2-cells precisely the opposite of the transformations of Kleisli.

Definition 2.8. Let (J, λ) and $(J', \lambda')\colon (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}', \mathbb{T}')$ be two Kl-morphisms of monads. A *transformation of Kleisli* or, for brevity, a *Kl-transformation*, from (J, λ) to (J', λ') is a natural transformation $\Xi\colon J' \Longrightarrow T'J$ making commutative the following diagram

$$\begin{array}{ccccc}
J'T & \xrightarrow{\ \Xi T\ } & T'JT & \xrightarrow{\ T'\lambda\ } & T'T'J \\
{\scriptstyle \lambda'}\downarrow & & & & \downarrow{\scriptstyle \mu' J} \\
T'J' & \xrightarrow[\ T'\Xi\ ]{} & T'T'J & \xrightarrow[\ \mu' J\ ]{} & T'J
\end{array}$$

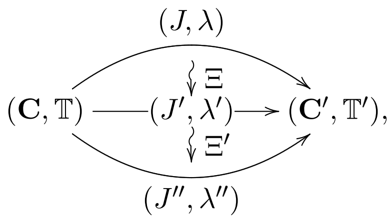
If Ξ is a Kl-transformation from (J, λ) to (J', λ'), then we will write Ξ_{k} for the unique natural transformation from $J'T$ to $T'J$ in the above diagram. Moreover, we will use $\Xi\colon (J, \lambda) \rightsquigarrow (J', \lambda')$ or a diagram as displayed in



to indicate that Ξ is a Kl-transformation from (J,λ) to (J',λ').

For every Kl-morphism (J,λ) from $(\mathbf{C},\mathbb{T})$ to $(\mathbf{C}',\mathbb{T}')$ the *Kl-identity at* (J,λ) is the natural transformation $J\eta'\colon J \Longrightarrow T'J$.

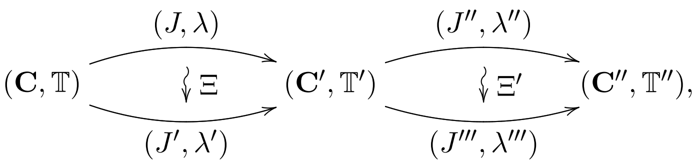
The *vertical composition* of two Kl-transformations as in the diagram



denoted by $\Xi' \mathbin{\tilde{\circ}} \Xi$, is the natural transformation

$$J'' \xrightarrow{\ \Xi'\ } T'J' \xrightarrow{\ T'\Xi\ } T'T'J \xrightarrow{\ \mu' J\ } T'J.$$

The *horizontal composition* of two Kl-transformations as in the diagram



denoted by $\Xi' \mathbin{\tilde{*}} \Xi$, is the natural transformation

$$J'''J' \xrightarrow{\ J'''\Xi\ } J'''T'J \xrightarrow{\ \Xi'_{\mathrm{k}}J\ } T'J''J.$$

We leave it to the reader to verify the following proposition.

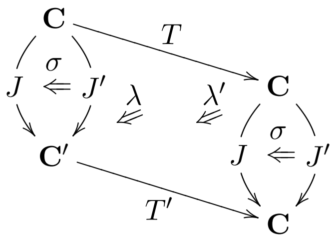
Proposition 2.9. *Monads,* Kl-*morphisms, and* Kl-*transformations between* Kl-*morphisms yield a* 2-*category, hereafter denoted by* $\mathbf{Mnd}_{\mathrm{Kl}}$.

We next give an example (really a bundle or schema of examples) of the concept of Kl-transformation that comes from the theory of closure spaces. We shall see later concrete algebraic and logical examples of the aforementioned concept.

EXAMPLE 2.10. Let (A, C) and (B, D) be two closure spaces. If j and j' are two continuous mappings from (A, C) to (B, D) and, for every $X \subseteq A$, it happens that $j'[X] \subseteq D(j[X])$, then we obtain a Kl-transformation from the Kl-morphism $(j[\cdot], \lambda)$ to the Kl-morphism $(j'[\cdot], \lambda')$.

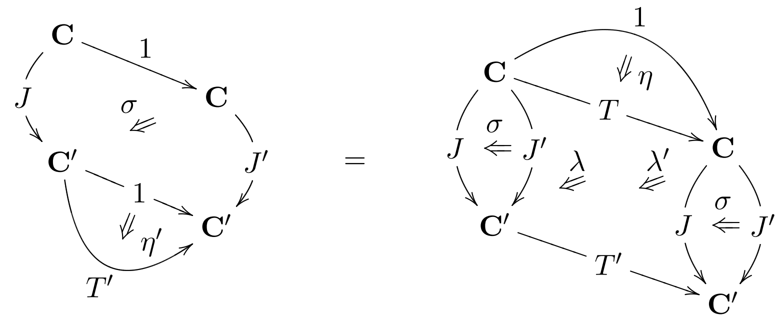
We notice that the 2-cells between morphisms of monads considered by Street in [30] are a particular case of the transformations treated here. In fact, the transformations of Street are characterized as those transformations of Kleisli which can be factorized through a natural transformation between the underlying functors of the Kl-morphisms.

DEFINITION 2.11. Let (J, λ) and (J', λ') be two Kl-morphisms from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$. A *transformation of Street* or, for brevity, an *St-transformation*, from (J, λ) to (J', λ') is a natural transformation σ from J' to J such that the following equality holds $\lambda \circ \sigma T = T'\sigma \circ \lambda'$, i.e., such that the following diagram commutes



To every St-transformation there corresponds a Kl-transformation as stated in the following proposition. However, as we will show below not every Kl-transformation can be obtained from an St-transformation—in other words, St-transformations are too strict to include what are felt should be examples.

PROPOSITION 2.12. *Let* (J, λ) *and* $(J', \lambda')\colon (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}', \mathbb{T}')$ *be two* Kl-*morphisms and let* σ *be an* St-*transformation from* (J, λ) *to* (J', λ')*. Then the natural transformation* $\eta' J \circ \sigma = \lambda\sigma\eta = T'\sigma \circ \lambda'(J'\eta)$ *is a* Kl-*transformation.*



In what follows we will say that a Kl-transformation $\Xi\colon (J,\lambda) \rightsquigarrow (J',\lambda')$ is a *transformation of Kleisli-Street* or, for brevity, a *KS-transformation*, if Ξ is obtained from an St-transformation $\sigma\colon J' \Longrightarrow J$ as indicated in the above proposition.

Since the set of all KS-transformations is, obviously, closed under the operation of composition, we obtain a sub-2-category of $\mathbf{Mnd}_{\mathrm{Kl}}$, hereafter denoted by $\mathbf{Mnd}_{\mathrm{KS}}$.

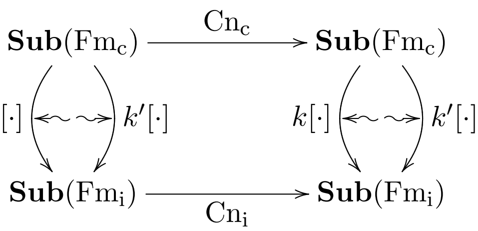
The following examples, the former algebraic and the latter logical, prove that not every Kl-transformation can be obtained from an St-transformation.

Example 2.13. For the closure space $(\mathbb{Z},\mathrm{Sg}_{\mathbb{Z}})$, where $\mathbb{Z}$ is the additive group of the integers and $\mathrm{Sg}_{\mathbb{Z}}$ the subalgebra generating operator which sends a subset X of $\mathbb{Z}$ to $\mathrm{Sg}_{\mathbb{Z}}(X)$, the additive subgroup of $\mathbb{Z}$ generated by X, and the endomorphisms $\mathrm{id}_{\mathbb{Z}}$ and μ_2 of $\mathbb{Z}$, where μ_2 is multiplication by 2, we have, for every subset X of $\mathbb{Z}$, that $\mu_2[X] \subseteq \mathrm{Sg}_{\mathbb{Z}}(\mathrm{id}_{\mathbb{Z}}[X]) = \mathrm{Sg}_{\mathbb{Z}}(X)$. Hence there exists a Kl-transformation from $(\mathrm{id}_{\mathbb{Z}}[\cdot],\lambda)$ to $(\mu_2[\cdot],\lambda')$, with λ and λ' trivial. However, there is not any St-transformation from $(\mathrm{id}_{\mathbb{Z}}[\cdot],\lambda)$ to $(\mu_2[\cdot],\lambda')$, since it is not true that, for every subset X of $\mathbb{Z}$, $\mu_2[X] \subseteq X$.

Example 2.14. For the interpretations t, of McKinsey-Tarski [26], and t', of Gödel [15], of the intuitionistic propositional logic into the modal propositional logic S4, we have that, for every formula $\varphi \in \mathrm{Fm}_{\mathrm{i}}$, with Fm_{i} the set of the intuitionistic formulas, from $\vdash_{\mathrm{i}} \varphi$, it follows that $\vdash_{\mathrm{S4}} t(\varphi)$ and $\vdash_{\mathrm{S4}} t'(\varphi)$, moreover, $\vdash_{\mathrm{S4}} t(\varphi) \leftrightarrow \Box t'(\varphi)$. Therefore, for every intuitionistic formula φ, we can assert that $\vdash_{\mathrm{S4}} t(\varphi) \to t'(\varphi)$, hence $t'[\mathrm{Cn}_{\mathrm{i}}(\varnothing)] \subseteq \mathrm{Cn}_{\mathrm{S4}}(t[\mathrm{Cn}_{\mathrm{i}}(\varnothing)])$, where Cn_{i} is the consequence operator for the intuitionistic propositional logic and $\mathrm{Cn}_{\mathrm{S4}}$ the consequence operator for the modal propositional logic S4. From this it follows that there exists a Kl-transformation from $(t[\cdot] \circ \kappa_{\mathrm{Cn}_{\mathrm{i}}(\varnothing)},\lambda)$ to

$(t'[\cdot]\circ\kappa_{\mathrm{Cn_i}(\varnothing)}, \lambda')$, where $t[\cdot]$ and $t'[\cdot]$ are the operators of direct image formation from $\mathrm{Sub}(\mathrm{Fm_i})$ to $\mathrm{Sub}(\mathrm{Fm_{S4}})$, with $\mathrm{Fm_{S4}}$ the set of the modal formulas, and $\kappa_{\mathrm{Cn_i}(\varnothing)}$ the mapping from $\mathrm{Sub}(\varnothing)$ to $\mathrm{Sub}(\mathrm{Fm_i})$ which picks $\mathrm{Cn_i}(\varnothing)$. However, there is not any St-transformation from $(t[\cdot]\circ\kappa_{\mathrm{Cn_i}(\varnothing)}, \lambda)$ to $(t'[\cdot]\circ\kappa_{\mathrm{Cn_i}(\varnothing)}, \lambda')$.

For the interpretations k, of Kolmogorov [21], and k', of Gentzen [14], of the classical propositional logic into the intuitionistic propositional logic, we have, for every set of formulas $\Gamma \subseteq \mathrm{Fm_c}$ and every formula $\varphi \in \mathrm{Fm_c}$, with $\mathrm{Fm_c}$ the set of the classical formulas, that $\Gamma \vdash_{\mathrm{c}} \varphi$ if and only if $k[\Gamma] \vdash_{\mathrm{i}} k(\varphi)$ and $\Gamma \vdash_{\mathrm{c}} \varphi$ if and only if $k'[\Gamma] \vdash_{\mathrm{i}} k'(\varphi)$. Moreover, for every $\varphi \in \mathrm{Fm_c}$, it happens that $\vdash_{\mathrm{i}} k(\varphi) \leftrightarrow k'(\varphi)$. Thus, for every $\Gamma \subseteq \mathrm{Fm_c}$, we have that $k'[\Gamma] \subseteq \mathrm{Cn_i}(k[\Gamma])$ and $k[\Gamma] \subseteq \mathrm{Cn_i}(k'[\Gamma])$. From this it follows that there are two mutually inverse Kl-transformations between $(k[\cdot], \lambda)$ and $(k'[\cdot], \lambda')$, as shown by the following diagram



However, there is not any St-transformation neither from $(k[\cdot], \lambda)$ to $(k'[\cdot], \lambda')$ nor from $(k'[\cdot], \lambda')$ to $(k[\cdot], \lambda)$, since, for every set $\Gamma \subseteq \mathrm{Fm_c}$, neither $k'[\Gamma] \subseteq k[\Gamma]$ nor $k[\Gamma] \subseteq k'[\Gamma]$.

Definition 2.15. We denote by $\mathbf{Kl}$ the 2-category which has as 0-cells the monads, as 1-cells from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ the pairs of functors (J, H), with J a functor from $\mathbf{C}$ to $\mathbf{C}'$ and H a functor from $\mathbf{Kl}(\mathbb{T})$ to $\mathbf{Kl}(\mathbb{T}')$, such that $H \circ F_{\mathbb{T}} = F_{\mathbb{T}'} \circ J$, as 2-cells from (J, H) to (J', H') the natural transformations from H to H', and as identities and compositions the obvious ones.

Proposition 2.16. *The 2-categories* $\mathbf{Mnd}_{\mathrm{Kl}}$ *and* $\mathbf{Kl}^{\mathrm{c}}$ *are 2-isomorphic.*

Proof. By Proposition 2.4 there exists a one-to-one correspondence between the 1-cells of $\mathbf{Mnd}_{\mathrm{Kl}}$ and those of $\mathbf{Kl}^{\mathrm{c}}$. Let (J, λ) and (J', λ') be two Kl-morphisms from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ and $\Xi\colon (J, \lambda) \rightsquigarrow (J', \lambda')$ a Kl-transformation. Then Ξ determines a 2-cell $\tau^{\Xi}\colon (J', H_{\lambda'}) \Longrightarrow (J, H_{\lambda})$ in $\mathbf{Kl}^{\mathrm{c}}$, where τ^{Ξ} is the natural transformation which sends an object X in $\mathbf{C}$ to the morphism in $\mathbf{Kl}(\mathbb{T}')$ that corresponds to the morphism Ξ_X from $J'(X)$ to $T'(J(X))$ in $\mathbf{C}$.

Reciprocally, if (J, H) and (J', H') are two 1-cells in $\mathbf{Kl}^{\mathrm{c}}$ from $\mathbf{Kl}(\mathbb{T})$ to $\mathbf{Kl}(\mathbb{T}')$ and $\vartheta\colon (J', H') \Longrightarrow (J, H)$ a 2-cell in $\mathbf{Kl}$, then the mapping Ξ^{ϑ} which sends a $\mathbf{C}$-object X to the morphism in $\mathbf{C}'$ that corresponds to ϑ_X in $\mathbf{Kl}(\mathbb{T}')$ is a Kl-transformation from (J, λ_H) to $(J', \lambda_{H'})$.

Both correspondences are clearly mutually inverse.

To complete the proof that the 2-categories $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Kl}^{\mathrm{c}}$ are 2-isomorphic, it only remains to verify the compatibility with the vertical and horizontal compositions and the compatibility with the identities. The details are left to the reader. ∎

We next define a 2-category whose conjugate is, as a consequence of the above proposition, 2-isomorphic to the 2-category $\mathbf{Mnd}_{\mathrm{KS}}$.

Definition 2.17. We denote by $\mathbf{Kl}_{\mathrm{St}}$ the 2-category which has as 0-cells the monads, as 1-cells from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ the pairs of functors (J, H), with $J\colon \mathbf{C} \longrightarrow \mathbf{C}'$ and $H\colon \mathbf{Kl}(\mathbb{T}) \longrightarrow \mathbf{Kl}(\mathbb{T}')$, such that the following equality holds $H \circ F_{\mathbb{T}} = F_{\mathbb{T}'} \circ J$, as 2-cells from (J, H) to (J', H') the pairs of natural transformations (σ, τ), with $\sigma\colon J \Longrightarrow J'$ and $\tau\colon H \Longrightarrow H'$ such that the following equality holds $\tau F_{\mathbb{T}} = F_{\mathbb{T}'}\sigma$, and as identities and compositions the obvious ones.

The 2-category $\mathbf{Kl}_{\mathrm{St}}$ can be identified to a sub-2-category of $\mathbf{Kl}$ by forgetting the first component of all the 2-cells. Furthermore, by restricting the 2-isomorphism between $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Kl}^{\mathrm{c}}$ we have the following corollary.

Corollary 2.18. *The 2-categories $\mathbf{Mnd}_{\mathrm{KS}}$ and $\mathbf{Kl}^{\mathrm{c}}_{\mathrm{St}}$ are 2-isomorphic.*

By inverting the direction of the natural transformation, but leaving invariant the direction of the functor, in the definition of the concept of Kl-morphism we obtain another concept of morphism of monads, that of morphism of Eilenberg-Moore. Let us notice that *since the direction of the functor does not change, the aforementioned concepts do not give rise to dual categories*. However, because of its relation with the algebraic morphisms between monads, defined later on, and which are one of the sources of this research (and since for us the theory of monads is not a purely formal but a substantial subject whose notions and constructions should be founded, ultimately, on mathematical examples), it is suitable to define the morphisms of Eilenberg-Moore by inverting the direction of the functor instead of that of the natural transformation.

We next turn to defining the morphisms of Eilenberg-Moore between monads, the identity at a monad and the composition of two composable morphisms of Eilenberg-Moore.

Definition 2.19. Consider two monads $(\mathbf{C}, \mathbb{T})$ and $(\mathbf{C}', \mathbb{T}')$. A *morphism of Eilenberg-Moore* or, for brevity, an *EM-morphism*, from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ is a pair (K, λ), where $K\colon \mathbf{C}' \longrightarrow \mathbf{C}$ is a functor and $\lambda\colon TK \Longrightarrow KT'$ a natural transformation such that the following diagrams commute

$$
\begin{array}{ccc}
K & \xrightarrow{\eta K} & TK \\
 & \searrow^{K\eta'} & \downarrow{\lambda} \\
 & & KT'
\end{array}
\qquad\qquad
\begin{array}{ccccc}
TTK & \xrightarrow{T\lambda} & TKT' & \xrightarrow{\lambda T'} & KT'T' \\
\downarrow{\mu K} & & & & \downarrow{K\mu'} \\
TK & & \xrightarrow[\lambda]{} & & KT'
\end{array}
$$

We write $(K, \lambda)\colon (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}', \mathbb{T}')$ to denote that (K, λ) is an EM-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$. For every monad $\mathbb{T}$ on $\mathbf{C}$ the *identity at* $(\mathbf{C}, \mathbb{T})$, denoted by $\mathrm{id}_{(\mathbf{C},\mathbb{T})}$, is the morphism $(\mathrm{Id}_{\mathbf{C}}, \mathrm{id}_T)$. If (K, λ) is an EM-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ and (K', λ') an EM-morphism from $(\mathbf{C}', \mathbb{T}')$ to $(\mathbf{C}'', \mathbb{T}'')$, then the *composition* of (K, λ) with (K', λ'), denoted by $(K', \lambda') \circ (K, \lambda)$, is

$$(K', \lambda') \circ (K, \lambda) = (K' \circ K, K\lambda' \circ \lambda K').$$

We leave it to the reader to verify the following proposition.

Proposition 2.20. *Monads and* EM-*morphisms yield a category, hereafter denoted by* $\mathbf{Mnd}_{\mathrm{EM}}$.

In the following proposition, for a given pair of monads, we prove that there exists a one-to-one correspondence between the set of all pairs of functors that (in the opposite direction) relate, respectively, the underlying categories of the monads and the categories of Eilenberg-Moore associated to the underlying monads on the underlying categories of the monads and satisfy, in addition, a suitable condition (specified below) and the set of all EM-morphisms between the given pair of monads.

