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On a New Relative Invariant Covering Dimension *

D.N. Georgiou, A.C. Megaritis

Department of Mathematics, University of Patras, 26500 *Patras, Greece*
georgiou@math.upatras.gr , megariti@master.math.upatras.gr

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Abstract: In [7] (see also [2, p. 35]) two relative covering dimensions, denoted by dim and $\dim^*$, defined and studied. In [3] and [4] we studied these dimensions and we gave some properties including subspace, sum, partition, compactification, and product theorems. Also, we gave partial answers for the questions which are given in [7]. Here we give and study a new relative covering dimension, denoted by r-dim, which is different from dim and $\dim^*$. Finally, we give some questions concerning the new relative dimension r-dim.

Key words: Covering dimension, relative dimension.

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1. Introduction and preliminaries

The first infinite cardinal is denoted by ω. We also consider two symbols, "-1" and "∞", for which we suppose that:

(i) $-1 < n < \infty$ for every $n \in \omega$;

(ii) $\infty + n = n + \infty = \infty$ and $-1 + n = n + (-1) = n$ for every $n \in \omega \cup \{-1, \infty\}$.

By a *class of subsets* we mean a class consisting of pairs (Q, X), where Q is a subset of a topological space X.

Let A and B be two disjoint subsets of a topological space X. We say that *a subset L of X is a partition between A and B* if there exist two open subsets U and W of X such that (1) $A \subseteq U$, $B \subseteq W$, (2) $U \cap W = \emptyset$, and (3) $X \setminus L = U \cup W$.

Let X be a topological space. A *cover* of X is a non-empty set of subsets of X, whose union is X. A cover c of X is said to be open (respectively, closed) if all elements of c is open (respectively, closed). A family $r = \{R_t : t \in T\}$ of subsets of X is said to be a *refinement* of a family $c = \{C_s : s \in S\}$ of

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subsets of X if each element of r is contained in an element of c, that is for every $t \in T$ there exists $s(t) \in S$ such that $R_t \subseteq C_{s(t)}$.

Define the *order* of a family r of subsets of a space X as follows:

(a) $\mathrm{ord}(r) = -1$ if and only if r consists the empty set only;

(b) $\mathrm{ord}(r) = n$, where $n \in \omega$, if and only if the intersection of any $n+2$ distinct elements of r is empty and there exist $n+1$ distinct elements of r, whose intersection is not empty;

(c) $\mathrm{ord}(r) = \infty$, if and only if for every $n \in \omega$ there exist n distinct elements of r, whose intersection is not empty.

The given below definitions are actually the definitions of dimensions dim and dim* given in [7] (see also [2]) for regular spaces.

Definition 1.1. We denote by dim the (unique) function with as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$, satisfying the following condition:

$$\dim(Q, X) \leq n\,, \qquad \text{where } n \in \{-1\} \cup \omega\,,$$

if and only if for every finite open cover c of the space X there exists a finite family r_Q of open subsets of Q refinement of c which is a cover of Q and $\mathrm{ord}(r_Q) \leq n$.

Definition 1.2. We denote by $\dim^*$ the (unique) function with as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$, satisfying the following condition:

$$\dim^*(Q, X) \leq n\,, \qquad \text{where } n \in \{-1\} \cup \omega\,,$$

if and only if for every finite open cover c of the space X there exists a finite family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$.

In [3] and [4] we studied the above dimensions and we gave some properties including subspace, sum, partition, compactification, and product theorems. Also, we gave partial answers for the questions which are given in [7]. In this paper, we give and study a new relative covering dimension.

2. The new relative covering dimension

Definition 2.1. We denote by r-dim the (unique) function that has as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$ satisfying the following condition

$$\text{r-dim}(Q, X) \leq n\,, \qquad \text{where } n \in \{-1\} \cup \omega\,,$$

if and only if for every finite family c of open subsets of X such that

$$Q \subseteq \cup\{U : U \in c\}\,,$$

there exists a finite family r of open subsets of X refinement of c such that

$$Q \subseteq \cup\{V : V \in r\}$$

and $\text{ord}(r) \leq n$.

