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On the Numerical Radius of the Truncated Adjoint Shift

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Abstract: A celebrated theorem of Fejer (1915) asserts that for a given positive trigonometric polynomial $\sum_{j=-n+1}^{n-1} c_j e^{ijt}$, we have $|c_1| \leqslant c_0 \cos \frac{\pi}{n+1}$. A more recent inequality due to U. Haagerup and P. de la Harpe [9] asserts that, for any contraction T such that $T^n = 0$, for some $n \geqslant 2$, the inequality $\omega_2(T) \leqslant \cos \frac{\pi}{n+1}$ holds, and $\omega_2(T) = \cos \frac{\pi}{n+1}$ when T is unitarily equivalent to the extremal operator $S_n^* = S^*{}_{|\mathbb{C}^n} = S^*{}_{|\mathrm{Ker}(u_n(S^*))}$ where $u_n(z) = z^n$ and S^* is the adjoint of the shift operator on the Hilbert space of all square summable sequences. Apparently there is no relationship between them. In this mathematical note, we show that there is a connection between Taylor coefficients of positive rational functions on the torus and numerical radius of the extremal operator $S^*(\phi) = S^*{}_{|\mathrm{Ker}(\phi(S^*))}$ for a precise inner function ϕ. This result completes a line of investigation begun in 2002 by C. Badea and G. Cassier [1]. An upper and lower bound of the numerical radius of $S^*(\phi)$ are given where ϕ is a finite Blashke product with unique zero.

Key words: Numerical radius, numerical range, truncated shift, eigenvalues, Toeplitz forms, inequalities for positive trigonometric polynomials.

AMS *Subject Class.* (2000): 47A12, 47B35.

References

[1] C. Badea, G. Cassier, Constrained von Neumann inequalities, *Adv. Math.* **166** (2) (2002), 260 – 297.

[2] A. Böttcher, S. Grudsky, "Spectral Properties of Banded Toeplitz Matrices", Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.

[3] G. Cassier, I. Chalendar, B. Chevreau, New examples of contractions illustrating membership and non-membership in the classes $A_{m,n}$, *Acta Sci. Math. (Szeged)* **64** (1998), 701 – 731.

[4] E.V. Egerváry, O. Szász, Einige Extremalprobleme in Bereiche der trigonometrischen Polynomen, *Math. Z.* **27** (1928), 641 – 652.

[5] H.L. Gau, P.Y. Wu, Numerical range and Poncelet property, *Taiwanese J. Math.* **7** (2) (2003), 173 – 193.

[6] H.L. Gau, P.Y. Wu, Numerical range of $S(\varphi)$, *Linear and Multilinear Algebra* **45** (1) (1998), 49 – 73.

[7] U. Grenander, G. Szegö, "Toeplitz Forms and their Applications", California Monographs in Mathematical Sciences, University of California Press, Berkeley-Los Angeles, 1958.

[8] K.E. Gustafson, D.K.M. Rao, "Numerical Range", Springer, New York, 1997.

[9] U. Haagerup, P. de la Harpe, The numerical radius of a nilpotent operator on a Hilbert space, *Proc. Amer. Math. Soc.* **115** (1992), 371 – 379.

[10] P.R. Halmos, "A Hilbert Space Problem Book, Second Edition", Graduate Texts in Mathematics, 19 (Encyclopedia of Mathematics and its Applications, 17), Springer-Verlag, New York-Berlin, 1982.

[11] R.A. Horn, C.R. Johnson, "Topics in Matrix Analysis" Cambridge University Press, Cambridge, 1991.

[12] M. Kac, W.L. Murdock, G. Szegö, On the eigenvalues of certain Hermitian forms, *J. Rational Mech. Anal.* **2** (1953), 767 – 800.

[13] B. Sz.-Nagy, C. Foias, "Harmonic Analysis of Operator on Hilbert Space", North-Holland Publishing Co., Amesterdam-London; Americain Elsevier Publishing Co., Inc., New York; Akadémiai Kiado, Budapest, 1970.

[14] W.F. Trench, Interlacement of the even and odd spectra of real symmetric Toeplitz matrices, *Linear Algebra Appl.* **195** (1993), 59 – 68.