But before stating the aforementioned proposition we recall that, for a category $\mathbf{C}$ and a monad $\mathbb{T}$ on $\mathbf{C}$, the *Eilenberg-Moore category* of $\mathbb{T}$, denoted from now on by $\mathbf{EM}(\mathbb{T})$, has as objects the $\mathbb{T}$-algebras, i.e., the ordered pairs (A, α) where A is an object of $\mathbf{C}$ and $\alpha\colon T(A) \longrightarrow A$ a morphism of $\mathbf{C}$ such that $\alpha \circ \eta_A = \mathrm{id}_A$ and $\alpha \circ T(\alpha) = \alpha \circ \mu_A$, and for two $\mathbb{T}$-algebras (A, α), (B, β),

$\mathrm{Hom}_{\mathbf{EM}(\mathbf{C})}((A,\alpha),(B,\beta))$ is the set of all morphisms $f\colon A \longrightarrow B$ such that $f\circ\alpha = \beta\circ T(\alpha)$. Moreover, let $F^{\mathbb{T}}$ denote the functor from $\mathbf{C}$ to $\mathbf{EM}(\mathbb{T})$ which sends an object X to $(T(X),\mu_X)$ and a morphism $f\colon X\longrightarrow Y$ to $T(f)$, and $G^{\mathbb{T}}$ the functor from $\mathbf{EM}(\mathbb{T})$ to $\mathbf{C}$ which sends a $\mathbb{T}$-algebra (A,α) to A and a morphism $f\colon (A,\alpha)\longrightarrow(B,\beta)$ to f.

PROPOSITION 2.21. *Let* $(\mathbf{C},\mathbb{T})$ *and* $(\mathbf{C}',\mathbb{T}')$ *be two monads. Then there exists a one-to-one correspondence between the* EM-*morphisms* (K,λ) *from* $(\mathbf{C},\mathbb{T})$ *to* $(\mathbf{C}',\mathbb{T}')$ *and the pairs* (K,H)*, where* K *is a functor from* $\mathbf{C}'$ *to* $\mathbf{C}$ *and* H *a functor from* $\mathbf{EM}(\mathbb{T}')$ *to* $\mathbf{EM}(\mathbb{T})$*, such that the following equality holds* $G^{\mathbb{T}}\circ H = K\circ G^{\mathbb{T}'}$.

Proof. Let $(K,\lambda)\colon(\mathbf{C},\mathbb{T})\longrightarrow(\mathbf{C}',\mathbb{T}')$ be an EM-morphism. Then the ordered pair (K,H^{λ}), where H^{λ} is the functor from $\mathbf{EM}(\mathbb{T}')$ to $\mathbf{EM}(\mathbb{T})$ which assigns to a $\mathbb{T}'$-algebra (A,α) the $\mathbb{T}$-algebra $(K(A),K(\alpha)\circ\lambda_A)$ and to a morphism f in $\mathbf{EM}(\mathbb{T}')$ the morphism $K(f)$ in $\mathbf{EM}(\mathbb{T})$, is such that $G^{\mathbb{T}}\circ H^{\lambda} = K\circ G^{\mathbb{T}'}$.

Reciprocally, if (K,H), where $K\colon\mathbf{C}'\longrightarrow\mathbf{C}$ and $H\colon\mathbf{EM}(\mathbb{T}')\longrightarrow\mathbf{EM}(\mathbb{T})$, is such that $G^{\mathbb{T}}\circ H = K\circ G^{\mathbb{T}'}$, then let κ be the conjugate natural transformation of the identity natural transformation from $K\circ G^{\mathbb{T}'}$ to $G^{\mathbb{T}}\circ H$ and λ^H the composition of κ with $G^{\mathbb{T}}$. Then

$$\lambda^H(=G^{\mathbb{T}}\kappa)\colon TK = G^{\mathbb{T}}F^{\mathbb{T}}K \Longrightarrow G^{\mathbb{T}}HF^{\mathbb{T}'} = KG^{\mathbb{T}'}F^{\mathbb{T}'} = KT'.$$

The pair (K,λ^H) is, obviously, an EM-morphism from $(\mathbf{C},\mathbb{T})$ to $(\mathbf{C}',\mathbb{T}')$. It is easy to check that both correspondences are mutually inverse. ∎

DEFINITION 2.22. The category $\mathbf{EM}$ has as objects the monads and as morphisms from $(\mathbf{C},\mathbb{T})$ to $(\mathbf{C}',\mathbb{T}')$ the pairs (K,H), where K is a functor from $\mathbf{C}'$ to $\mathbf{C}$ and H a functor from $\mathbf{EM}(\mathbb{T}')$ to $\mathbf{EM}(\mathbb{T})$, such that the following equality holds

$$G^{\mathbb{T}}\circ H = K\circ G^{\mathbb{T}'}.$$

From the above definition it is easy to verify the following proposition.

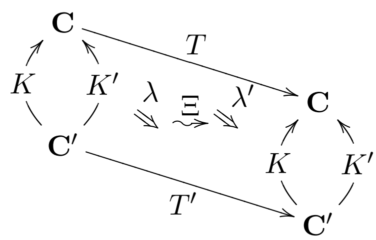
PROPOSITION 2.23. *The categories* $\mathbf{Mnd}_{\mathrm{EM}}$ *and* $\mathbf{EM}^{\mathrm{op}}$ *are isomorphic.*

We have already seen how to compare the Kl-morphisms by means of the Kl-transformations. The same can be done for the EM-morphisms of monads, but in this case by means of the so-called transformations of Eilenberg-Moore (that, for a given pair of EM-morphisms of monads, are in a one-to-one correspondence with a suitable subset of the set of all natural transformations between the functors on the categories of Eilenberg-Moore associated to the given pair of EM-morphisms).

DEFINITION 2.24. Consider two EM-morphisms of monads (K, λ) and (K', λ') from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$. A *transformation of Eilenberg-Moore* or, for brevity, an *EM-transformation*, from (K, λ) to (K', λ') is a natural transformation Ξ from K to $K'T'$ making commutative the following diagram

$$\begin{array}{ccccc} TK & \xrightarrow{\lambda} & KT' & \xrightarrow{\Xi T'} & K'T'T' \\ {\scriptstyle T\Xi}\downarrow & & & & \downarrow{\scriptstyle K'\mu'} \\ TK'T' & \xrightarrow[\lambda' T']{} & K'T'T' & \xrightarrow[K'\mu']{} & K'T' \end{array}$$

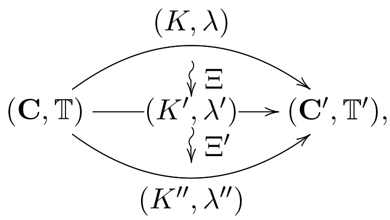
If Ξ is an EM-transformation from (K, λ) to (K', λ'), then we will write Ξ_{e} for the unique natural transformation from TK to $K'T'$ in the above diagram. Moreover, we will use $\Xi\colon (K, \lambda) \rightsquigarrow (K', \lambda')$ or a diagram as displayed in



to indicate that Ξ is an EM-transformation from (K, λ) to (K', λ').

For every EM-morphism (K, λ) from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ the *EM-identity at* (K, λ) is the natural transformation $K\eta'\colon K \Longrightarrow KT'$.

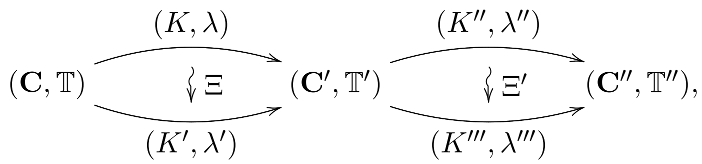
The *vertical composition* of two EM-transformations as in the diagram



denoted by $\Xi' \,\tilde{\circ}\, \Xi$, is the natural transformation

$$K \xrightarrow{\ \Xi\ } K'T' \xrightarrow{\ \Xi' T'\ } K''T'T' \xrightarrow{\ K''\mu'\ } K''T'.$$

The *horizontal composition* of two EM-transformations as in the diagram



denoted by $\Xi' \tilde{*} \Xi$, is the natural transformation

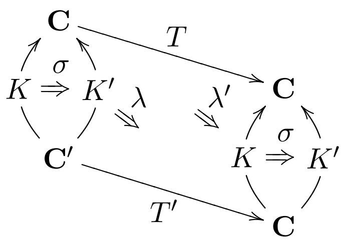
$$KK'' \xrightarrow{\Xi K''} K'T'K'' \xrightarrow{K'\Xi'_{\mathrm{e}}} K'K'''T''.$$

We leave it to the reader to verify the following proposition.

Proposition 2.25. *Monads,* EM-*morphisms, and* EM-*transformations yield a* 2*-category, hereafter denoted by* $\mathbf{Mnd}_{\mathrm{EM}}$.

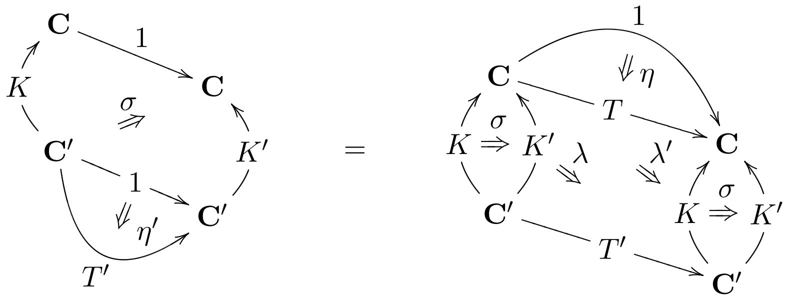
As was the case for the Kl-transformations, there are also EM-transformations which have the additional property of factorizing through a natural transformation between the underlying functors of the EM-morphisms.

Definition 2.26. Let (K,λ) and (K',λ') be two EM-morphisms from $(\mathbf{C},\mathbb{T})$ to $(\mathbf{C}',\mathbb{T}')$. A *transformation of Street* or, for brevity, an *St-transformation*, from (K,λ) to (K',λ') is a natural transformation σ from K to K' such that the following equality holds $\sigma T' \circ \lambda = \lambda' \circ T\sigma$, i.e., such that tal the following diagram commutes



To every St-transformation there corresponds an EM-transformation as stated in the following proposition.

Proposition 2.27. *Let* (K,λ) *and* (K',λ') *be two* EM-*morphisms from* $(\mathbf{C},\mathbb{T})$ *to* $(\mathbf{C}',\mathbb{T}')$ *and* σ *an* St-*transformation from* (K,λ) *to* (K',λ'). *Then* *the natural transformation* $K'\eta' \circ \sigma = \sigma T' \circ \lambda \circ \eta K = \lambda' \circ \eta \circ \sigma$ *is an* EM-*transformation.*



From now on, we will say that an EM-transformation $\Xi\colon (K,\lambda) \rightsquigarrow (K',\lambda')$ is a *transformation of Eilenberg-Moore-Street* or, for brevity, an *EMS-transformation*, if Ξ is obtained from an St-transformation $\sigma\colon K \Longrightarrow K'$ as indicated in the above proposition.

Since the set of all EMS-transformations is, obviously, closed under the operation of composition, we obtain a sub-2-category of $\mathbf{Mnd}_{\mathrm{EM}}$, hereafter denoted by $\mathbf{Mnd}_{\mathrm{EMS}}$.

Let us notice that, as was the case for the Kl-transformations, not every EM-transformation can be obtained from an St-transformation.

Definition 2.28. We denote by $\mathbf{EM}$ the 2-category which has as 0-cells the monads, as 1-cells from $(\mathbf{C},\mathbb{T})$ to $(\mathbf{C}',\mathbb{T}')$ the pairs of functors (K,H), with K a functor from $\mathbf{C}'$ to $\mathbf{C}$ and H a functor from $\mathbf{EM}(\mathbb{T}')$ to $\mathbf{EM}(\mathbb{T})$, such that $G^{\mathbb{T}} \circ H = K \circ G^{\mathbb{T}'}$, as 2-cells from (K,H) to (K',H') the natural transformations from H to H', and as identities and compositions the obvious ones.

Proposition 2.29. *The* 2*-categories* $\mathbf{Mnd}_{\mathrm{EM}}$ *and* $\mathbf{EM}^{\mathrm{t}}$ *are* 2*-isomorphic.*

Proof. By Proposition 2.21 there exists a one-to-one correspondence between the 1-cells of $\mathbf{Mnd}_{\mathrm{EM}}$ and those of $\mathbf{EM}^{\mathrm{op}}$. Let (K,λ) and (K',λ') be two EM-morphisms from $(\mathbf{C},\mathbb{T})$ to $(\mathbf{C}',\mathbb{T}')$ and Ξ an EM-transformation from (K,λ) to (K',λ'). Then Ξ determines a 2-cell $\tau_{\Xi}\colon (K,H^{\lambda}) \Longrightarrow (K',H^{\lambda'})$ in $\mathbf{EM}^{\mathrm{t}}$, where τ_{Ξ} is the natural transformation which sends a $\mathbb{T}'$-algebra (A,α) in $\mathbf{EM}(\mathbb{T}')$ to the morphism $K'(\alpha) \circ \Xi_A$ in $\mathbf{EM}(\mathbb{T})$.

Reciprocally, if (K,H) and K',H' are two 1-cells in $\mathbf{EM}^{\mathrm{t}}$ from $\mathbf{EM}(\mathbb{T}')$ to $\mathbf{EM}(\mathbb{T})$ and ζ a 2-cell from (K,H) to (K',H') in $\mathbf{EM}$, then the mapping Ξ_{ζ}

which sends an object A in $\mathbf{C}$ to $\zeta_{(T'(A),\mu'_A)} \circ K(\eta'_A)$ is an EM-transformation from (K, λ^H) to $(K', \lambda^{H'})$.

Both correspondences are mutually inverse.

To complete the proof that the 2-categories $\mathbf{Mnd}_{\mathrm{EM}}$ and $\mathbf{EM}^{\mathrm{t}}$ are 2-isomorphic, it only remains to verify the compatibility with the vertical and horizontal compositions and the compatibility with the identities. The details are left to the reader. ∎

We next define a 2-category whose transpose is, as a consequence of the above proposition, 2-isomorphic to the 2-category $\mathbf{Mnd}_{\mathrm{EMS}}$.

Definition 2.30. We denote by $\mathbf{EM}_{\mathrm{St}}$ the 2-category which has as 0-cells the monads, as 1-cells from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ the pairs of functors (K, H), with K a functor from $\mathbf{C}'$ to $\mathbf{C}$ and H a functor from $\mathbf{EM}(\mathbb{T}')$ to $\mathbf{EM}(\mathbb{T})$, such that the following equality holds $G^{\mathbb{T}} \circ H = K \circ G^{\mathbb{T}'}$, as 2-cells from (K, H) to (K', H') the pairs of natural transformations (σ, τ), with $\sigma\colon K \Longrightarrow K'$ and $\tau\colon H \Longrightarrow H'$, such that the following equality holds $G^{\mathbb{T}}\sigma = \tau G^{\mathbb{T}'}$, and as identities and compositions the obvious ones.

The 2-category $\mathbf{EM}_{\mathrm{St}}$ can be identified to a sub-2-category of $\mathbf{EM}$ by forgetting the first component of all the 2-cells. Furthermore, by restricting the 2-isomorphism between $\mathbf{Mnd}_{\mathrm{EM}}$ and $\mathbf{EM}^{\mathrm{t}}$ we have the following corollary.

Corollary 2.31. *The 2-categories $\mathbf{Mnd}_{\mathrm{EMS}}$ and $\mathbf{EM}^{\mathrm{t}}_{\mathrm{St}}$ are 2-isomorphic.*

Our next aim is to construct, by using the concept of adjoint square, a new 2-category, with the same 0-cells that $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$, denoted by $\mathbf{Mnd}_{\mathrm{alg}}$ (because its 1-cells and 2-cells will be called, respectively, algebraic morphisms and algebraic transformations). By definition, the 2-category $\mathbf{Mnd}_{\mathrm{alg}}$, as we will see, is isomorphic to the sub-2-category of $\mathbf{Mnd}_{\mathrm{Kl}}$ for which the underlying functors of the 1-cells have a right adjoint, and to the sub-2-category of $\mathbf{Mnd}_{\mathrm{EM}}$ for which the underlying functors of the 1-cells have a left adjoint. But before doing that, since, as we have said previously, it will be used afterwards to define $\mathbf{Mnd}_{\mathrm{alg}}$, we explain what the adjoint squares in the sense of [17] are precisely.

Definition 2.32. (Cf., [17, pp. 144–145]) An *adjoint square* is an ordered triple $(F \dashv G, (J, \lambda, H), F' \dashv G')$, also denoted by $(J, \lambda, H)\colon F \dashv G \longrightarrow F' \dashv G'$, where the adjoints $F \dashv G$ and $F' \dashv G'$ and the functors J and H are related as in the diagram

$$\begin{array}{ccc} \mathbf{C} & \underset{F}{\overset{G}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{D} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbf{C}' & \underset{F'}{\overset{G'}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{D}' \end{array}$$

and λ is a matrix

$$\lambda = \begin{pmatrix} \lambda_0 \colon F'J \Longrightarrow HF & \lambda_1 \colon J \Longrightarrow G'HF \\ \lambda_2 \colon F'JG \Longrightarrow H & \lambda_3 \colon JG \Longrightarrow G'H \end{pmatrix}$$

of compatible 2-cells (our notation is slightly different from that of [17]), i.e., a matrix of natural transformations as indicated such that the following equations are fulfilled

$$\begin{aligned} \lambda_0 &= (\lambda_2 F)(F'J\eta) = (\varepsilon' HF)(F'\lambda_1) = (\varepsilon' HF)(F'\lambda_3 F)(F'J\eta)\,, \\ \lambda_1 &= (G'\lambda_0)(\eta' J) = (G'\lambda_2 F)(\eta' J\eta) = (\lambda_3 F)(J\eta)\,, \\ \lambda_2 &= (H\varepsilon)(\lambda_0 G) = (\varepsilon' H\varepsilon)(F'\lambda_1 G) = (\varepsilon' H)(F'\lambda_3)\,, \\ \lambda_3 &= (G'H\varepsilon)(G'\lambda_0 G)(\eta' JG) = (G'\lambda_2)(\eta' JG) = (G'H\varepsilon)(\lambda_1 G)\,, \end{aligned}$$

where $\eta \colon 1 \Longrightarrow GF$ and $\varepsilon \colon FG \Longrightarrow 1$ are the unit and counit respectively of $F \dashv G$, whereas $\eta' \colon 1 \Longrightarrow G'F'$ and $\varepsilon' \colon F'G' \Longrightarrow 1$ are the unit and counit respectively of $F' \dashv G'$.

We next turn to recalling one of the fundamental facts about the concept of adjoint square, specifically that the adjoint squares are equipped with a structure of double category. We do not give a proof of it, since one by Gray can be found in [17, pp. 146 – 149]. However, following the proposition we recall the definition of the data that occur in the double category under consideration (referring the reader to the original sources [17], [25], and [27] for more details) since some of them will be necessary later on when defining the 1-cells and the 2-cells of the 2-category $\mathbf{Mnd}_{\mathrm{alg}}$.

PROPOSITION 2.33. *Adjoint squares yield a double category, hereafter denoted by* **AdFun**.