Remark. We observe that if $\text{r-dim}(Q, X) \leq n$, where $n \in \omega$, then for every finite family c of open subsets of X such that $Q \subseteq \cup\{U : U \in c\}$ there exists a finite family r_Q of open subsets of Q refinement of c which is a cover of Q and $\text{ord}(r_Q) \leq n$.

Proposition 2.2. *Let Q be a subset of a topological space X. The following statements are true:*

(a)

$$\dim(Q) \leq \text{r-dim}(Q, X)\,,$$

where $\dim(Q)$ is the covering dimension of the subset Q of X. Moreover, if the subset Q of X is open, then

$$\dim(Q) = \text{r-dim}(Q, X)\,.$$

(b) $\dim(Q, X) \leq \dim^*(Q, X) \leq \text{r-dim}(Q, X)$.

(c) *If the subset Q of X is closed, then*

$$\dim^*(Q, X) = \text{r-dim}(Q, X) \leq \dim(X)\,,$$

where $\dim(X)$ is the covering dimension of X.

Proof. (a) Let r-dim$(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $\dim(Q) \leq n$. Let $c_Q = \{U_1^Q, \ldots, U_m^Q\}$ be a finite open cover of the space Q. For every $i = 1, \ldots, m$ there exists an open subset U_i of X such that $U_i^Q = Q \cap U_i$. We consider the family $c = \{U_1, \ldots, U_m\}$. Then, the family c is a finite family of open subsets of X such that $Q \subseteq \cup_{i=1}^m U_i$. Since r-dim$(Q, X) = n$, there exists a finite family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$. We set $r_Q = \{Q \cap V : V \in r\}$. Then, the family r_Q is a finite open cover of Q refinement of c_Q such that $\mathrm{ord}(r_Q) \leq n$. Thus, $\dim(Q) \leq n$.

Now, we suppose that the subset Q of X is open. Clearly, it suffices to prove the inequality

$$\text{r-dim}(Q, X) \leq \dim(Q)\,. \tag{1}$$

Let $\dim(Q) = n \in \omega \cup \{-1, \infty\}$. The inequality (1) is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that r-dim$(Q, X) \leq n$. Let c be a finite family of open subsets of X such that $Q \subseteq \cup\{U : U \in c\}$. Then, the family $c_Q = \{Q \cap U : U \in c\}$ is a finite open cover of the space Q. Since $\dim(Q) = n$, there exists a finite open cover r_Q of Q refinement of c_Q such that $\mathrm{ord}(r_Q) \leq n$. Obviously, the family r_Q is a refinement of c. Also, since the subspace Q of X is open, every element of the family r_Q is open subset of X. Thus, r-dim$(Q, X) \leq n$.

(b) It is known that $\dim(Q, X) \leq \dim^*(Q, X)$ (see [7]). So it suffices to prove the inequality

$$\dim^*(Q, X) \leq \text{r-dim}(Q, X)\,. \tag{2}$$

Let r-dim$(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality (2) is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $\dim^*(Q, X) \leq n$. Let c be a finite open cover of the space X. Obviously, $Q \subseteq \cup\{U : U \in c\}$. Since r-dim$(Q, X) = n$ there exists a finite family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$. Thus, $\dim^*(Q, X) \leq n$.

(c) Suppose that the subset Q of X is closed. By (b) it suffices to prove the inequality

$$\text{r-dim}(Q, X) \leq \dim^*(Q, X)\,. \tag{3}$$

Let $\dim^*(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality (3) is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that r-dim$(Q, X) \leq n$. Let c be a finite family of open subsets of X such that $Q \subseteq \cup\{U : U \in c\}$. Since the subspace Q of X is closed, the family $c \cup \{X \setminus Q\}$ is a finite open cover of the space X. Also, since $\dim^*(Q, X) = n$, there exists a finite family r of open subsets of X refinement of $c \cup \{X \setminus Q\}$ such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$.

Then, the family $r' = r \setminus \{V \in r : V \subseteq X \setminus Q\}$ is a refinement of c such that $Q \subseteq \cup\{V : V \in r'\}$ and $\mathrm{ord}(r') \leq n$. Thus, $\text{r-dim}(Q, X) \leq n$.