Proof. See [17, pp. 146 – 149]. ∎

Let $(J,\lambda,H)\colon F\dashv G \longrightarrow F'\dashv G'$ be an adjoint square, then its *Ad-domain* and *Ad-codomain* in **AdFun** are $F\dashv G$ and $F'\dashv G'$, respectively, and its Fun-*domain* and Fun-*codomain* in **AdFun** are J and H, respectively. The *Ad-identities* and *Fun-identities* are represented by the following adjoint squares

$$
\begin{array}{ccc}
\mathbf{C} & \overset{1}{\underset{1}{\overset{\longleftarrow}{\underset{\longrightarrow}{\top}}}} & \mathbf{C} \\
{\scriptstyle J}\big\downarrow & \begin{pmatrix} J & J \\ J & J \end{pmatrix} & \big\downarrow{\scriptstyle J} \\
\mathbf{C}' & \overset{1}{\underset{1}{\overset{\longleftarrow}{\underset{\longrightarrow}{\top}}}} & \mathbf{C}'
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathbf{C} & \overset{G}{\underset{F}{\overset{\longleftarrow}{\underset{\longrightarrow}{\top}}}} & \mathbf{D} \\
{\scriptstyle 1}\big\downarrow & \begin{pmatrix} F & \eta \\ \varepsilon & G \end{pmatrix} & \big\downarrow{\scriptstyle 1} \\
\mathbf{C} & \overset{G}{\underset{F}{\overset{\longleftarrow}{\underset{\longrightarrow}{\top}}}} & \mathbf{D}
\end{array}
$$

The *Ad-composition* of two adjoint squares

$$(J,\lambda,H)\colon F\dashv G \longrightarrow F'\dashv G' \quad \text{and} \quad (H,\delta,M)\colon L\dashv R \longrightarrow L'\dashv R'$$

is the adjoint square

$$(LF\dashv GR, (J, \delta \overset{\text{ad}}{\circ} \lambda, M), L'F'\dashv G'R'),$$

where $\delta \overset{\text{ad}}{\circ} \lambda$ is the matrix

$$\delta \overset{\text{ad}}{\circ} \lambda = \begin{pmatrix} (\delta_0 F)(L'\lambda_0) & (G'\delta_1 F)\lambda_1 \\ \delta_2(L'\lambda_2 R) & (G'\delta_3)(\lambda_3 R) \end{pmatrix}.$$

And, finally, the *Fun-composition* of two adjoint squares

$$(J,\lambda,H)\colon F\dashv G \longrightarrow F'\dashv G' \quad \text{and} \quad (J',\lambda',H')\colon F'\dashv G' \longrightarrow F''\dashv G''$$

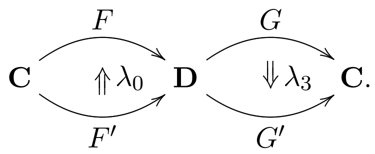
is the adjoint square

$$(F\dashv G, (J'J, \lambda' \overset{\text{fn}}{\circ} \lambda, H'H), F''\dashv G''),$$

where $\lambda' \overset{\text{fn}}{\circ} \lambda$ is the matrix

$$\lambda' \overset{\text{fn}}{\circ} \lambda = \begin{pmatrix} (H'\lambda_0)(\lambda'_0 J) & (G'H'\varepsilon' HF)(\lambda'_1\lambda_1) \\ (\lambda'_2\lambda_2)(F''J'\eta' JG) & (\lambda'_3 H)(J'\lambda_3) \end{pmatrix}.$$

If in the double category **AdFun** we take as adjunctions the identities, then we obtain the ordinary 2-category **Cat**. If, on the other hand, in **AdFun** we take as functors the identities, then we obtain the 2-category **Adj** which has as 0-cells categories, as 1-cells from $\mathbf{C}$ to $\mathbf{D}$ adjunctions $F \dashv G$, and as 2-cells from $F \dashv G$ to $F' \dashv G'$ adjoint squares $(1, \lambda, 1)\colon F \dashv G \longrightarrow F' \dashv G'$, or, what is equivalent, conjugate pairs (λ_0, λ_3) (for this concept see [23, pp. 99–100]), which we represent as



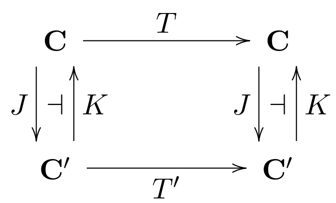
Observe that the 2-category **Adj** is the conjugate (in the sense of [2]) of the 2-category of categories, adjunctions, and conjugate pairs in [23, p. 104].

Before defining the concept of compatible pair with a pair of adjoint squares which will be used below, we notice that to give a natural transformation σ from J to J' is equivalent to give an adjoint square where the involved adjunctions are identities.

Definition 2.34. Let (J, λ, H) and (J', λ', H') be two adjoint squares from $F \dashv G$ to $F' \dashv G'$ and $\sigma\colon J \Longrightarrow J'$, $\tau\colon H \Longrightarrow H'$ a pair of natural transformations. Then we will say that the pair (σ, τ) is *compatible with* λ *and* λ' if the following equation is fulfilled $\lambda' \overset{\mathrm{ad}}{\circ} \sigma = \tau \overset{\mathrm{ad}}{\circ} \lambda$.

Next, since it will be used afterwards to define the algebraic morphisms from a monad to another, we state, for a pair of monads and an adjunction between the underlying categories of the monads, the existence of a commutative square of bijections between four sets of natural transformations obtained from the monads and the adjunction, as well as conditions of compatibility on the matrices of natural transformations arranged in the pattern of the just named commutative square of bijections.

Proposition 2.35. *Let* $(\mathbf{C}, \mathbb{T})$ *and* $(\mathbf{C}', \mathbb{T}')$ *be two monads and* $(J, K, \overline{\eta}, \overline{\varepsilon})$ *an adjunction from* $\mathbf{C}$ *to* $\mathbf{C}'$*. Then for the following diagram*



there exists, by Corollary I,6.6 *stated by Gray in* [17, p. 143], *a commutative square of bijections*

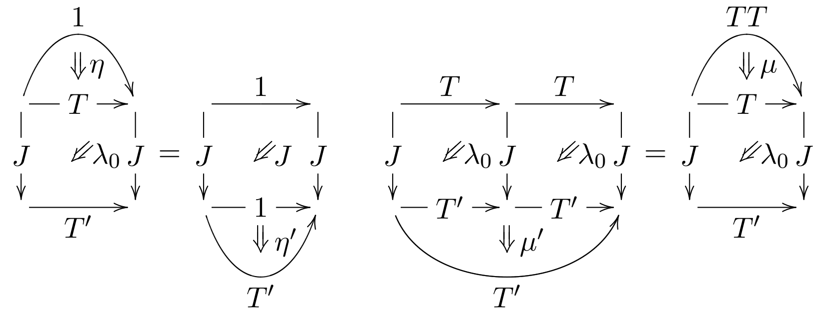
$$\begin{array}{ccc}
\mathrm{Nat}(JT, T'J) & \xrightarrow{\ \cong\ } & \mathrm{Nat}(T, KT'J) \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
\mathrm{Nat}(JTK, T') & \xrightarrow[\ \cong\ ]{} & \mathrm{Nat}(TK, KT')
\end{array}$$

Furthermore, the following conditions on the natural transformations in the matrix

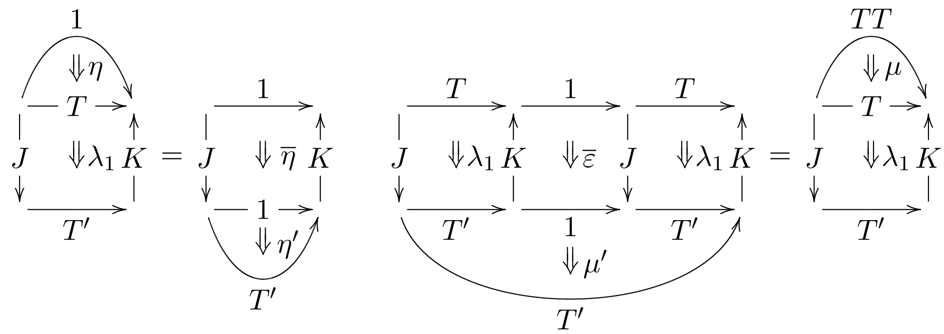
$$\lambda = \begin{pmatrix} \lambda_0 \colon JT \Longrightarrow T'J & \lambda_1 \colon T \Longrightarrow KT'J \\ \lambda_2 \colon JTK \Longrightarrow T' & \lambda_3 \colon TK \Longrightarrow KT' \end{pmatrix}$$

are compatible with the above bijections:

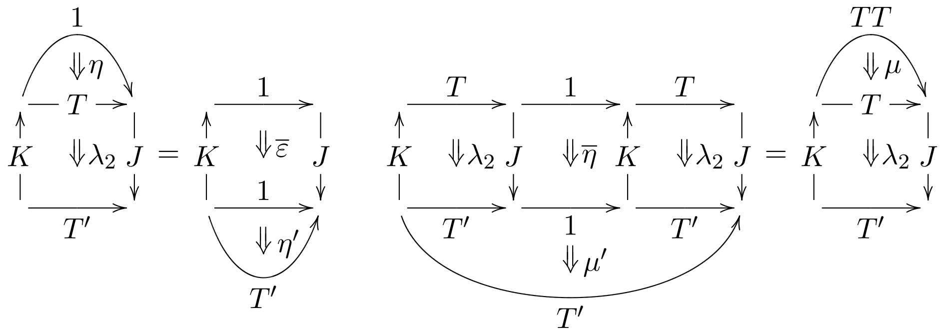
(1) *The natural transformations* $\lambda_0 \colon JT \Longrightarrow T'J$ *such that*



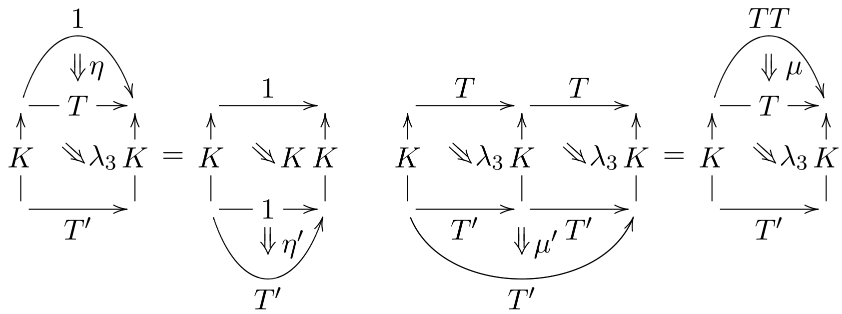
(2) *The natural transformations* $\lambda_1 \colon T \Longrightarrow KT'J$ *such that*



(3) *The natural transformations* $\lambda_2 \colon JTK \Longrightarrow T'$ *such that*

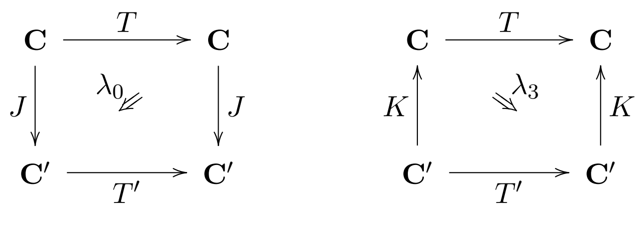


(4) *The natural transformations* $\lambda_3 \colon TK \Longrightarrow KT'$ *such that*



Definition 2.36. Let $(\mathbf{C}, \mathbb{T})$ and $(\mathbf{C}', \mathbb{T}')$ be two monads. An *algebraic morphism* or, to abbreviate, an *alg-morphism*, from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ is an adjoint square $(T, \lambda, T') \colon J \dashv K \longrightarrow J \dashv K$, also denoted in what follows by $(J \dashv K, \lambda) \colon (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}', \mathbb{T}')$, such that its components are compatible with the conditions in Proposition 2.35. *Identities* and *compositions* of alg-morphisms are defined, respectively, as the Ad-identities and the Ad-compositions of its underlying adjoint squares.

From the above definition it follows, immediately, that if $(J \dashv K, \lambda)$ is an alg-morphism, then (J, λ_0) is a Kl-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ and (K, λ_3) an EM-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$



Reciprocally, if (J, λ) is a Kl-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbf{T}')$ and the functor J has a right adjoint K, then λ determines an alg-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$. In the same way, if (K, λ) is an EM-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ and K has a left adjoint J, then λ determines an alg-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$.

We now give an example of the concept of alg-morphism which has to do with the theory of closure spaces.

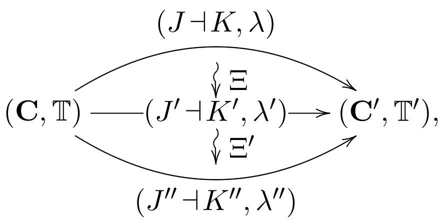
Example 2.37. As we know, to every continuous mapping j from (A, C) to (B, D) there corresponds a morphism of Kleisli $(j[\cdot], \lambda)$ from $(\mathbf{Sub}(A), \mathbb{C})$ to $(\mathbf{Sub}(B), \mathbb{D})$. But the functor $j[\cdot]$ from $\mathbf{Sub}(A)$ to $\mathbf{Sub}(B)$ has a right adjoint, precisely $j^{-1}[\cdot]$, i.e., the formation of j-inverse images, therefore λ gives rise to an alg-morphism from $(\mathbf{Sub}(A), \mathbb{C})$ to $(\mathbf{Sub}(B), \mathbb{D})$. At the end of this section, we provide additional examples of the concept of alg-morphism connected with the fields of many-sorted universal algebra and of many-sorted closure spaces.

Definition 2.38. Let $(J \dashv K, \lambda)$ and $(J' \dashv K', \lambda')$ be two alg-morphisms from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$. An *algebraic transformation* or, to abbreviate, an *alg-transformation*, from $(J \dashv K, \lambda)$ to $(J' \dashv K', \lambda')$ is an adjoint square $(1, \Xi, 1)\colon J \dashv K \longrightarrow J \dashv K$ such that $\mu' \overset{\mathrm{ad}}{\circ} (\Xi \overset{\mathrm{fn}}{\circ} \lambda) = \mu' \overset{\mathrm{ad}}{\circ} (\lambda' \overset{\mathrm{fn}}{\circ} \Xi)$. We will use the notation $\Xi\colon (J \dashv K, \lambda) \rightsquigarrow (J' \dashv K', \lambda')$ to indicate that the algebraic square $(1, \Xi, 1)\colon J \dashv K \longrightarrow J \dashv K$ is an alg-transformation from $(J \dashv K, \lambda)$ to $(J' \dashv K', \lambda')$.

For every alg-morphism $(J \dashv K, \lambda)\colon (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}', \mathbb{T}')$, the *identity at* $(J \dashv K, \lambda)$ is the adjoint square determined by the matrix

$$\begin{pmatrix} J\eta' & K\eta' J \circ \eta \\ \eta' \circ \varepsilon & \eta' K \end{pmatrix}.$$

The *vertical composition* of two alg-transformations as in the diagram



denoted by $\Xi' \tilde{\circ} \Xi$, is the adjoint square $\mu' \overset{\mathrm{ad}}{\circ} (\Xi' \overset{\mathrm{fn}}{\circ} \Xi)$.

The *horizontal composition* of two alg-transformations as in the diagram

$$(\mathbf{C},\mathbb{T}) \underset{(J'\dashv K',\lambda')}{\overset{(J\dashv K,\lambda)}{\rightrightarrows}} \;\Downarrow\Xi\; (\mathbf{C}',\mathbb{T}') \underset{(J'''\dashv K''',\lambda''')}{\overset{(J''\dashv K'',\lambda'')}{\rightrightarrows}} \;\Downarrow\Xi'\; (\mathbf{C}'',\mathbb{T}''),$$

denoted by $\Xi' \mathbin{\tilde{*}} \Xi$, is the adjoint square

$$\mu' \overset{\mathrm{ad}}{\circ} (\lambda'' \overset{\mathrm{fn}}{\circ} \Xi') \overset{\mathrm{ad}}{\circ} \Xi = \mu' \overset{\mathrm{ad}}{\circ} (\Xi' \overset{\mathrm{fn}}{\circ} \lambda''') \overset{\mathrm{ad}}{\circ} \Xi.$$

As was the case for the alg-morphisms, $\Xi\colon (J \dashv K, \lambda) \rightsquigarrow (J' \dashv K', \lambda')$ is an alg-transformation if, and only if, Ξ_0 is a Kl-transformation, or Ξ_3 is an EM-transformation.

At the end of this section, we give examples of alg-transformations which come from the fields of many-sorted universal algebra and of many-sorted closure spaces.

Definition 2.39. Let $(J \dashv K, \lambda)$ and $(J' \dashv K', \lambda')$ be two alg-morphisms from $(\mathbf{C},\mathbb{T})$ to $(\mathbf{C}',\mathbb{T}')$. An *Street transformation* or, to abbreviate, an *St-transformation*, from $(J \dashv K, \lambda)$ to $(J' \dashv K', \lambda')$ is an adjoint square $(1,\Xi,1)\colon J \dashv K \longrightarrow J \dashv K$ such that

$$\begin{array}{ccccc} \mathbf{C} & \xrightarrow{T} & \mathbf{C} & \xrightarrow{1} & \mathbf{C} \\ J\downarrow\dashv\uparrow K & \lambda & J\downarrow\dashv\uparrow K & \Xi & J'\downarrow\dashv\uparrow K' \\ \mathbf{C}' & \xrightarrow[T']{} & \mathbf{C}' & \xrightarrow[1]{} & \mathbf{C}' \end{array} = \begin{array}{ccccc} \mathbf{C} & \xrightarrow{1} & \mathbf{C} & \xrightarrow{T} & \mathbf{C} \\ J\downarrow\dashv\uparrow K & \Xi & J'\downarrow\dashv\uparrow K' & \lambda' & J'\downarrow\dashv\uparrow K' \\ \mathbf{C}' & \xrightarrow[1]{} & \mathbf{C}' & \xrightarrow[T']{} & \mathbf{C}' \end{array}$$

To give a Street transformation is equivalent to give a pair of natural transformations (σ,τ), with $\sigma\colon J' \Longrightarrow J$ and $\tau\colon K \Longrightarrow K'$, such that σ is a KS-transformation and τ an EMS-transformation. It is immediate that each Street transformation gives rise to one algebraic transformation, although not every algebraic transformation can be obtained from a Street transformation.

The Street transformations are natural transformations between the underlying functors of the corresponding alg-morphisms that have the additional property of being compatible with the structures of the involved monads, but, unlike the algebraic transformations, they do not make any essential use of the monad structure of which is equipped the codomain.

Proposition 2.40. *Monads, alg-morphisms, and alg-transformations yield a 2-category, hereafter denoted by* $\mathbf{Mnd}_{\mathrm{alg}}$*, and there are canonical 2-embeddings* J_{Kl} *from* $\mathbf{Mnd}_{\mathrm{alg}}$ *into* $\mathbf{Mnd}_{\mathrm{KL}}$ *and* J_{EM} *from* $\mathbf{Mnd}_{\mathrm{alg}}$ *into* $\mathbf{Mnd}_{\mathrm{EM}}$*. Moreover, the Street transformations between alg-morphisms yield a sub-2-category* $\mathbf{Mnd}_{\mathrm{alg,St}}$ *of* $\mathbf{Mnd}_{\mathrm{alg}}$*.*

If we leave out the 2-cells, then it happens that the category $\mathbf{Mnd}_{\mathrm{alg}}$ of monads and algebraic morphisms is isomorphic to $\int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$, the category obtained by applying the Ehresmann-Grothendieck construction (see [8, pp. 89 – 91] and [18, pp. (sub.) 175 – 177]) to a contravariant functor, $G \circ \mathrm{Mnd}$, from the category $\mathbf{Adj}$, of categories and adjunctions (an extensive treatment of the category $\mathbf{Adj}$ can be found in [23, pp. 103 – 104] to $\mathbf{Cat}$. But before proving it, since it will be used afterwards to define Mnd (a key step in obtaining the proof itself), we next recall that each category $\mathbf{C}$ gives rise to a 2-category, $\mathbf{Mnd}(\mathbf{C})$, of monads on $\mathbf{C}$. Concretely, $\mathbf{Mnd}(\mathbf{C})$ has (1) as objects the monads on $\mathbf{C}$, (2) for two monads $\mathbb{T} = (T, \eta, \mu)$ and $\mathbb{T}' = (T', \eta', \mu')$ on $\mathbf{C}$, as morphism of monads from $\mathbb{T} = (T, \eta, \mu)$ to $\mathbb{T}' = (T', \eta', \mu')$ those natural transformation $\lambda\colon T \Longrightarrow T'$ such that $\lambda \circ \eta = \eta'$ and $\lambda \circ \mu = \mu' \circ (\lambda * \lambda)$, and (3) for two monads $\mathbb{T}$, $\mathbb{T}'$ on $\mathbf{C}$ and two morphisms of monads λ, λ' from $\mathbb{T}$ to $\mathbb{T}'$, as transformations from λ to λ' those natural transformations $\Xi\colon 1_{\mathbf{C}} \Longrightarrow T'$, denoted by $\Xi\colon \lambda \rightsquigarrow \lambda'$, making commutative the following diagram

$$\begin{array}{ccc} T & \xrightarrow{\ \Xi\lambda\ } & T'T' \\ {\scriptstyle \lambda'\Xi}\downarrow & & \downarrow{\scriptstyle \mu'} \\ T'T' & \xrightarrow[\ \mu'\ ]{} & T' \end{array}$$

Before stating the result to be proved, i.e., that $\mathbf{Mnd}_{\mathrm{alg}} \cong \int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$, we give an example of an interesting 2-category of groups which is 2-embedded into a 2-category of the type $\mathbf{Mnd}(\mathbf{C})$.

Example 2.41. Let $\mathbf{G}$ be a group. Then $\mathbf{G}$ determines a monad $\mathbb{G} = (G \times (\cdot), \eta, \mu)$ on $\mathbf{Set}$ where: (1) $G \times (\cdot)$ is the functor from $\mathbf{Set}$ to $\mathbf{Set}$ that sends X to $G \times X$ and $\varphi\colon X \longrightarrow Y$ to $\mathrm{id}_G \times \varphi\colon G \times X \longrightarrow G \times Y$, (2) η the natural transformation from $\mathrm{Id}_{\mathbf{Set}}$ to $G \times (\cdot)$ that sends a set X to the mapping $\eta_X\colon X \longrightarrow G \times X$ that to $x \in X$ assigns $(1, x)$, and (3) μ the natural transformation from $(G \times (\cdot)) \circ (G \times (\cdot))$ to $G \times (\cdot)$ that to a set X associates the mapping $\mu_X\colon G \times (G \times X) \longrightarrow G \times X$ that to $(a,(b,x)) \in G \times (G \times X)$ assigns $(ab, x) \in G \times X$.

On the other hand, if $f\colon \mathbf{G} \longrightarrow \mathbf{H}$ is a homomorphism of groups, then f determines a natural transformation λ^f from the functor $G \times (\cdot)$ to the functor $H \times (\cdot)$ that to a set X assigns the mapping $\lambda^f_X = f \times \mathrm{id}_X$ from $G \times X$ to $H \times X$, and λ^f is, in fact, a morphism from the monad $\mathbb{G}$ to the monad $\mathbb{H} = (H \times (\cdot), \eta', \mu')$. Finally, if $f, g\colon \mathbf{G} \longrightarrow \mathbf{H}$ are two conjugate morphisms of groups, i.e., if there exists an $a \in H$ such that, for every $x \in G$, $af(x) = g(x)a$, then a determines a natural transformation Ξ^a from $\mathrm{Id}_{\mathbf{Set}}$ to $H \times (\cdot)$, by associating to a set X the mapping Ξ^a_X from X to $H \times X$ that sends $x \in X$ to (a, x). Notice that Ξ^a is, in fact, a transformation from λ^f to λ^g since the following equality holds

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\Xi^a_{G\times X}} & H \times (G \times X) \\
\downarrow{\scriptstyle \lambda^f_X} & & \downarrow{\scriptstyle \mathrm{id}_H \times \lambda^f_X} \\
H \times X & \xrightarrow[\Xi^a_{H\times X}]{} & H \times (H \times X) \\
 & & \downarrow{\scriptstyle \mu'_X} \\
 & & H \times X
\end{array}
\quad = \quad
\begin{array}{ccc}
G \times X & \xrightarrow{\lambda^g_X} & H \times X \\
\downarrow{\scriptstyle \mathrm{id}_G \times \Xi^a_X} & & \downarrow{\scriptstyle \mathrm{id}_H \times \Xi^a_X} \\
G \times (H \times X) & \xrightarrow[\lambda^g_{H\times X}]{} & H \times (H \times X) \\
 & & \downarrow{\scriptstyle \mu'_X} \\
 & & H \times X
\end{array}
$$

Let us denote by $\mathbf{Grp}_{\mathrm{inn}}$ the 2-category which has as objects groups, as 1-cells morphisms of groups, and as 2-cells from f to g, with $f, g\colon \mathbf{G} \longrightarrow \mathbf{H}$, those inner automorphisms of $\mathbf{H}$ transforming f into g. Then it is easy to check that there is a 2-embedding of $\mathbf{Grp}_{\mathrm{inn}}$ into $\mathbf{Mnd}(\mathbf{Set})$.

It may be readily verified, after a ghastly but wholly straightforward set of computations, the following lemma.

LEMMA 2.42. *Let $\mathbf{J} = (J, K, \overline{\eta}, \overline{\varepsilon})$ be an adjunction from $\mathbf{C}$ to $\mathbf{C}'$. Then $\mathbf{J}$ gives rise to a 2-functor*

$$\mathrm{Mnd}(\mathbf{J})\colon \mathbf{Mnd}(\mathbf{C}') \longrightarrow \mathbf{Mnd}(\mathbf{C})$$

defined by setting: (1) *if $\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathbf{C}'$, then*

$$\mathrm{Mnd}(\mathbf{J})(\mathbb{T}) = (KTJ, K\eta J, K\mu J \circ KT\overline{\varepsilon}TJ),$$

(2) *if* $\lambda\colon \mathbb{T} \longrightarrow \mathbb{T}'$ *is a morphism of monads on* $\mathbf{C}'$, *then* $\mathrm{Mnd}(\mathbf{J})(\lambda) = K\lambda J$, *and* (3) *if* $\Xi\colon \lambda \rightsquigarrow \lambda'$ *is a transformation, with* λ, $\lambda'\colon \mathbb{T} \longrightarrow \mathbb{T}'$ *morphisms of monads on* $\mathbf{C}'$, *then* $\mathrm{Mnd}(\mathbf{J})(\Xi) = K\Xi J \circ \overline{\eta}$.

Now we extend the above construction, Mnd, to a contravariant functor from the category **Adj**, of categories and adjunctions (recall that a detailed exposition of the category **Adj** is given in [23, pp. 103 − 104], to the category **2-Cat**, of 2-categories and 2-functors.

Proposition 2.43. *There exists a contravariant functor* Mnd *from* **Adj** *to* **2-Cat** *defined by assigning to a category* $\mathbf{C}$ *the* 2*-category* $\mathbf{Mnd}(\mathbf{C})$ *and to an adjunction* $\mathbf{J} = (J, K, \overline{\eta}, \overline{\varepsilon})$ *from* $\mathbf{C}$ *to* $\mathbf{C}'$ *the* 2*-functor*

$$\mathrm{Mnd}(\mathbf{J})\colon \mathbf{Mnd}(\mathbf{C}') \longrightarrow \mathbf{Mnd}(\mathbf{C}).$$

Proof. It is immediate that the identities are preserved. Concerning the composition of adjunctions, if $\mathbf{J} = (J, K, \overline{\eta}, \overline{\varepsilon})$ is an adjunction from $\mathbf{C}$ to $\mathbf{C}'$, $\mathbf{J}' = (J', K', \overline{\eta}', \overline{\varepsilon}')$ an adjunction from $\mathbf{C}'$ to $\mathbf{C}''$, and $\mathbb{T}'' = (T'', \eta'', \mu'')$ a monad on $\mathbf{C}''$, then it is clear that $(\mathrm{Mnd}(\mathbf{J}) \circ \mathrm{Mnd}(\mathbf{J}'))(\mathbb{T}'')$ is equal to $\mathrm{Mnd}(\mathbf{J}' \circ \mathbf{J})(\mathbb{T}'')$. For example, for the multiplication, we have that

$$\begin{aligned}
\mu^{(\mathrm{Mnd}(\mathbf{J})\circ\mathrm{Mnd}(\mathbf{J}'))(\mathbb{T}'')} &= K(K'\mu J' \circ K'T\overline{\varepsilon}'TJ')J \circ KK'TJ'\overline{\varepsilon}K'TJ'J \\
&= KK'\mu J'J \circ KK'T\overline{\varepsilon}'TJ'J \circ KK'TJ'\overline{\varepsilon}K'TJ'J \\
&= KK'\mu J'J \circ KK'T(\overline{\varepsilon}' \circ J'\overline{\varepsilon}K')TJ'J \\
&= \mu^{\mathrm{Mnd}(\mathbf{J}'\circ\mathbf{J})(\mathbb{T}'')}.
\end{aligned}$$

∎

As a consequence of the foregoing results, we obtain, by applying the construction of Ehresmann-Grothendieck to the composition of Mnd with the forgetful functor G from **2-Cat** to **Cat**, the category $\int^{\mathbf{Adj}}(G\circ\mathrm{Mnd})$. Its objets are all monads $(\mathbf{C}, \mathbb{T})$. Its morphisms from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ are all pairs $(\mathbf{J}, \lambda)$ where $\mathbf{J} = (J, K, \overline{\eta}, \overline{\varepsilon})$ is an adjunction from $\mathbf{C}$ to $\mathbf{C}'$ and $\lambda\colon \mathbb{T} \longrightarrow \mathrm{Mnd}(\mathbf{J})(\mathbb{T}')$ a morphism of monads in $\mathbf{Mnd}(\mathbf{C})$. This category has, in addition, an obvious projection functor $\pi_{\mathbf{Adj}}$ from $\int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$ to **Adj**.

Following these preliminary results, we now prove that $\mathbf{Mnd}_{\mathrm{alg}}$ is isomorphic to $\int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$.

Proposition 2.44. *The category* $\int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$ *is isomorphic to the category* $\mathbf{Mnd}_{\mathrm{alg}}$ *of monads and algebraic morphisms.*

Proof. Both categories have the same objects. Moreover, a morphism $(\mathbf{J}, \lambda)$ from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}, \mathbb{T}')$ in the category $\int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$ gives rise to an adjoint square by means of the transposes of λ. By Proposition 2.35, the conjugate pairs of such adjoint squares are, respectively, morphisms of Kleisli and of Eilenberg-Moore, hence, the adjoint square is an algebraic morphism.

Reciprocally, given a morphism in $\mathbf{Mnd}_{\mathrm{alg}}$, its underlying adjunction together with the 1-th component of its underlying adjoint square, give rise to a morphism in $\int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$. ∎

From here it follows immediately the following corollary.

COROLLARY 2.45. *The forgetful functor from* $\mathbf{Mnd}_{\mathrm{alg}}$ *to* $\mathbf{Adj}$ *(or, what is equivalent, the projection functor* $\pi_{\mathbf{Adj}}$ *from* $\int^{\mathbf{Adj}}(G \circ \mathrm{Mnd})$ *to* $\mathbf{Adj}$*) which sends a monad* $(\mathbf{C}, \mathbb{T})$ *to* $\mathbf{C}$ *and an alg-morphism* $(J \dashv K, \lambda)$ *from* $(\mathbf{C}, \mathbb{T})$ *to* $(\mathbf{C}', \mathbb{T}')$ *to its underlying adjunction, is a fibration.*

Remark 2.46. It does not seem to exist, however, any 2-category structure on $\mathbf{Adj}$ such that the construction of Ehresmann-Grothendieck for 2-functors in $\mathbf{2\text{-}Cat}$ yields the 2-category of monads, alg-morphisms, and alg-transformations (or, in particular, transformations of Street).

Since it will used in the following example, we agree to denote, for a Grothendieck universe $\boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$, by $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ the 2-category with objects the monads $(\mathbf{C}, \mathbb{T})$ such that $\mathbf{C}$ is in $\boldsymbol{\mathcal{V}}$, 1-cells the alg-morphisms, and 2-cells the alg-transformations between alg-morphisms.

EXAMPLE 2.47. There exists a natural embedding of the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$, of signatures, polyderivors, and transformations between polyderivors, defined in [6], into the 2-category $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$. The embedding sends: (1) a many-sorted signature $\boldsymbol{\Sigma}$ to the monad $(\mathbf{Set}^S, \mathbb{T}_{\boldsymbol{\Sigma}})$, where, we recall, $\mathbb{T}_{\boldsymbol{\Sigma}} = (\mathrm{T}_{\boldsymbol{\Sigma}}, \eta, \mu)$ is the standard monad derived from the adjunction $\mathbf{T}_{\boldsymbol{\Sigma}} \dashv \mathrm{G}_{\boldsymbol{\Sigma}}$ between the category $\mathbf{Alg}(\boldsymbol{\Sigma})$ and the category $\mathbf{Set}^S$, with $\mathrm{T}_{\boldsymbol{\Sigma}} = \mathrm{G}_{\boldsymbol{\Sigma}} \circ \mathbf{T}_{\boldsymbol{\Sigma}}$, (2) a polyderivor $\mathbf{d}$ from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ to the alg-morphism

$$\begin{array}{ccc} \mathbf{Set}^S & \xrightarrow{\ \mathrm{T}_{\boldsymbol{\Sigma}}\ } & \mathbf{Set}^S \\ \coprod^{\dagger}_{\varphi} \downarrow \dashv \uparrow \Delta^{\natural}_{\varphi} & \lambda & \coprod^{\dagger}_{\varphi} \downarrow \dashv \uparrow \Delta^{\natural}_{\varphi} \\ \mathbf{Set}^T & \xrightarrow[\ \mathrm{T}_{\boldsymbol{\Lambda}}\ ]{} & \mathbf{Set}^T \end{array}$$

also denoted by $\mathbb{T}_{\mathbf{d}} = (\coprod_{\varphi}^{\dagger} \dashv \Delta_{\varphi}^{\natural}, \lambda)$, from $(\mathbf{Set}^{S}, \mathbb{T}_{\mathbf{\Sigma}})$ to $(\mathbf{Set}^{T}, \mathbb{T}_{\mathbf{\Lambda}})$, where the component λ_1 of the matrix λ at X is the underlying many-sorted mapping of $\big(\theta_{\varphi}^{\dagger\natural}(\eta_{\coprod_{\varphi}^{\dagger} X})\big)^{\sharp}$, the canonical extension to $\mathbf{T}_{\mathbf{\Sigma}}(X)$ of the many-sorted mapping

$$\Delta_{\varphi}^{\natural}(\eta_{\coprod_{\varphi}^{\dagger} X}) \circ (\eta_{\varphi}^{\dagger\natural})_X \colon X \longrightarrow \Delta_{\varphi}^{\natural}(\mathrm{T}_{\mathbf{\Lambda}}(\textstyle\coprod_{\varphi}^{\dagger} X)),$$

as stated in [6], and (3) a transformation ξ from $\mathbf{d}$ to $\mathbf{d}'$ to the alg-transformation

$$\begin{array}{ccc} \mathbf{Set}^{S} & \xrightarrow{\quad 1 \quad} & \mathbf{Set}^{S} \\ \coprod_{\varphi}^{\dagger} \downarrow \dashv \uparrow \Delta_{\varphi}^{\natural} & \xi & \coprod_{\varphi'}^{\dagger} \downarrow \dashv \uparrow \Delta_{\varphi'}^{\natural} \\ \mathbf{Set}^{T} & \xrightarrow[\quad \mathrm{T}_{\mathbf{\Lambda}} \quad]{} & \mathbf{Set}^{T} \end{array}$$

also denoted by $\mathbb{T}_{\xi}$, from $\mathbb{T}_{\mathbf{d}}$ to $\mathbb{T}_{\mathbf{d}'}$, where the component ξ_0 of the matrix ξ at X is the many-sorted mapping ξ_X, as stated in [6].

Let us notice that since there exists a forgetful 2-functor from the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications, $\mathfrak{pd}$-specification morphisms, and transformations between $\mathfrak{pd}$-specification morphisms, defined in [6], to $\mathbf{Sig}_{\mathfrak{pd}}$ and a 2-embedding of $\mathbf{Sig}_{\mathfrak{pd}}$ into $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$, we have that $\mathbf{Spf}_{\mathfrak{pd}}$ and $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ are connected by a faithful 2-functor.

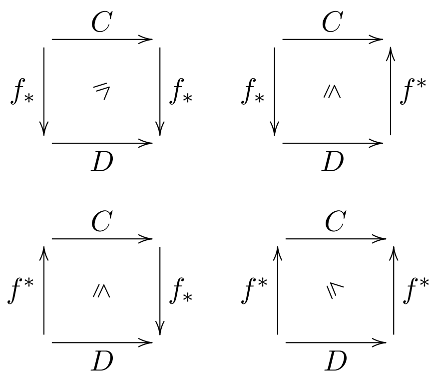
From this 2-embedding and taking into account the work by Street in [30], it follows that the polyderivors together with the transformations between polyderivors are a concrete foundation for a two-dimensional many-sorted universal algebra.

Remark 2.48. The semantical equivalence of any two many-sorted specifications, understood, by convention, as meaning the categorical equivalence of the canonically associated categories of models, can not be properly reflected at the purely syntactical level of the many-sorted specifications and many-sorted specification morphisms, i.e., can not be mathematically defined in the category **Spf**. And this is so, essentially, as a consequence of the fact of not having actually equipped **Spf** with a (non trivial) structure of 2-category. Thus, if one remains anchored in the tradition of viewing **Spf** as being, simply, a category, then the only reasonable way of classifying many-sorted specifications from within the category **Spf** is through the categorical concept of isomorphism, and not, due to structural impossibility, by means of some other notion of equivalence between many-sorted specifications, itself being strictly weaker than that of isomorphism (as it would be the case if instead of having a category, we had a 2-category). Therefore, what is really needed to settle the problem of the equivalence between many-sorted specifications (i.e., the problem of determining whether or not two many-sorted specifications yield equivalent categories) is to have at one's disposal some way of comparing many-sorted specifications that goes, strictly, beyond the mere isomorphisms, in the same way as equivalences go beyond the isomorphisms when comparing categories among them. An adequate way of providing a solution to the just mentioned problem is by constructing suitable 2-categories of many-sorted signatures and many-sorted specifications, through the appropriate definitions of the 2-cells between the 1-cells, e.g., $\mathbf{Sig}_{\mathfrak{pd}}$ and $\mathbf{Spf}_{\mathfrak{pd}}$. This two-dimensionality, by supplying one additional degree of freedom, generates a richer world, that opens the possibility to deal not only with isomorphic but also with adjoint and equivalent many-sorted specifications. Thus carrying further the previous development which was incomplete because of its restriction to categories.

We close this section by showing that **MClSp**, the category of many-sorted closure spaces and continuous mappings between many-sorted closure spaces (also called morphisms between many-sorted closure spaces), defined in [7] (into which is embedded the category **ClSp**, of closure spaces and continuous mappings between closure spaces) is a subcategory of the underlying category of the 2-category $\mathbf{Mnd}_{\mathrm{alg}}$ (and also a sub-2-category of the 2-category $\mathbf{Mnd}_{\mathrm{alg}}$, since the concept of category falls under that of 2-category).

Example 2.49. Let (S, A, C) be a many-sorted closure space, where S is a set of sorts, $A = (A_s)_{s\in S}$ an S-sorted set, i.e., an object of $\mathbf{Set}^S$ and C an S-closure operator on A (see [7] for more details). In the sequel, $(\mathbf{Sub}(A), \mathbb{C})$ denotes the monad associated to (S, A, C), where $\mathbf{Sub}(A)$ is the category determined by the ordered set $(\mathrm{Sub}(A), \subseteq)$, with $\mathrm{Sub}(A) = \{X \in \boldsymbol{\mathcal{U}}^S : X \subseteq A\}$ the set of all sub-S-sorted sets of A, where $X \subseteq A$ means, in this context, that, for all $s \in S$, $X_s \subseteq A_s$, and $\mathbb{C}$ the monad on $\mathbf{Sub}(A)$ obtained from C. Let (S, A, C) and (T, B, D) be many-sorted closure spaces. Then an alg-morphism from $(\mathbf{Sub}(A), \mathbb{C})$ to $(\mathbf{Sub}(B), \mathbb{D})$ is an adjoint square

$$\begin{array}{ccc}
\mathbf{Sub}(A) & \xrightarrow{\quad C \quad} & \mathbf{Sub}(A) \\
f_* \big\downarrow \dashv \big\uparrow f^* & & f_* \big\downarrow \dashv \big\uparrow f^* \\
\mathbf{Sub}(B) & \xrightarrow[\quad D \quad]{} & \mathbf{Sub}(B)
\end{array}$$



i.e., an adjunction $f_* \dashv f^*$ from $\mathbf{Sub}(A)$ to $\mathbf{Sub}(B)$ such that one of the following four equivalent conditions is fulfilled:

(1) for each $X \subseteq A$, $f_*(C(X)) \subseteq D(f_*(X))$;

(2) for each $X \subseteq A$, $C(X) \subseteq f^*(D(f_*(X)))$;

(3) for each $Y \subseteq B$, $f_*(C(f^*(Y))) \subseteq D(Y)$;

(4) for each $Y \subseteq B$, $C(f^*(Y)) \subseteq f^*(D(Y))$.