Also, it is clear that $\text{r-dim}(Q, X) \leq \dim(X)$. ∎

Examples. (1) Let (X, τ) be a topological space, where $X = \{a, b, c, d\}$ and

$$\tau = \big\{\emptyset\,, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\big\}\,.$$

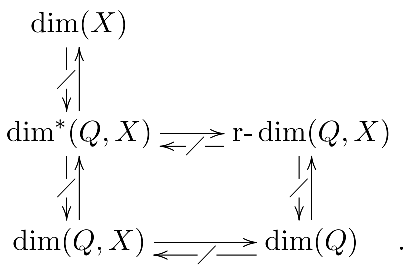
Let $Q = \{a, c\}$. We observe that $\text{r-dim}(Q, X) = 1$ and

$$\dim(Q, X) = \dim^*(Q, X) = \dim(X) = 0\,.$$

(2) Let X be the space of the real numbers and $Q = \{0\}$. Then, $\text{r-dim}(Q, X) = 0$ and $\dim(X) = 1$.

(3) Let $X = [-1, 1]$ and $Q = \{-1, 1\}$. The family consisting of all sets of the form $[-1, b)$ for $b > 0$, $(a, 1]$ for $a < 0$, and (a, b) is a basis for some topology in X. It is easy to see that $\dim(Q) = 0$ and $\text{r-dim}(Q, X) = 1$.

The relations between the dimension-like functions of the type dim are summarized in the following diagram, where "$\rightarrow$" means "$\leq$" and "$\not\rightarrow$" means that "in general $\not\leq$":



It is known that (see [3], [4] and [7]) there exist examples such that in the above diagram the invariants $\dim(X)$, $\dim(Q, X)$, $\dim^*(Q, X)$, and $\dim(Q)$ to be different.

Proposition 2.3. *For every subset Q of a space X the following conditions are equivalent:*

(1) $\text{r-dim}(Q, X) \leq n$.

(2) *For every finite family c of open subsets of X with $Q \subseteq \cup\{U : U \in c\}$ there exists a family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$.*

(3) *For every finite family* $\{U_1, U_2, \ldots, U_m\}$ *of open subsets of* X *with* $Q \subseteq \cup_{i=1}^{m} U_i$ *there exists a family* $\{V_1, V_2, \ldots, V_m\}$ *of open subsets of* X *such that*

$$V_i \subseteq U_i \quad \textit{for } i = 1, \ldots, m\,,$$
$$Q \subseteq \cup_{i=1}^{m} V_i \qquad \textit{and} \qquad \mathrm{ord}(\{V_1, V_2, \ldots, V_m\}) \leq n\,.$$

(4) *For every family* $\{U_1, U_2, \ldots, U_{n+2}\}$ *of open subsets of* X *with* $Q \subseteq \cup_{i=1}^{n+2} U_i$ *there exists a family* $\{V_1, V_2, \ldots, V_{n+2}\}$ *of open subsets of* X *such that*

$$V_i \subseteq U_i \quad \textit{for } i = 1, \ldots, n+2\,, \qquad Q \subseteq \cup_{i=1}^{n+2} V_i \qquad \textit{and} \qquad \cap_{i=1}^{n+2} V_i = \emptyset\,.$$

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Let $c = \{U_1, \ldots, U_m\}$ be a finite family of open subsets of X such that $Q \subseteq \cup_{i=1}^{m} U_i$. By assumption there exists a family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$. For every $V \in r$ we choose an element $l(V)$ of $\{1, \ldots, m\}$ such that $V \subseteq U_{l(V)}$ and we set

$$V_i = \cup\{V \in r : l(V) = i\}\,, \qquad i = 1, \ldots, m\,.$$

It is clear that $\{V_1, \ldots, V_m\}$ is a family of open subsets of X such that $V_i \subseteq U_i$ for $i = 1, \ldots, m$, $Q \subseteq \cup_{i=1}^{m} V_i$, and $\mathrm{ord}(\{V_1, \ldots, V_m\}) \leq n$.