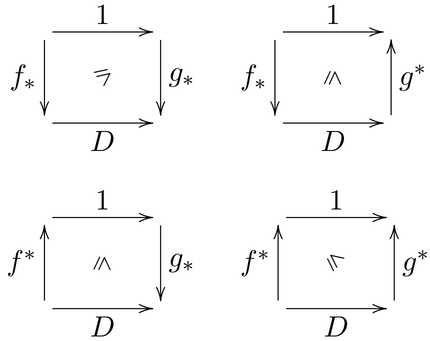
In the sequel, (f_*, f^*) stands for an alg-morphism from $(\mathbf{Sub}(A), \mathbb{C})$ to $(\mathbf{Sub}(B), \mathbb{D})$. From this it follows that a continuous mapping between many-sorted closure spaces (see [7] for the definition of the concept of continuous mapping) is a particular case of the concept of alg-morphism. In fact, if (φ, j) is a continuous mapping from (S, A, C) to (T, B, D), then the adjunctions $j[\cdot] \dashv j^{-1}[\cdot]$ from $\mathbf{Sub}(A)$ to $\mathbf{Sub}((B_{\varphi(s)})_{s\in S})$ and $\bigcup_{\varphi,B} \dashv \Delta_{\varphi,B}$ from $\mathbf{Sub}((B_{\varphi(s)})_{s\in S})$ to $\mathbf{Sub}(B)$ (reference for the latter adjunction is [7]) determine an alg-morphism

$$((\varphi, j)_*, (\varphi, j)^*) = (\textstyle\bigcup_{\varphi,B} \circ j[\cdot], j^{-1}[\cdot] \circ \Delta_{\varphi,B})$$

from $(\mathbf{Sub}(A), \mathbb{C})$ to $(\mathbf{Sub}(B), \mathbb{D})$. Therefore the category **MClSp** is a subcategory of the underlying category of the 2-category $\mathbf{Mnd}_{\mathrm{alg}}$. Let us notice that *not every alg-morphism between the monads associated to many-sorted closure spaces is obtained from pairs of adjunctions of the form* $j[\cdot] \dashv j^{-1}[\cdot]$ *and* $\bigcup_{\varphi,B} \dashv \Delta_{\varphi,B}$. One obtains examples of this phenomenon by means of the alg-morphisms determined by the consequence operators of Hall and of Bénabou (concerning many-sorted equational logic and defined in [5]).

EXAMPLE 2.50. Since $\mathbf{Mnd}_{\mathrm{alg}}$ is a 2-category, we can also consider the concept of alg-transformation between alg-morphisms from the monad associated to a many-sorted closure space to the monad associated to another many-sorted closure space. An alg-transformation from an alg-morphism (f_*, f^*) to another alg-morphism (g_*, g^*), both from $(\mathbf{Sub}(A), \mathbb{C})$ to $(\mathbf{Sub}(B), \mathbb{D})$, is, simply, an adjoint square

$$\begin{array}{ccc} \mathbf{Sub}(A) & \xrightarrow{\quad 1 \quad} & \mathbf{Sub}(A) \\ f_* \big\downarrow \dashv \big\uparrow f^* & & g_* \big\downarrow \dashv \big\uparrow g^* \\ \mathbf{Sub}(B) & \xrightarrow[\quad D \quad]{} & \mathbf{Sub}(B) \end{array}$$



Thus from (f_*, f^*) to (g_*, g^*) there exists an alg-transformation if, and only if, one of the following four equivalent conditions is fulfilled:

(1) for each $X \subseteq A$, $g_*(X) \subseteq D(f_*(X))$;

(2) for each $X \subseteq A$, $X \subseteq g^*(D(f_*(X)))$;

(3) for each $Y \subseteq B$, $g_*(f^*(Y)) \subseteq D(Y)$;

(4) for each $Y \subseteq B$, $f^*(Y) \subseteq g^*(D(Y))$.

Remark 2.51. The forgetful functor from **MClSp** to **MSet**, the category of many-sorted sets and many-sorted mappings, i.e., the category with objects all pairs (S, A), where S is a set and A an S-sorted set and morphisms from (S, A) to (T, B) all pairs (φ, f), where $\varphi\colon S \longrightarrow T$ and $f\colon A \longrightarrow (B_{\varphi(s)})_{s\in S}$, has left and right adjoints and constructs limits and colimits. Therefore all of the results stated by Feitosa and D'Ottaviano in [10] (compare with those stated a long time ago by Brown in [3], by Brown and Suszko in [4], and by Porte in [28], especially those in Chapter 12, pp. 83–96) that have to do with closure spaces, continuous mappings, optimal and co-optimal lifts, and completeness and co-completeness of the category **ClSp** fall, as a very particular case, under those for the category **MClSp**, since their "logics" are nothing more nor less than ordinary (not many-sorted) closure spaces. Besides, by defining the appropriate subcategories of **MClSp**, the many-sorted counterparts of the remaining results in [10] are also, easily, provable from the above generalized theory about many-sorted closure spaces and morphisms between them.

3. Adjunctions and monads

Our main concern in this section is to obtain from a 2-category **Ad**, of adjunctions, two new 2-categories of adjunctions, $\mathbf{Ad}_{\mathrm{Kl}}$ and $\mathbf{Ad}_{\mathrm{EM}}$, which will allow us to extend to two 2-functors the classical, and well-known, construction that assigns to an adjunction a monad, and all in such a way that the classical constructions of Kleisli and Eilenberg-Moore are left and right biadjoints, respectively, for these 2-functors. Moreover, the morphisms and transformations of Kleisli and of Eilenberg-Moore between monads will be characterized, respectively, as the image of morphisms and transformations of Kleisli and of Eilenberg-Moore between the adjunctions. Finally, we define a 2-category $\mathbf{Ad}_{\mathrm{alg}}$, of adjunctions, algebraic squares, and algebraic transformations, and prove that there exists a canonical 2-functor $\mathrm{Md}_{\mathrm{alg}}$ from $\mathbf{Ad}_{\mathrm{alg}}$ to $\mathbf{Mnd}_{\mathrm{alg}}$.

Definition 3.1. Let $F \dashv G$ be an adjunction from $\mathbf{C}$ to $\mathbf{D}$, $F' \dashv G'$ an adjunction from $\mathbf{C}'$ to $\mathbf{D}'$, and (J, δ, H), (J', δ', H') two adjoint squares from $F \dashv G$ to $F' \dashv G'$. Then a *transformation* from (J, δ, H) to (J', δ', H') is a natural transformation τ from H to H'.

Proposition 3.2. *Adjunctions, adjoint squares, and transformations between adjoint squares yield a 2-category, hereafter denoted by* **Ad**.

Proof. It is sufficient to define the identities as

$$
\begin{array}{ccc}
\mathbf{C} & \underset{F}{\overset{G}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{D} \\
{\scriptstyle 1}\big\downarrow & \begin{pmatrix} F & \eta \\ \varepsilon & G \end{pmatrix} & \big\downarrow{\scriptstyle 1} \\
\mathbf{C} & \underset{F}{\overset{G}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{D}
\end{array}
$$

and the composition in **Ad** of two adjoint squares

$$(J, \lambda, H)\colon F \dashv G \longrightarrow F' \dashv G' \quad \text{and} \quad (J', \lambda', H')\colon F' \dashv G' \longrightarrow F'' \dashv G'',$$

as the adjoint square

$$(F \dashv G, (J'J, \lambda' \overset{\mathrm{fn}}{\circ} \lambda, H'H), F'' \dashv G''),$$

where $\lambda' \overset{\mathrm{fn}}{\circ} \lambda$ is the matrix

$$\lambda' \overset{\mathrm{fn}}{\circ} \lambda = \begin{pmatrix} (H'\lambda_0)(\lambda'_0 J) & (G'H'\varepsilon' HF)(\lambda'_1\lambda_1) \\ (\lambda'_2\lambda_2)(F''J'\eta' JG) & (\lambda'_3 H)(J'\lambda_3) \end{pmatrix}.$$

Moreover, for the transformations of adjoint squares there are identities, horizontal compositions, and vertical compositions, defined like those of its underlying natural transformations. ∎

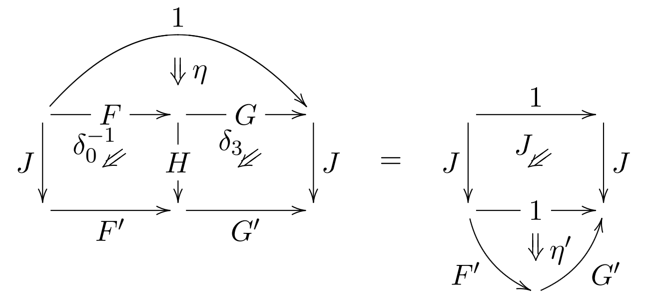
Definition 3.3. Let $F \dashv G$ be an adjunction from $\mathbf{C}$ to $\mathbf{D}$, $F' \dashv G'$ an adjunction from $\mathbf{C}'$ to $\mathbf{D}'$, and (J, δ, H), (J', δ', H') two adjoint squares from $F \dashv G$ to $F' \dashv G'$. Then a transformation τ from (J, δ, H) to (J', δ', H') is a *transformation of Street* if there exists a natural transformation $\sigma\colon J \Longrightarrow J'$ such that the pair (σ, τ) is compatible with the respective adjoint squares. Since the transformations of Street are stable under composition, we obtain the corresponding sub-2-category $\mathbf{Ad}_{\mathrm{St}}$ of **Ad** determined by the transformations of Street.

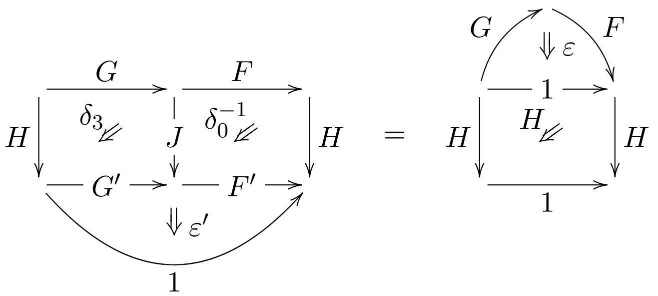
Not every adjoint square, understood as a morphism of adjunctions, gives rise to a morphism between the monads associated to the corresponding adjunctions. However, for a definite class of adjoint squares such an association is possible.

Definition 3.4. We say that an adjoint square (J, δ, H) from $F \dashv G$ to $F' \dashv G'$ is an *adjoint square of Kleisli* or, for brevity, a *Kl-square*, if its 0-th component, δ_0, is a natural isomorphism. Since the Kl-squares are stable under composition, we obtain the sub-2-category $\mathbf{Ad}_{\mathrm{Kl}}$ of **Ad** which has as 0-cells those of **Ad**, as 1-cells the Kl-squares, and as 2-cells those of **Ad**.

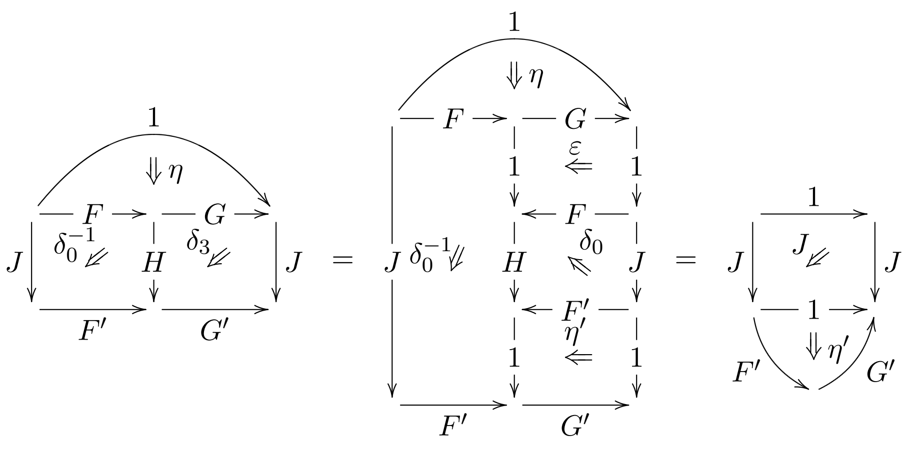
We next prove that from the 2-category $\mathbf{Ad}_{\mathrm{Kl}}$ to $\mathbf{Mnd}^{\mathrm{c}}_{\mathrm{Kl}}$, the conjugate 2-category of $\mathbf{Mnd}_{\mathrm{Kl}}$, there exists a 2-functor which assigns to an adjunction its corresponding monad, to a Kl-square a Kl-morphism of monads, and to a transformation of Kl-squares a Kl-transformation. But before doing that we state the following lemma.

Lemma 3.5. *Let* $(J, \delta, H)\colon F \dashv G \longrightarrow F' \dashv G'$ *be a* Kl-*square. Then the following diagrammatic equations are fulfilled*





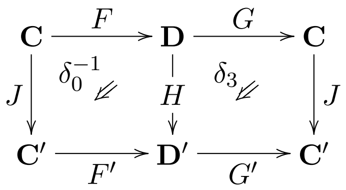
Proof. For the first equation it is sufficient to remark that



The proof of the second equation is formally identical. ∎

Proposition 3.6. *There exists a 2-functor* $\mathrm{Md}_{\mathrm{Kl}}$ *from the 2-category* $\mathbf{Ad}_{\mathrm{Kl}}$ *to the 2-category* $\mathbf{Mnd}^{\mathrm{c}}_{\mathrm{Kl}}$ *which sends:* (1) *an adjunction* $(F \dashv G, \eta, \varepsilon)$ *to the monad* $(G \circ F, \eta, G\varepsilon F)$, (2) *a* Kl-*square* (J, δ, H) *to the* Kl-*morphism* (J, λ_δ), *where* λ_δ *is* $G\delta_0^{-1} \circ \delta_3 F$, *and* (3) *a transformation* τ *from* (J', δ', H') *to* (J, δ, H) *to the* Kl-*transformation* $\Xi^\tau = G'\delta_0^{-1} \circ G'\tau F \circ \delta'_3 F \circ J'\eta$.

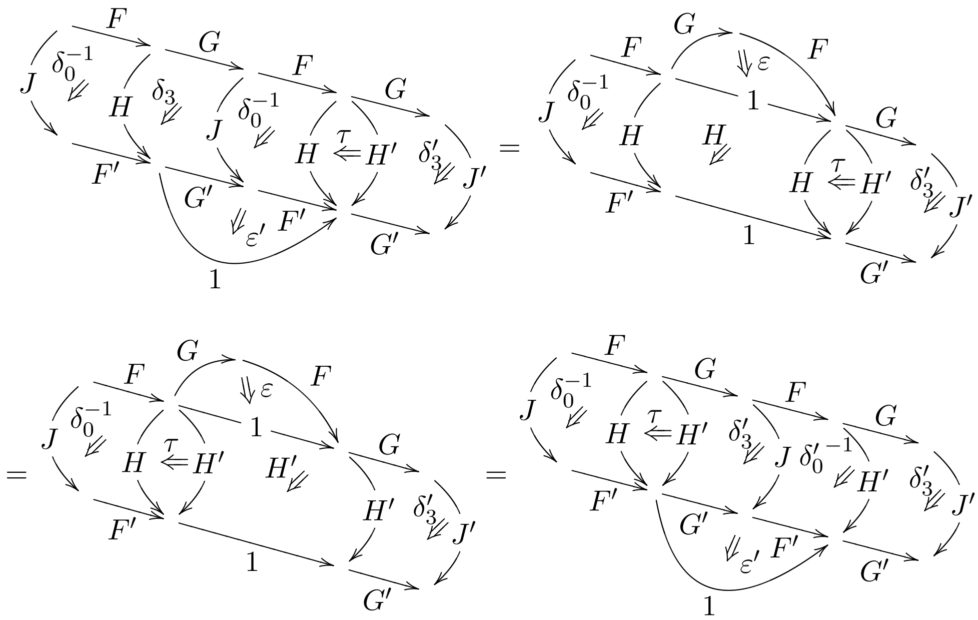
Proof. Let $(J, \delta, H)\colon F \dashv G \longrightarrow F' \dashv G'$ be a Kl-square. From Lemma 3.5 it is a simple matter to verify that the natural transformation



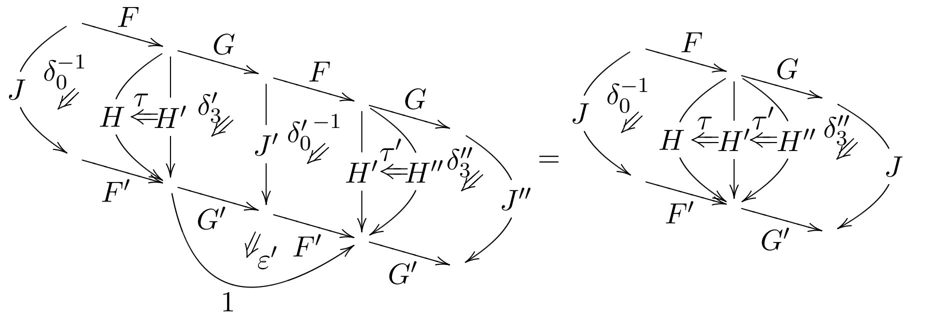
is a Kl-morphism of monads.

The compatibility with the identity and the compositions is immediate.

By Lemma 3.5, Ξ^τ is a transformation, since

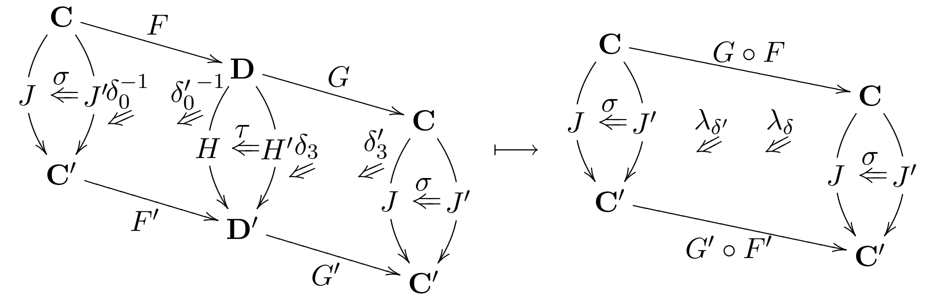


The compatibility with the 2-identities is immediate. Also the compatibility with the horizontal composition is immediate, making use of the alternative definition of the horizontal composition of Kl-transformations. For the vertical composition, we have that



∎

The transformations of Street between Kl-squares are carried into Kl-transformations of monads under the action of the 2-functor $\mathrm{Md}_{\mathrm{Kl}}$, and we denote by $\mathrm{Md}_{\mathrm{KS}}$ the bi-restriction of $\mathrm{Md}_{\mathrm{Kl}}$ to $\mathbf{Ad}_{\mathrm{KS}}$ and $\mathbf{Mnd}_{\mathrm{KS}}$. The action of $\mathrm{Md}_{\mathrm{KS}}$ on a transformation of Street (σ, τ) is the mapping



Remark 3.7. The 2-functor $\mathrm{Md}_{\mathrm{Kl}}$ is obtained by composing the 2-functor from $\mathbf{Ad}_{\mathrm{Kl}}$ to $\mathbf{Kl}$ that forgets all components of the Kl-squares with the exception of the first one, and the 2-isomorphism between $\mathbf{Kl}$ and $\mathbf{Mnd}^{\mathrm{c}}_{\mathrm{Kl}}$.