(3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (1) Let $c = \{U_1, \ldots, U_m\}$ be a finite family of open subsets of X such that $Q \subseteq \cup_{i=1}^{m} U_i$. We prove that there exists a family $\{V_1, \ldots, V_m\}$ of open subsets of X such that

$$V_i \subseteq U_i \quad \text{for } i = 1, \ldots, m\,, \qquad Q \subseteq \cup_{i=1}^{m} V_i \qquad \text{and} \qquad \mathrm{ord}(\{V_1, \ldots, V_m\}) \leq n\,.$$

If $m \leq n+1$, then the required family $\{V_1, \ldots, V_m\}$ of open subsets of X is the family $\{U_1, \ldots, U_m\}$. Let us suppose that $m \geq n+2$. We consider the family $g = \{G_1, \ldots, G_{n+2}\}$, where $G_i = U_i$ for $i = 1, \ldots, n+1$ and $G_{n+2} = \cup_{i=n+2}^{m} U_i$. Obviously, $Q \subseteq \cup_{i=1}^{n+2} G_i$. Therefore, by assumption there exists a family $\{H_1, \ldots, H_{n+2}\}$ of open subsets of X such that $H_i \subseteq G_i$ for $i = 1, \ldots, n+2$, $Q \subseteq \cup_{i=1}^{n+2} H_i$, and $\cap_{i=1}^{n+2} H_i = \emptyset$. We consider the family $w = \{W_1, \ldots, W_m\}$, where $W_i = H_i$ for $i = 1, \ldots, n+1$ and $W_i = U_i \cap H_{n+2}$ for $i = n+2, \ldots, m$. It is clear that w is a family of open subsets of X such that

$$W_i \subseteq U_i \quad \text{for } i = 1, \ldots, m\,, \qquad Q \subseteq \cup_{i=1}^{m} W_i \qquad \text{and} \qquad \cap_{i=1}^{n+2} W_i = \emptyset\,.$$

If the intersection of any $n+2$ distinct elements of w is empty, then the required family $\{V_1,\dots,V_m\}$ of open subsets of X is the family w. We suppose that there exists a subset $A=\{i_1,\dots,i_{n+2}\}$ of $\{1,\dots,m\}$ such that $\cap_{i\in A}W_i\neq\emptyset$ and let $p=\{P_1,\dots,P_m\}$ be the same family w reordered so that

$$P_1=W_{i_1},\dots,P_{n+2}=W_{i_{n+2}}\,.$$

Since $p=w$ and $\cap_{i=1}^{n+2}W_i=\emptyset$, there exists a subset $B_1\neq\{1,\dots,n+2\}$ of $\{1,\dots,m\}$ with $n+2$ elements such that $\cap_{i\in B_1}P_i=\emptyset$. Applying the above construction to p we find a family $w'=\{W'_1,\dots,W'_m\}$ of open subsets of X such that

$$W'_i\subseteq P_i \quad \text{for}\ \ i=1,\dots,m\,,\qquad Q\subseteq\cup_{i=1}^m W'_i \qquad \text{and} \qquad \cap_{i=1}^{n+2}W'_i=\emptyset\,.$$

We observe that

$$W'_1\subseteq W_{i_1},\dots,W'_{n+2}\subseteq W_{i_{n+2}}\,.$$

If the intersection of any $n+2$ distinct elements of w' is empty, then the required family $\{V_1,\dots,V_m\}$ of open subsets of X is the family w'. We suppose that there exists a subset $A'=\{i'_1,\dots,i'_{n+2}\}$ of $\{1,\dots,m\}$ such that $\cap_{i\in A'}W'_i\neq\emptyset$ and let $p'=\{P'_1,\dots,P'_m\}$ be the same family w' reordered so that

$$P'_1=W'_{i'_1},\dots,P'_{n+2}=W'_{i'_{n+2}}.$$

Since $p'=w'$ and $\cap_{i=1}^{n+2}W'_i=\emptyset$, there exists a subset $B_2\neq\{1,\dots,n+2\}$ of $\{1,\dots,m\}$ with $n+2$ elements such that $\cap_{i\in B_2}P'_i=\emptyset$. Applying the above construction to p' we find a family $w''=\{W''_1,\dots,W''_m\}$ of open subsets of X such that

$$W''_i\subseteq P'_i \quad \text{for}\ \ i=1,\dots,m\,,\qquad Q\subseteq\cup_{i=1}^m W''_i \qquad \text{and} \qquad \cap_{i=1}^{n+2}W''_i=\emptyset\,.$$

We observe that

$$W''_1\subseteq W'_{i'_1},\dots,W''_{n+2}\subseteq W'_{i'_{n+2}}\,.$$

Since the family $\{U_1,\dots,U_m\}$ is finite, after a finite number of repetitions of the above process we find a family $\{V_1,\dots,V_m\}$ of open subsets of X such that

$$V_i\subseteq U_i \quad \text{for}\ \ i=1,\dots,m\,,\quad Q\subseteq\cup_{i=1}^m V_i \quad \text{and} \quad \mathrm{ord}(\{V_1,\dots,V_m\})\leq n\,.$$

Thus, r-dim$(Q,X)\leq n$. ▮

3. Subspace theorems

In this section we give subspace theorems for the dimension r-dim.

Proposition 3.1. *Let K and Q be two subspaces of a space X with $K \subseteq Q$. If K is a closed subspace of X or $Q \setminus K$ is an open subspace of X, then*

$$\text{r-dim}(K, X) \leq \text{r-dim}(Q, X)\,.$$

Proof. Suppose that the subset $Q \setminus K$ of X is open. Let

$$\text{r-dim}(Q, X) = n \in \omega \cup \{-1, \infty\}\,.$$

The inequality is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that r-dim$(K, X) \leq n$. Let c be a finite family of open subsets of X such that $K \subseteq \cup\{U : U \in c\}$. Since the subspace $Q \setminus K$ of X is open, the family $c \cup \{Q \setminus K\}$ consists of open subsets of X such that

$$Q \subseteq \cup\{U : U \in c\} \cup \{Q \setminus K\}\,.$$

Also, since r-dim$(Q, X) = n$, there exists a finite family r of open subsets of X refinement of $c \cup \{Q \setminus K\}$ such that $Q \subseteq \cup\{V : V \in r\}$ and ord$(r) \leq n$. Then, the family

$$r' = r \setminus \{V \in r : V \subseteq Q \setminus K\}$$

is a refinement of c such that $K \subseteq \cup\{V : V \in r'\}$ and ord$(r') \leq n$. Thus, r-dim$(K, X) \leq n$. ∎

Proposition 3.2. *Let Y be a subspace of a space X and $Q \subseteq Y$. Then,*

$$\text{r-dim}(Q, Y) \leq \text{r-dim}(Q, X)\,.$$

Proof. Let r-dim$(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that r-dim$(Q, Y) \leq n$. Let $c_Y = \{U_1^Y, \ldots, U_m^Y\}$ be a finite family of open subsets of Y such that $Q \subseteq \cup_{i=1}^m U_i^Y$. For every $i = 1, \ldots, m$, there exists an open subset U_i of X such that $U_i^Y = Y \cap U_i$. We set $c = \{U_1, \ldots, U_m\}$. The family c is a finite family of open subsets of X such that $Q \subseteq \cup_{i=1}^m U_i$. Since r-dim$(Q, X) = n$, there exists a finite family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and ord$(r) \leq n$. We consider the family

$$r_Y = \left\{V^Y \equiv Y \cap V : V \in r\right\}.$$

Since $Q \subseteq Y$, the family r_Y is a finite family of open subsets of Y refinement of c_Y such that

$$Q \subseteq \cup\{V^Y : V^Y \in r_Y\}.$$

Also, since the family r_Y is refinement of the family r and $\text{ord}(r) \leq n$, we have that $\text{ord}(r_Y) \leq n$. Thus, $\text{r-dim}(Q, Y) \leq n$. ∎

4. Sum theorems

In this section we give sum theorems for the dimension r-dim.