Our next objective is to prove that the 2-functor $\mathrm{Md}_{\mathrm{Kl}}$ has a left biadjoint which is, essentially, obtained by composing the 2-isomorphism from $\mathbf{Mnd}^{\mathrm{c}}_{\mathrm{Kl}}$ to $\mathbf{Kl}$ with the 2-functor which embeds $\mathbf{Kl}$ into $\mathbf{Ad}_{\mathrm{Kl}}$ by assigning: (1) to an object of $\mathbf{Kl}$ its corresponding adjunction of Kleisli, (2) to every 1-cell the Kl-square obtained through the natural transformations transpose of the identity of the commutative square corresponding to the 1-cell, and (3) leaving invariant the 2-cells. From this it follows that the full sub-2-category of $\mathbf{Ad}_{\mathrm{Kl}}$ determined by the adjunctions of Kleisli is a co-reflective sub-2-category of $\mathbf{Ad}_{\mathrm{Kl}}$.

Proposition 3.8. *There exists a* 2*-functor* Kl *from the* 2*-category* $\mathbf{Mnd}^{c}_{\mathrm{Kl}}$ *to the* 2*-category* $\mathbf{Ad}_{\mathrm{Kl}}$ *which sends:* (1) *a monad* $(\mathbf{C}, \mathbb{T})$ *to the canonical adjunction* $(F_{\mathbb{T}}, G_{\mathbb{T}})$, (2) *a* Kl*-morphism* (J, λ) *of monads to the* Kl*-square* $(J, \delta_\lambda, H_\lambda)$, *where* H_λ *is the functor associated to* λ *by the bijection in Proposition* 2.4 *and* δ_λ *the adjoint square determined by the corresponding commutative square, and* (3) *a* Kl*-transformation* $\Xi\colon (J, \lambda) \rightsquigarrow (J', \lambda')$ *to the transformation* τ^{Ξ} *associated to* Ξ *by the bijection in Proposition* 2.16*.*

The transformations of Street between Kl-morphisms of monads are carried into transformations of Street between Kl-squares under the action of the 2-functor Kl, and we denote by $\mathrm{Kl}_{\mathrm{St}}$ the bi-restriction of Kl to $\mathbf{Mnd}_{\mathrm{KS}}$ and $\mathbf{Ad}_{\mathrm{KS}}$.

Proposition 3.9. *The* 2*-functor* Kl *is a left biadjoint for the* 2*-functor* $\mathrm{Md}_{\mathrm{Kl}}$.

$$\mathbf{Ad}_{\mathrm{Kl}} \underset{\mathrm{Kl}}{\overset{\mathrm{Md}_{\mathrm{Kl}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Mnd}^{c}_{\mathrm{Kl}}$$

Proof. We want to prove that for every adjunction there exists a universal morphism from the 2-functor Kl to it, i.e., that if $F \dashv G$ is an adjunction, with associated monad $\mathbb{T}$, then there exists a Kl-square

$$\varepsilon_{F \dashv G}\colon F_{\mathbb{T}} \dashv G_{\mathbb{T}} \longrightarrow F \dashv G$$

such that, for every monad $(\mathbf{A}, \mathbb{M})$ and every Kl-square

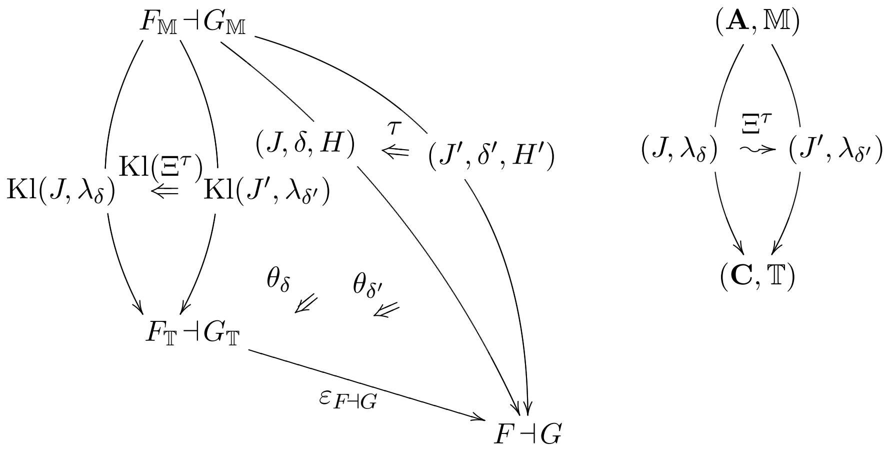
$$(J, \delta, H)\colon F_{\mathbb{M}} \dashv G_{\mathbb{M}} \longrightarrow F \dashv G,$$

the Kl-morphism of monads $(J, \lambda_\delta)\colon (\mathbf{A}, \mathbb{M}) \longrightarrow (\mathbf{C}, \mathbb{T})$ is, up to isomorphism, the unique for which there exists an invertible transformation

$$\theta_\delta\colon (J, \delta, H) \Longrightarrow \varepsilon_{F \dashv G} \circ \mathrm{Kl}(J, \lambda_\delta)$$

$$\begin{array}{ccc} F_{\mathbb{M}} \dashv G_{\mathbb{M}} & & \\ {\scriptstyle \mathrm{Kl}(J,\lambda_\delta)}\downarrow & \theta_\delta \Swarrow & \searrow {\scriptstyle (J,\delta,H)} \\ F_{\mathbb{T}} \dashv G_{\mathbb{T}} & \xrightarrow[\varepsilon_{F \dashv G}]{} & F \dashv G \end{array} \qquad \qquad \begin{array}{c} (\mathbf{A}, \mathbb{M}) \\ \downarrow {\scriptstyle (J, \lambda_\delta)} \\ (\mathbf{C}, \mathbb{T}) \end{array}$$

and that, for every transformation $\tau\colon (J', \delta', H') \longrightarrow (J, \delta, H)$, the Kl-transformation Ξ^τ from (J, λ_δ) to $(J', \lambda_{\delta'})$ is the unique that makes the left-hand side diagram of the following figure commutative

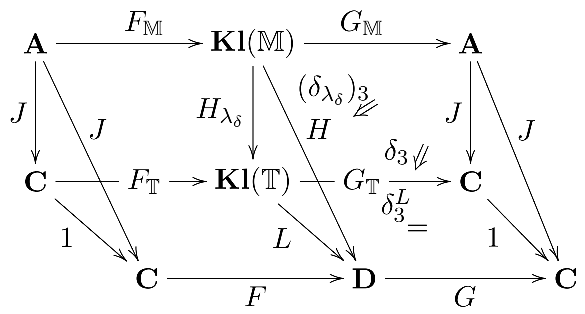


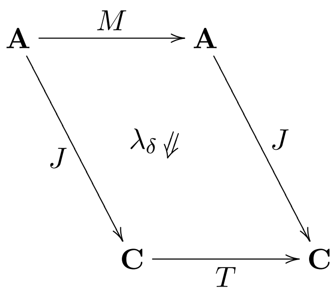
i.e., such that $\theta_\delta \circ \tau = \varepsilon_{F \dashv G}\mathrm{Kl}(\Xi^\tau) \circ \theta_{\delta'}$.

Let $F \dashv G$ be an adjunction from $\mathbf{C}$ to $\mathbf{D}$ and $\mathbb{T}$ its canonically associated monad. Then, from the functor $L\colon \mathbf{Kl}(\mathbb{T}) \longrightarrow \mathbf{D}$, the comparison functor of Kleisli, we get the Kl-square $(1, \delta^L, L)$ from $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ to $F \dashv G$, by the commutativity of the following diagram

$$\begin{array}{ccccc}
\mathbf{C} & \xrightarrow{F_{\mathbb{T}}} & \mathbf{Kl}(\mathbb{T}) & \xrightarrow{G_{\mathbb{T}}} & \mathbf{C} \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle 1} \\
\mathbf{C} & \xrightarrow[F]{} & \mathbf{D} & \xrightarrow[G]{} & \mathbf{C}
\end{array}$$

and the fact that the identity natural transformations in the squares of the above diagram are mutually conjugate. The Kl-square $(1, \delta^L, L)$ is the value of the counit of the biadjunction looked for on $F \dashv G$. Let $\mathbb{M}$ be a monad on $\mathbf{A}$ and (J, δ, H) a Kl-square from $F_{\mathbb{M}} \dashv G_{\mathbb{M}}$ to $F \dashv G$. Then $\mathrm{Md}_{\mathrm{Kl}}(J, \delta, H) = (J, \lambda_\delta)$ is a Kl-morphism of monads. Let $(J, \delta_{\lambda_\delta}, H_{\lambda_\delta})$ be its image under the functor Kl. Then we have the situation described by the following diagram





Let θ_δ be the mapping that to a $\mathbf{Kl}(\mathbb{M})$-object a assigns the $\mathbf{D}$-morphism

$$(\delta_0^{-1})_a \colon HF_{\mathbb{M}}(a) \longrightarrow FJ(a).$$

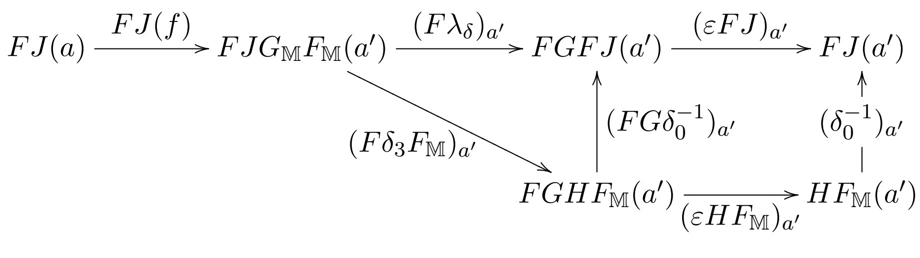
Thus defined θ_δ is a natural isomorphism between the functors H and $L \circ H_{\lambda_\delta}$. Let us verify that it is an invertible transformation in the 2-category $\mathbf{Ad}_{\mathrm{Kl}}$. Let $f \colon a \longrightarrow a'$ be a $\mathbf{Kl}(\mathbb{M})$-morphism. The functor H_{λ_δ} assigns to f the $\mathbf{Kl}(\mathbb{T})$-morphism which corresponds to the following $\mathbf{C}$-morphism

$$J(a) \xrightarrow{J(f)} JG_{\mathbb{M}}F_{\mathbb{M}}(a') \xrightarrow{(\lambda_\delta)_{a'}} GFJ(a'),$$

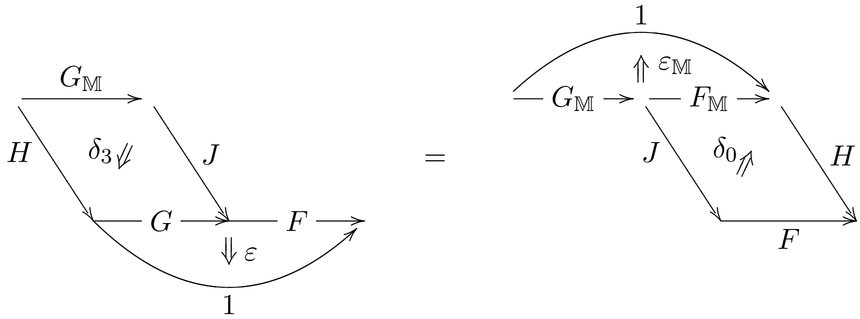
and the comparison functor of Kleisli L assigns to every $\mathbf{Kl}(\mathbb{T})$-morphism g from c to c' the $\mathbf{D}$-morphism $L(g) = \varepsilon_{F(c')} \circ F(g) \colon F(c) \longrightarrow F(c')$, as depicted in the following diagram

$$F(c) \xrightarrow{F(g)} FGF(c') \xrightarrow{\varepsilon_{F(c')}} F(c').$$

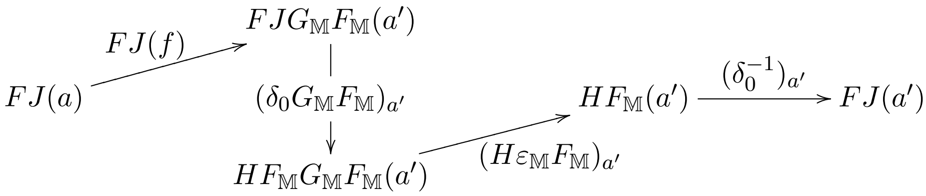
Therefore $L \circ H_{\lambda_\delta}(f)$ is the $\mathbf{D}$-morphism from $FJ(a)$ to $FJ(a')$ in the commutative diagram



But it happens that $\varepsilon H \circ F\delta_3 = H\varepsilon_{\mathbb{M}} \circ \delta_0 G_{\mathbb{M}}$



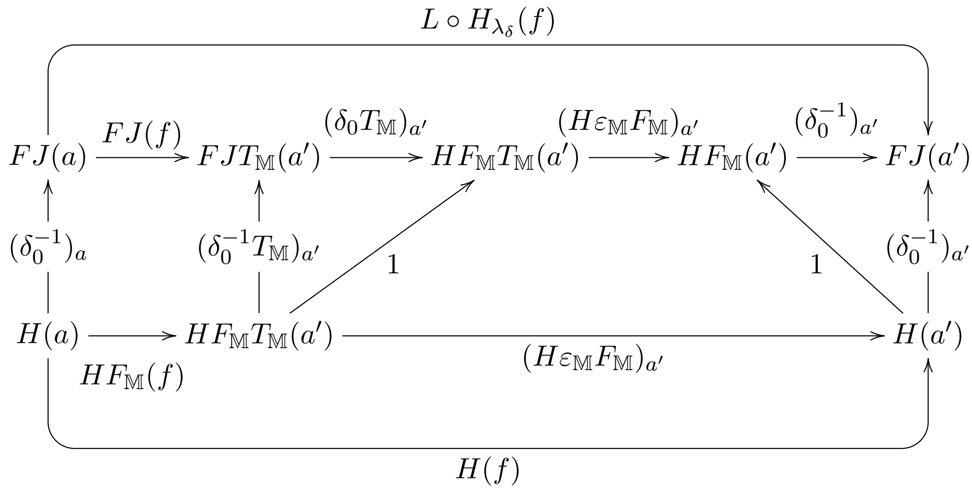
therefore $L \circ H_{\lambda_\delta}(f)$ is



On the other hand, we have that $F_{\mathbb{M}}(f) = (\eta_{\mathbb{M}})_{M(a')} \circ f = F_{\mathbb{M}}((\eta_{\mathbb{M}})_{a'}) \diamond f$, and, therefore, that

$$\begin{aligned}(H\varepsilon_{\mathbb{M}}F_{\mathbb{M}})_{a'} \circ HF_{\mathbb{M}}(f) &= (H\varepsilon_{\mathbb{M}}F_{\mathbb{M}})_{a'} \circ HF_{\mathbb{M}}((\eta_{\mathbb{M}})_{a'}) \circ H(f) \\ &= \mathrm{id}_{HF_{\mathbb{M}}(a')} \circ H(f) = \mathrm{id}_{H(a')} \circ H(f) = H(f)\,.\end{aligned}$$

Consequently the following diagram commutes



where $T_{\mathbb{M}}$ stands for $G_{\mathbb{M}}F_{\mathbb{M}}$ and, by the definition of the functor $F_{\mathbb{M}}$ from $\mathbf{A}$ to $\mathbf{Kl}(\mathbb{M})$, for each $\mathbf{A}$-object a, it happens that $H(a) = HF_{\mathbb{M}}(a)$. Thus θ_δ is an invertible transformation from (J, δ, H) to $(1, \delta^L, L) \circ (J, \delta_{\lambda_\delta}, H_{\lambda_\delta})$. Furthermore, if $(J', \lambda') \colon (\mathbf{A}, \mathbb{M}) \longrightarrow (\mathbf{C}, \mathbb{T})$ is a Kl-morphism and θ' an invertible transformation from (J, δ, H) to $(1, \delta^L, L) \circ (J', \delta_{\lambda'}, H_{\lambda'})$, then (J, λ_δ) and (J', λ') are isomorphic. This is because $\mathrm{Md}_{\mathrm{Kl}}(\theta' \circ \theta_\delta^{-1})$ is an invertible Kl-transformation in $\mathbf{Mnd}_{\mathrm{Kl}}$ from (J, λ_δ) to (J', λ') as shown by the following diagram

$$(J, \lambda_\delta) = (J, \lambda_{H_{\lambda_\delta}}) = \mathrm{Md}_{\mathrm{Kl}}((1, \delta^L, L) \circ (J, \delta_{\lambda_\delta}, H_{\lambda_\delta}))$$

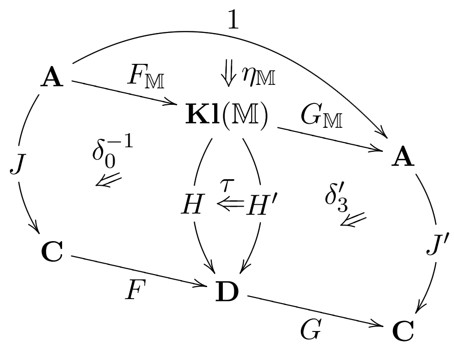
$$\Big\downarrow \mathrm{Md}_{\mathrm{Kl}}(\theta_\delta^{-1})$$

$$\mathrm{Md}_{\mathrm{Kl}}(J, \delta, H)$$

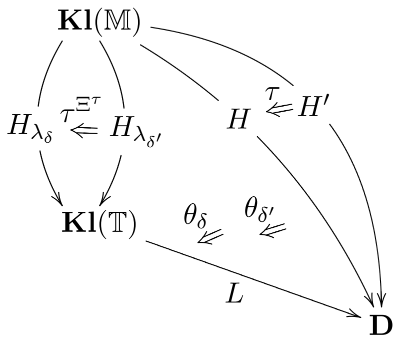
$$\Big\downarrow \mathrm{Md}_{\mathrm{Kl}}(\theta')$$

$$\mathrm{Md}_{\mathrm{Kl}}((1, \delta^L, L) \circ (J', \delta_{\lambda'}, H_{\lambda'})) = (J', \lambda_{H_{\lambda'}}) = (J', \lambda')$$

Next, let $\tau \colon (J', \delta', H') \longrightarrow (J, \delta, H)$ be a transformation. Then $\mathrm{Md}_{\mathrm{Kl}}(\tau)$ is, precisely, the Kl-transformation $\Xi^\tau = G\delta_0^{-1} \circ G\tau F_{\mathbb{M}} \circ \delta_3' F_{\mathbb{M}} \circ J'\eta_{\mathbb{M}}$, obtained as shown in the following diagram



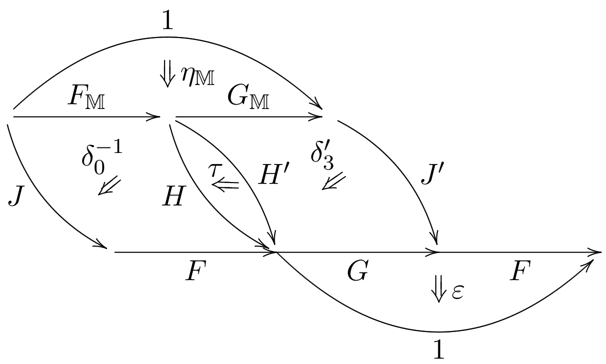
Let us put $\tau^{\Xi^\tau} = \mathrm{Kl}(\mathrm{Md}_{\mathrm{Kl}}(\tau))$. We claim that $\theta_\delta \circ \tau = \varepsilon_{F \dashv G} \tau^{\Xi^\tau} \circ \theta_{\delta'}$. However, to state this equation it is sufficient to verify that $\tau \circ \theta_\delta = L\tau^{\Xi^\tau} \circ \theta_{\delta'}$, i.e., that the following diagram commutes