PROPOSITION 4.1. *Let Q be a subspace of a space X. If $X = X_1 \cup X_2$, where $Q \subseteq X_1 \cap X_2$, $\text{r-dim}(Q, X_1) \leq n$, and $\text{r-dim}(Q, X_2) \leq n$, then $\text{r-dim}(Q, X) \leq n$.*

Proof. Let $c = \{U_1, \ldots, U_m\}$ be a finite family of open subsets of X with $Q \subseteq \cup_{i=1}^m U_i$. By Proposition 2.3 (3) it suffices to prove that there exists a finite family s of open subsets of X shrinking of c such that $Q \subseteq \cup\{V : V \in s\}$ and $\text{ord}(s) \leq n$. Since the family $\{X_1 \cap U_1, \ldots, X_1 \cap U_m\}$ is a finite family of open subsets of X_1 with $Q \subseteq \cup_{i=1}^m (X_1 \cap U_i)$ and $\text{r-dim}(Q, X_1) \leq n$, by Proposition 2.3 (3) there exists a family $\{V_1^1, \ldots, V_m^1\}$ of open subsets of X_1 such that $V_i^1 \subseteq X_1 \cap U_i$ for $i = 1, \ldots, m$, $Q \subseteq \cup_{i=1}^m V_i^1$, and $\text{ord}(\{V_1^1, \ldots, V_m^1\}) \leq n$. For $i = 1, \ldots, m$ there exists an open subset V_i of X such that $V_i^1 = X_1 \cap V_i$. We set

$$W_i = U_i \cap V_i, \qquad i = 1, \ldots, m.$$

Obviously, we have $W_i \subseteq U_i$ for $i = 1, \ldots, m$ and $Q \subseteq \cup_{i=1}^m W_i$. Moreover, since $X_1 \cap W_i = U_i \cap V_i^1 \subseteq V_i^1$ for $i = 1, \ldots, m$ and $\text{ord}(\{V_1^1, \ldots, V_m^1\}) \leq n$, we have

$$\text{ord}(\{X_1 \cap W_1, \ldots, X_1 \cap W_m\}) \leq n. \tag{4}$$

The family $\{X_2 \cap W_1, \ldots, X_2 \cap W_m\}$ is a finite family of open subsets of X_2 with $Q \subseteq \cup_{i=1}^m (X_2 \cap W_i)$. Also, since $\text{r-dim}(Q, X_2) \leq n$, by Proposition 2.3 (3) there exists a family $\{V_1^2, \ldots, V_m^2\}$ of open subsets of X_2 such that $V_i^2 \subseteq X_2 \cap W_i$ for $i = 1, \ldots, m$, $Q \subseteq \cup_{i=1}^{m+1} V_i^2$, and $\text{ord}(\{V_1^2, \ldots, V_m^2\}) \leq n$. For $i = 1, \ldots, m$ there exists an open subset V_i' of X such that $V_i^2 = X_2 \cap V_i'$. We consider the family $s = \{H_1, \ldots, H_m\}$, where $H_i = W_i \cap V_i'$, $i = 1, \ldots, m$. Obviously, we have $H_i \subseteq W_i \subseteq U_i$ for $i = 1, \ldots, m$ and $Q \subseteq \cup_{i=1}^m H_i$. Moreover, since $X_2 \cap H_i = W_i \cap V_i^2 \subseteq V_i^2$ for $i = 1, \ldots, m$ and $\text{ord}(\{V_1^2, \ldots, V_m^2\}) \leq n$, we have

$$\text{ord}(\{X_2 \cap H_1, \ldots, X_2 \cap H_m\}) \leq n. \tag{5}$$

We prove that $\mathrm{ord}(s) \leq n$. Let $H_{i_1}, \ldots, H_{i_{n+2}}$ be pairwise distinct elements of s, and $x \in H_{i_1} \cap \ldots \cap H_{i_{n+2}} \neq \emptyset$. Since $X = X_1 \cup X_2$, $x \in X_1$ or $x \in X_2$. If $x \in X_1$, then

$$x \in (X_1 \cap H_{i_1}) \cap \ldots \cap (X_1 \cap H_{i_{n+2}}) \subseteq (X_1 \cap W_{i_1}) \cap \ldots \cap (X_1 \cap W_{i_{n+2}}),$$

which contradicts the relation (4). If $x \in X_2$, then

$$x \in (X_2 \cap H_{i_1}) \cap \ldots \cap (X_2 \cap H_{i_{n+2}}),$$

which contradicts the relation (5). Thus, $\mathrm{ord}(s) \leq n$ and, therefore, $\text{r-dim}(Q, X) \leq n$. ∎