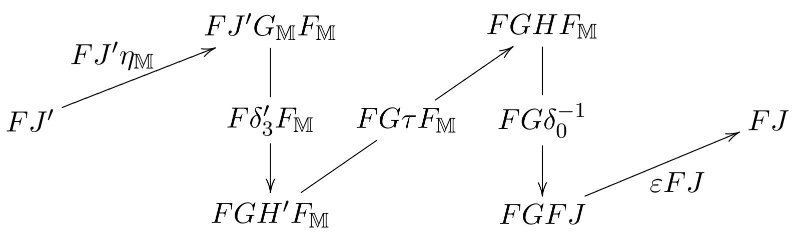
For every $\mathbf{Kl}(\mathbb{M})$-object a, $\tau_a^{\Xi^\tau}$ is the $\mathbf{Kl}(\mathbb{T})$-morphism that corresponds to the $\mathbf{C}$-morphism Ξ_a^τ, hence we have that

$$L\tau^{\Xi^\tau}(a) = L(\Xi_a^\tau) = L(G\delta_0^{-1} \circ G\tau F_{\mathbb{M}} \circ \delta_3' F_{\mathbb{M}} \circ J'\eta_{\mathbb{M}})_a,$$

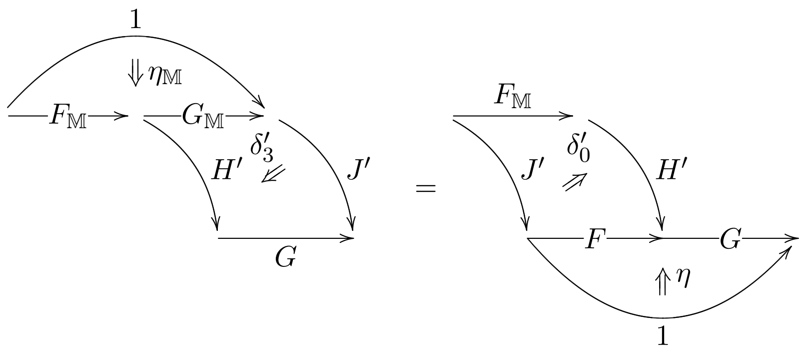
i.e., the action at a of the natural transformation of the diagram



and that, therefore, is equal to



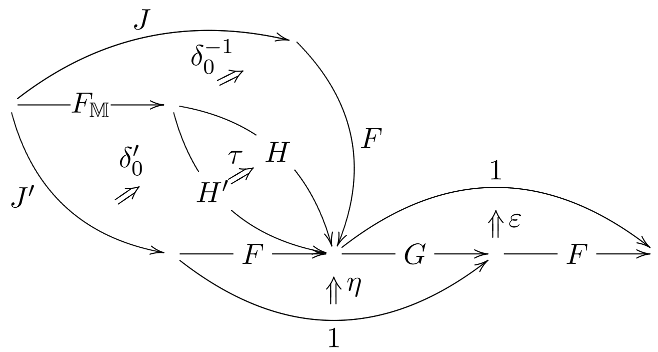
But $\delta'_3 F_{\mathbb{M}} \circ J'\eta_{\mathbb{M}} = G\delta_0 \circ \eta J'$, since



hence the considered natural transformation is

$$\begin{array}{l} FJ' \xrightarrow{F\eta J'} FGFJ' \xrightarrow{FG\delta_0} FGH'F_{\mathbb{M}} \\ \qquad\qquad\qquad\qquad\qquad \downarrow FG\tau F_{\mathbb{M}} \\ \qquad\qquad\qquad\qquad\qquad FGHF_{\mathbb{M}} \xrightarrow{FG\delta_0^{-1}} FGFJ \xrightarrow{\varepsilon FJ} FJ \end{array}$$

i.e., the natural transformation of the diagram



which is equal to $\delta_0^{-1} \circ \tau F_{\mathbb{M}} \circ \delta'_0$. Then we have that

$$\begin{aligned} (\theta_\delta \circ \tau)_a &= (\delta_0^{-1})_a \circ \tau_a = (\delta_0^{-1})_a \circ (\tau F_{\mathbb{M}})_a \\ &= (\delta_0^{-1} \circ \tau F_{\mathbb{M}} \circ \delta'_0 \circ {\delta'}_0^{-1})_a \\ &= (L\tau^{\Xi^\tau} \circ \theta_{\delta'})_a. \end{aligned}$$

Let us verify, finally, that the uniqueness is also fulfilled. If Ξ is a Kl-transformation from (J, λ_δ) to $(J', \lambda_{\delta'})$ such that $\theta_\delta \circ \tau = L\tau^{\Xi} \circ \theta_{\delta'}$, then

$$\begin{aligned} \mathrm{Md}_{\mathrm{Kl}}(\tau^{\Xi^\tau}) \circ \mathrm{Md}_{\mathrm{Kl}}(\theta_{\delta'}) &= \mathrm{Md}_{\mathrm{Kl}}(1_L \circ \tau^{\Xi^\tau} \circ \theta_{\delta'}) \\ &= \mathrm{Md}_{\mathrm{Kl}}(1_L \circ \tau^{\Xi} \circ \theta_{\delta'}) \\ &= \mathrm{Md}_{\mathrm{Kl}}(\tau^{\Xi}) \circ \mathrm{Md}_{\mathrm{Kl}}(\theta_{\delta'}), \end{aligned}$$

but $\mathrm{Md}_{\mathrm{Kl}}(\theta_{\delta'})$ is an isomorphism and, consequently, we have that

$$\Xi^\tau = \mathrm{Md}_{\mathrm{Kl}}(\tau^{\Xi^\tau}) = \mathrm{Md}_{\mathrm{Kl}}(\tau^{\Xi}) = \Xi\,.$$

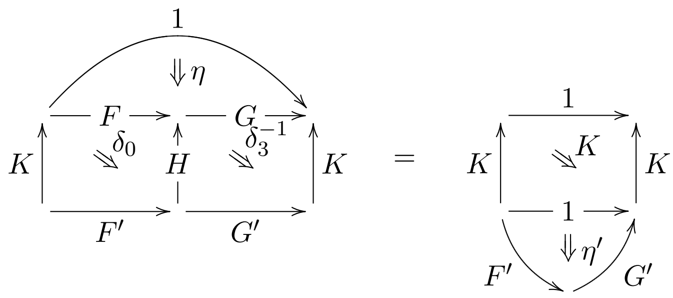
∎

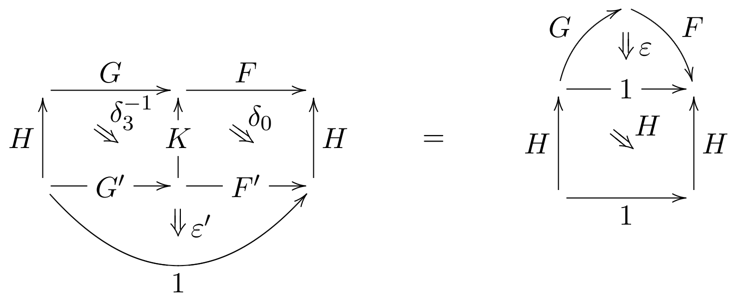
On the ground of an argument by symmetry it is obvious that everything we have done in this section based on the Kleisli construction, from Definition 3.4, about the concept of Kl-square, to Proposition 3.9, about the fact that the 2-functor Kl is a left biadjoint for the 2-functor $\mathrm{Md}_{\mathrm{Kl}}$, has a parallel development founded on the Eilenberg-Moore construction. For this reason we next restrict ourselves to state the counterparts of the above concepts and constructions and to leave it to the reader to verify the corresponding propositions.

Definition 3.10. We say that an adjoint square (K, δ, H) from $F' \dashv G'$ to $F \dashv G$ is an *adjoint square of Eilenberg-Moore* or, for brevity, an *EM-square*, if its 3-th component, δ_3, is a natural isomorphism. Since the EM-squares are stable under composition, we obtain the sub-2-category $\mathbf{Ad}_{\mathrm{EM}}$ of $\mathbf{Ad}$ which has as 0-cells those of $\mathbf{Ad}$, as 1-cells the EM-squares, and as 2-cells those of $\mathbf{Ad}$.

We next show that from the 2-category $\mathbf{Ad}_{\mathrm{EM}}$ to $\mathbf{Mnd}^{\mathrm{t}}_{\mathrm{EM}}$, the transpose 2-category of $\mathbf{Mnd}_{\mathrm{EM}}$, there exists a 2-functor which assigns to an adjunction its corresponding monad, to an EM-square an EM-morphism of monads, and to a transformation of EM-squares an EM-transformation. But before doing that we state without proof the following lemma (which is, for the EM-squares, the counterpart of Lemma 3.5).

Lemma 3.11. *Let* $(K, \delta, H)\colon F' \dashv G' \longrightarrow F \dashv G$ *be an* EM-*square. Then the following diagrammatic equations are fulfilled*





PROPOSITION 3.12. *There exists a* 2*-functor* $\mathrm{Md}_{\mathrm{EM}}$ *from the* 2*-category* $\mathbf{Ad}_{\mathrm{EM}}$ *to the* 2*-category* $\mathbf{Mnd}^{\mathrm{t}}_{\mathrm{EM}}$ *which sends:* (1) *an adjunction* $(F \dashv G, \eta, \varepsilon)$ *to the monad* $(G \circ F, \eta, G\varepsilon F)$, (2) *an* EM*-square* (K, δ, H) *to the* EM*-morphism* (K, λ^{δ}), *where* $\lambda^{\delta} = \delta_3^{-1} F' \circ G\delta_0$, *and* (3) *a transformation* τ *from* (J, δ, H) *to* (J', δ', H') *to the* EM*-transformation* $\Xi_{\tau} = {\delta'_3}^{-1} F' \circ G\tau F' \circ G\delta_0 \circ \eta K$.

Proof. Since the method of proof is formally identical to that we have already used, for the case of Kleisli, in Proposition 3.6, the proof is left to the reader. ∎

The transformations of Street between EM-squares are carried into EM-transformations of monads under the action of the 2-functor $\mathrm{Md}_{\mathrm{EM}}$, and we denote by $\mathrm{Md}_{\mathrm{EMS}}$ the bi-restriction of $\mathrm{Md}_{\mathrm{EM}}$ to $\mathbf{Ad}_{\mathrm{KS}}$ and $\mathbf{Mnd}_{\mathrm{EMS}}$.

Remark 3.13. The 2-functor $\mathrm{Md}_{\mathrm{EM}}$ is obtained by composing the 2-functor from $\mathbf{Ad}_{\mathrm{EM}}$ to $\mathbf{EM}$ that forgets all components of the EM-squares with the exception of the first one, and the 2-isomorphism between $\mathbf{EM}$ and $\mathbf{Mnd}^{\mathrm{t}}_{\mathrm{EM}}$.

The 2-functor $\mathrm{Md}_{\mathrm{EM}}$ has a right biadjoint, EM, which is, essentially, obtained by composing the 2-isomorphism from $\mathbf{Mnd}^{\mathrm{t}}_{\mathrm{EM}}$ to $\mathbf{EM}$ with the

2-functor which embeds **EM** into $\mathbf{Ad}_{\mathrm{EM}}$ by assigning to an object of **EM** its corresponding adjunction of Eilenberg-Moore, to every 1-cell the EM-square obtained through the natural transformations transpose of the identity of the commutative square corresponding to the 1-cell, and leaving invariant the 2-cells. From this it follows that the full sub-2-category of $\mathbf{Ad}_{\mathrm{EM}}$ determined by the adjunctions of Eilenberg-Moore is a reflective sub-2-category of $\mathbf{Ad}_{\mathrm{EM}}$.

PROPOSITION 3.14. *There exists a* 2*-functor* EM *from the* 2*-category* $\mathbf{Mnd}^{\mathrm{t}}_{\mathrm{EM}}$ *to the* 2*-category* $\mathbf{Ad}_{\mathrm{EM}}$ *which sends:* (1) *a monad* $(\mathbf{C}, \mathbb{T})$ *to the canonical adjunction* $(F^{\mathbb{T}}, G^{\mathbb{T}})$, (2) *an* EM*-morphism of monads* (K, λ) *to the* EM*-square* $(K, \delta^{\lambda}, H^{\lambda})$*, where* H^{λ} *is the functor associated to* λ *by the bijection in Proposition* 2.21 *and* δ^{λ} *the adjoint square determined by the corresponding commutative square, and* (3) *an* EM*-transformation* Ξ *from* (K, λ) *to* (K', λ') *to the transformation* τ_{Ξ} *associated to* Ξ *by the bijection in Proposition* 2.29.

The transformations of Street between EM-morphisms of monads are carried into transformations of Street between EM-squares, under the action of the 2-functor EM, and we denote by $\mathrm{EM}_{\mathrm{St}}$ the bi-restriction of EM to $\mathbf{Mnd}_{\mathrm{EMS}}$ and $\mathbf{Ad}_{\mathrm{EMS}}$.

PROPOSITION 3.15. *The* 2*-functor* EM *is a right biadjoint for the* 2*-functor* $\mathrm{Md}_{\mathrm{EM}}$.

$$\mathbf{Ad}_{\mathrm{EM}} \underset{\mathrm{Md}_{\mathrm{EM}}}{\overset{\mathrm{EM}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathbf{Mnd}^{\mathrm{t}}_{\mathrm{EM}}$$

Proof. Since the method of proof is formally identical to that we have already used, for the case of Kleisli, in Proposition 3.9, the proof is left to the reader. ∎

An adjoint square $(J, \lambda, H)\colon F \dashv G \longrightarrow F' \dashv G'$ can simultaneously be a Kl-square and an EM-square, in which case we call it a *KlEM-square*. Let us notice that then the following diagram commutes

$$\begin{array}{ccccc}
\mathbf{C} & \xrightarrow{F} & \mathbf{D} & \xrightarrow{G} & \mathbf{C} \\
{\scriptstyle J}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle J} \\
\mathbf{C}' & \xrightarrow[F']{} & \mathbf{D}' & \xrightarrow[G']{} & \mathbf{C}'
\end{array}$$

and that the pair (J, H) is a transformation of adjunctions as defined by Mac Lane in [23]. Adjunctions, KlEM-squares, and transformations yield a 2-category, hereafter denoted by $\mathbf{Ad}_{\mathrm{tn}}$, and it is the common sub-2-category of $\mathbf{Ad}_{\mathrm{Kl}}$ and $\mathbf{Ad}_{\mathrm{EM}}$.

For the concept of monad we also have a corresponding notion of KlEM-square as stated in the following definition.

Definition 3.16. Let $(\mathbf{C}, \mathbb{T})$ and $(\mathbf{C}', \mathbb{T}')$ be two monads. A *KlEM-square* from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ is a functor $J\colon \mathbf{C} \longrightarrow \mathbf{C}'$ such that: (1) the following square commutes

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\;T\;} & \mathbf{C} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathbf{C}' & \xrightarrow[\;T'\;]{} & \mathbf{C}' \end{array}$$

(2) $J\eta = \eta' J$, and (3) $\mu' J = J\mu$.

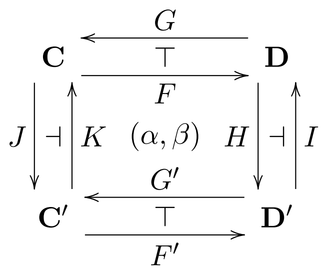
Such a KlEM-square is a Kl-morphism from $(\mathbf{C}, \mathbb{T})$ to $(\mathbf{C}', \mathbb{T}')$ and also an EM-morphism from $(\mathbf{C}', \mathbb{T}')$ to $(\mathbf{C}, \mathbb{T})$. Moreover, the KlEM-squares yield a 2-category, hereafter denoted by $\mathbf{Mnd}_{\mathrm{tn}}$, which is the common sub-2-category of $\mathbf{Mnd}^{\mathrm{c}}_{\mathrm{Kl}}$ and $\mathbf{Mnd}^{\mathrm{t}}_{\mathrm{EM}}$.

From $\mathbf{Ad}_{\mathrm{tn}}$ to $\mathbf{Mnd}_{\mathrm{tn}}$ there exists a 2-functor $\mathrm{Md}_{\mathrm{tn}}$ (obtained by bi-restriction to $\mathbf{Ad}_{\mathrm{tn}}$ and $\mathbf{Mnd}_{\mathrm{tn}}$). Likewise, it is easy to check that the morphism of adjunctions of Kleisli (respectively, of Eilenberg-Moore) determined by a KlEM-square between monads is a transformation of adjunctions, therefore the 2-functors Kl and EM can both be bi-restricted, respectively, to 2-functors $\mathrm{Kl}_{\mathrm{tn}}$ and $\mathrm{EM}_{\mathrm{tn}}$ from $\mathbf{Mnd}_{\mathrm{tn}}$ to $\mathbf{Ad}_{\mathrm{tn}}$, and that, consequently, $\mathrm{Kl}_{\mathrm{tn}}$ is a left biadjoint and $\mathrm{EM}_{\mathrm{tn}}$ a right biadjoint for the 2-functor $\mathrm{Md}_{\mathrm{tn}}$.

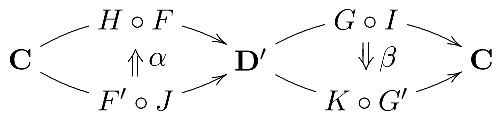
The existence of transformations, as defined by Mac Lane in [23], between adjunctions (associated to algebraic theories) is not, however, the only possible in algebraic contexts. It often happens that there are pairs of adjunctions such that their underlying categories are also, in its turn, mutually related by adjunctions. In this connection we point out that the following three statements are equivalent: (1) there exist a Kl-square from $F \dashv G$ to $F' \dashv G'$ and an EM-square from $F' \dashv G'$ to $F \dashv G$, such that the underlying functors are, two by two, mutually adjoints, (2) there exists a Kl-square from $F \dashv G$ to $F' \dashv G'$ such that the underlying functors have right adjoints, and (3) there exists an EM-square from $F' \dashv G'$ to $F \dashv G$ such that the underlying functors have left adjoints.

The above situation has a more concise description in terms of an square composed by adjunctions, and it is, in fact, equivalent to the existence of a certain natural isomorphism in such an square.

Definition 3.17. An *algebraic square* is a diagram of categories and adjunctions as in



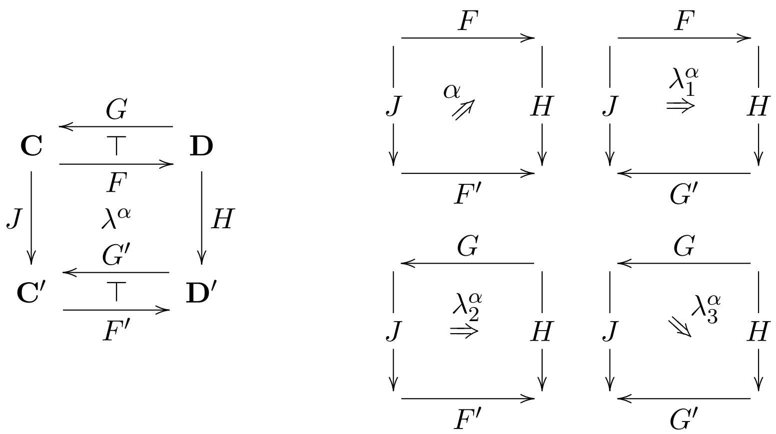
where (α, β) is a conjugate pair of natural isomorphisms from $H \circ F \dashv G \circ I$ to $F' \circ J \dashv K \circ G'$.

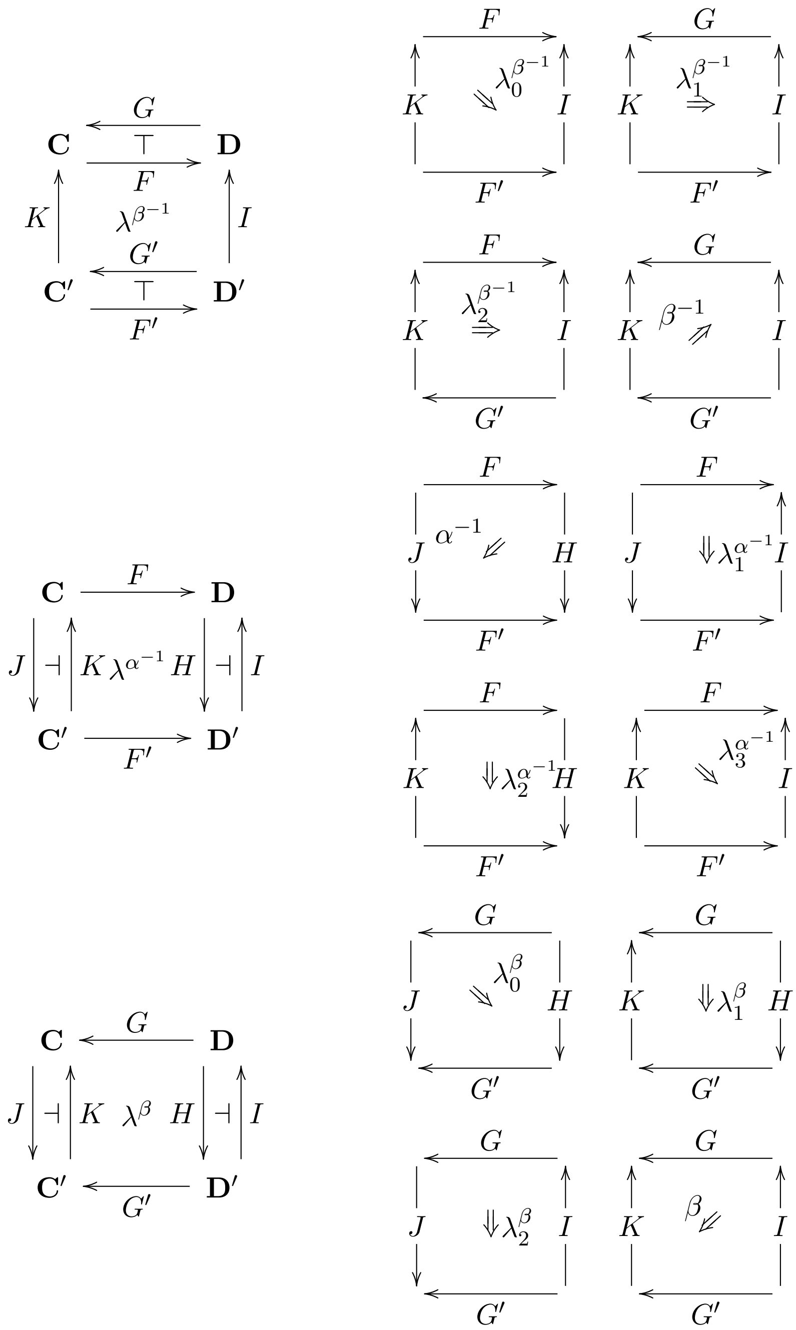


Proposition 3.18. *Given an algebraic square as in Definition* 3.17, *each one of the natural isomorphisms*

$$\alpha\colon F'J \Longrightarrow HF,\ \beta^{-1}\colon KG' \Longrightarrow GI,\ \alpha^{-1}\colon HF \Longrightarrow F'J, \text{ and } \beta\colon GI \Longrightarrow KG'$$

gives rise to an adjoint square





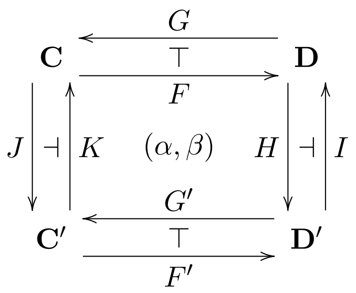
Moreover, each one of the natural transformations in the above diagrams univocally determines the remaining natural transformations.

Remark 3.19. The concept of algebraic squares can be defined alternative, but equivalently, as a diagram of adjunctions and functors as in 3.17, together with a matrix

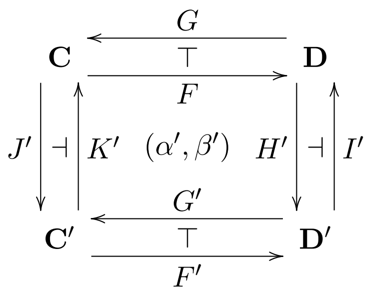
$$\lambda = \begin{pmatrix} \lambda_0 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{pmatrix}$$

of adjoint squares which is compatible with the involved adjunctions, where the compatibility is defined by stipulating that $(\lambda_{0,0}, \lambda_{2,3})$ and $(\lambda_{1,3}, \lambda_{3,3})$ are inverse natural isomorphisms and $(\lambda_{0,0}, \lambda_{1,3})$ and $(\lambda_{2,3}, \lambda_{3,3})$ conjugate pairs.

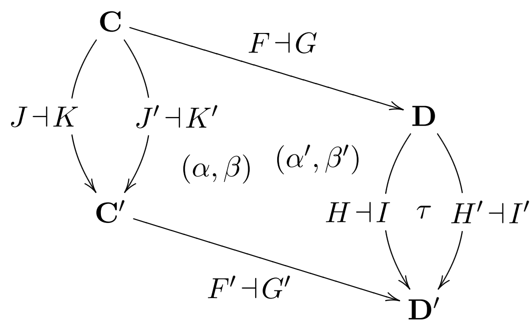
Definition 3.20. Let



and



be two algebraic squares. An *algebraic transformation* from the former to the latter is a conjugate pair $\tau = (\tau_0 \colon H' \Longrightarrow H, \tau_1 \colon I \Longrightarrow I')$ from $H \dashv I$ to $H' \dashv I'$. We agree that a diagram of the shape

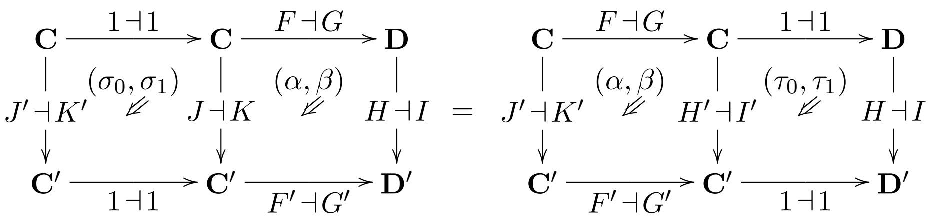


represents an algebraic transformation between two algebraic squares as above.

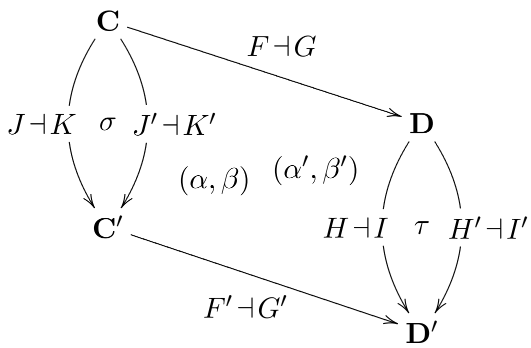
Proposition 3.21. *Adjunctions, algebraic squares, and algebraic transformations yield a 2-category, hereafter denoted by* $\mathbf{Ad}_{\mathrm{alg}}$.

Proof. Identities and compositions of transformations are defined as those of its conjugate pairs. ▮

DEFINITION 3.22. Let us consider two algebraic squares as in Definition 3.20. Then a *transformation of Street* from the former to the latter is a pair (σ, τ), where $\sigma = (\sigma_0, \sigma_1)$ is a conjugate pair of $J \dashv K$ in $J' \dashv K'$ and $\tau = (\tau_0, \tau_1)$ a conjugate pair of $H \dashv I$ in $H' \dashv I'$, compatible with the algebraic squares, i.e., such that



We agree that a diagram of the shape



represents a transformation of Street between two algebraic squares as above.
Identities and compositions of transformations of Street are defined by means of those of its conjugate pairs.

From the above definition it follows, immediately, that to every transformation of Street there corresponds an algebraic transformation (by forgetting its first component). The sub-2-category of $\mathbf{Ad}_{\mathrm{alg}}$ determined by the transformations of Street is denoted by $\mathbf{Ad}_{\mathrm{alg,St}}$.

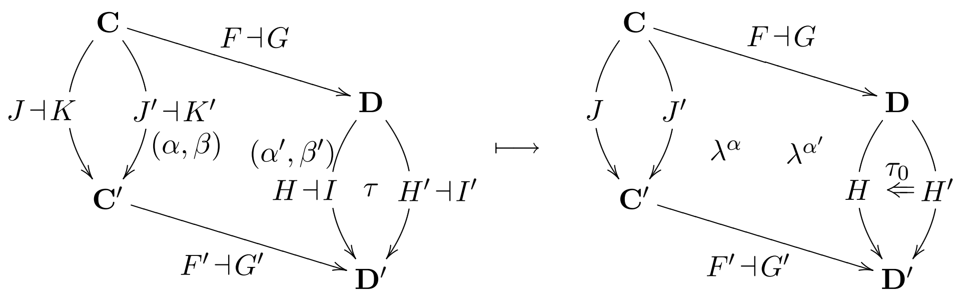
We notice that to every algebraic square there correspond two adjoint squares: one of Kleisli and another of Eilenberg-Moore. In addition, every transformation between algebraic squares determines two transformation: one between the associated squares of Kleisli and another between the associated Eilenberg-Moore squares.

It also happens that to every Kl-square such that its underlying functors have right adjoints there corresponds an algebraic square, and if between two such Kl-squares we have a transformation, then it, in its turn, gives rise to a transformation between the associated algebraic squares.

We point out that the situation for the EM-squares is, abstractly, identical to that of the Kl-squares, i.e., to every EM-square such that its underlying functors have left adjoints there corresponds an algebraic square, and if between two such EM-squares we have a transformation, then it, in its turn, gives rise to a transformation between the associated algebraic squares.

PROPOSITION 3.23. *From the* 2*-category* $\mathbf{Ad}_{\mathrm{alg}}$ *to the* 2*-category* $\mathbf{Ad}^{\mathrm{c}}_{\mathrm{Kl}}$ *there exists a* 2*-functor*

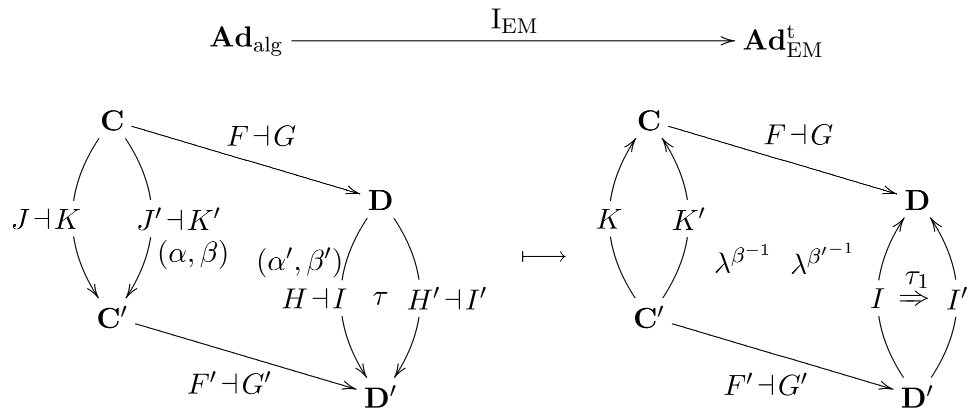
$$\mathbf{Ad}_{\mathrm{alg}} \xrightarrow{\quad \mathrm{I}_{\mathrm{Kl}} \quad} \mathbf{Ad}^{\mathrm{c}}_{\mathrm{Kl}}$$



where λ^{α} *and* $\lambda^{\alpha'}$ *are, respectively, the adjoint squares determined by* α *and* α'. *The* 2*-functor* I_{Kl} *is injective on the objects, pseudo-injective on the morphisms, i.e., for every* Kl*-square its fiber consists of isomorphic algebraic squares, and faithful and full on the* 2*-cells.*

Proof. The 2-functor I_{Kl} is: (1) pseudo-injective on the morphisms since for adjunctions $J \dashv K$ and $J \dashv K'$, we have that $K \cong K'$ and, therefore, $J \dashv K$ and $J \dashv K'$ are isomorphic in $\mathbf{Ad}_{\mathrm{alg}}$, (2) faithful on the 2-cells since the conjugate pairs are unique, and (3) full on the 2-cells since every transformation between algebraic morphisms gives rise to a corresponding conjugate pair. ∎

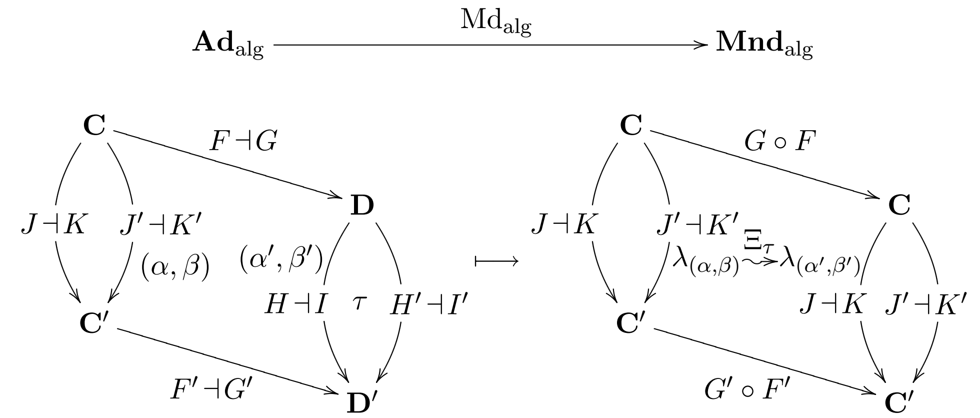
PROPOSITION 3.24. *From the* 2*-category* $\mathbf{Ad}_{\mathrm{alg}}$ *to the* 2*-category* $\mathbf{Ad}^{\mathrm{t}}_{\mathrm{EM}}$ *there exists a* 2*-functor*



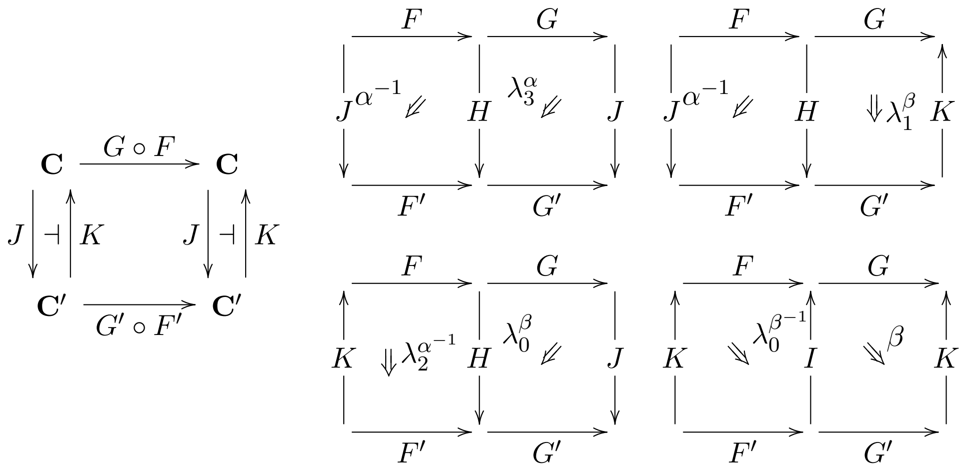
where $\lambda^{\beta^{-1}}$ *and* $\lambda^{\beta'^{-1}}$ *are, respectively, the adjoint squares determined by* β^{-1} *and* β'^{-1}*. The* 2*-functor* I_{EM} *is injective on the objects, pseudo-injective on the morphisms, and faithful and full on the* 2*-cells.*

Finally, we have the following proposition.

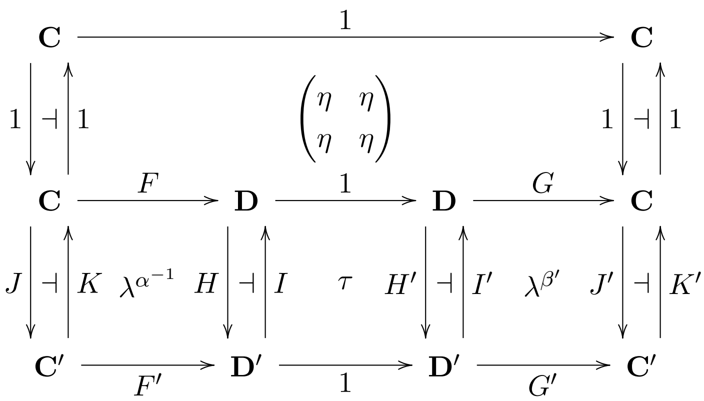
PROPOSITION 3.25. *From the* 2*-category* $\mathbf{Ad}_{\mathrm{alg}}$ *to the* 2*-category* $\mathbf{Mnd}_{\mathrm{alg}}$ *there exists a* 2*-functor*



where $\lambda_{(\alpha,\beta)}$ *is the adjoint square*



$\lambda_{(\alpha',\beta')}$ the corresponding adjoint square, and Ξ_τ the algebraic transformation from $\lambda_{(\alpha,\beta)}$ to $\lambda_{(\alpha',\beta')}$ defined as follows $\Xi_\tau = (\lambda^{\beta'} \overset{\mathrm{fn}}{\circ} \tau \overset{\mathrm{fn}}{\circ} \lambda^{\alpha^{-1}}) \overset{\mathrm{ad}}{\circ} \eta$, and obtained as shown in the following diagram



Proof. By Proposition 3.18, the natural transformations in the image of an algebraic square are mutually transpose and constitute, therefore, an algebraic morphism of monads. Alternatively, an easy verification shows that $\lambda_{(\alpha,\beta)} = \lambda^{\beta} \overset{\mathrm{ad}}{\circ} \lambda^{\alpha^{-1}}$.

The proof that Ξ_τ is, effectively, an algebraic transformation is formally identical to the proofs that Ξ^{τ_0} and Ξ_{τ_3} are, respectively, transformations of monads of Kleisli and of Eilenberg-Moore.

The preservation of identities and compositions follows likewise from those of its components. ∎

Remark 3.26. The 2-functor $\mathrm{Md}_{\mathrm{alg}}$ has not, generally, a left adjoint. However, for some sub-2-categories of $\mathbf{Mnd}_{\mathrm{alg}}$, $\mathrm{Md}_{\mathrm{alg}}$ has a left adjoint. For example, for the full sub-2-category of $\mathbf{Mnd}_{\mathrm{alg}}$ determined by the categories of the form $\mathbf{Set}^S$ where S is a set (of sorts), $\mathrm{Md}_{\mathrm{alg}}$ has a left adjoint, as stated in [6].

EXAMPLE 3.27. To every many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$ there corresponds the adjunction $\mathbf{T}_{\mathbf{\Sigma}} \dashv \mathrm{G}_{\mathbf{\Sigma}}$ from $\mathbf{Set}^S$ to $\mathbf{Alg}(\mathbf{\Sigma})$. To every polyderivor $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, which, as stated in [6], yields an adjunction $\mathbf{d}^{\mathfrak{pd}}_{*} \dashv \mathbf{d}^{*}_{\mathfrak{pd}}$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Alg}(\mathbf{\Lambda})$, there corresponds an algebraic square between the associated adjunctions. Finally, to every transformation $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ from a polyderivor $\mathbf{d}$ to another $\mathbf{e}$, both from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, there corresponds a transformation between the algebraic squares associated to the polyderivors. This follows from the fact, stated in [6], that to every transformation $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ there corresponds a natural transformation $\mathrm{Alg}_{\mathfrak{pd}}(\xi)$ from the functor $\mathbf{d}^{*}_{\mathfrak{pd}}$ to the functor $\mathbf{e}^{*}_{\mathfrak{pd}}$, which, in its turn, leads to the corresponding 2-cell in $\mathbf{Ad}_{\mathrm{alg}}$. Therefore we have two canonical 2-embeddings of the 2-categories $\mathbf{Sig}_{\mathfrak{pd}}$ and $\mathbf{Spf}_{\mathfrak{pd}}$ into the 2-category $\mathbf{Ad}_{\mathrm{alg}}$.

Remark 3.28. We think that our work can be transferred, without any conceptual problems, to the setting of 2-categories. However, we have preferred to formulate the results of this article in the present way, since this presentation might be clearer, more accessible, and even in this generality it covers already enough examples and applications.

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. . . And, although she forsook life,
Great consolation we find
In her memory.

Jorge Manrique

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