Corollary 4.2. *Let Q be a subspace of a space X. For every subset A of X such that $Q \subseteq A$ we have*

$$\text{r-dim}(Q, X) \leq \max\left\{\text{r-dim}(Q, A), \text{r-dim}(Q, (X \setminus A) \cup Q)\right\}.$$

Proof. Follows by Proposition 4.1 for $X_1 = A$ and $X_2 = (X \setminus A) \cup Q$. ∎

Corollary 4.3. *Let Q be a subspace of a space X. If $X = X_1 \cup X_2$, where $Q \subseteq X_1 \cap X_2$, then*

$$\text{r-dim}(Q, X) = \max\left\{\text{r-dim}(Q, X_1), \text{r-dim}(Q, X_2)\right\}.$$

Proof. Let $\text{r-dim}(Q, X_1) = n_1$ and $\text{r-dim}(Q, X_2) = n_2$, where $n_1, n_2 \in \omega \cup \{\infty\}$. We set $n = \max\{n_1, n_2\}$. Then, $\text{r-dim}(Q, X_1) \leq n$ and $\text{r-dim}(Q, X_2) \leq n$. By Proposition 4.1 we have $\text{r-dim}(Q, X) \leq n$. Also, by Proposition 3.2, $n_1 \leq \text{r-dim}(Q, X)$ and $n_2 \leq \text{r-dim}(Q, X)$. Thus, $n \leq \text{r-dim}(Q, X)$. By the above, $\text{r-dim}(Q, X) = n$. ∎

Proposition 4.4. *Let Q_1 and Q_2 be two subsets of a space X. Then,*

$$\text{r-dim}(Q_1 \cup Q_2, X) \leq \text{r-dim}(Q_1, X) + \text{r-dim}(Q_2, X) + 1.$$

Proof. Let

$$\text{r-dim}(Q_1, X) = n_1 \qquad \text{and} \qquad \text{r-dim}(Q_2, X) = n_2.$$

We prove that

$$\text{r-dim}(Q_1 \cup Q_2, X) \leq n_1 + n_2 + 1.$$

Let c be a finite family of open subsets of X with $Q_1 \cup Q_2 \subseteq \cup\{U : U \in c\}$. Since r-dim$(Q_1, X) = n_1$, there exists a finite family r_1 of open subsets of X refinement of c such that $Q_1 \subseteq \cup\{U : U \in r_1\}$ and $\mathrm{ord}(r_1) \leq n_1$. Moreover, since r-dim$(Q_2, X) = n_2$, there exists a finite family r_2 of open subsets of X refinement of c such that $Q_2 \subseteq \cup\{U : U \in r_2\}$ and $\mathrm{ord}(r_2) \leq n_2$. We set $r = r_1 \cup r_2$. Then, r is a family of open subsets of X refinement of c such that

$$Q_1 \cup Q_2 \subseteq \cup\{U : U \in r\} \qquad \text{and} \qquad \mathrm{ord}(r) \leq n_1 + n_2 + 1\,.$$

∎

5. Partition and product theorems

In this section we give partition, product, and compactification theorems for the dimension r-dim.

Proposition 5.1. *Let Q be a normal subspace of a space X. If for every family $\{(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})\}$ of $n+1$ pairs of disjoint subsets of X, where A_i's are closed in X and B_i's are closed in Q, there exist partitions L_i between A_i and B_i such that $Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset$, then* r-dim$(Q, X) \leq n$.

Proof. By Proposition 2.3 (4) it suffices to show that for any family $\{U_1, \ldots, U_{n+2}\}$ of open subsets of X with $Q \subseteq \cup_{i=1}^{n+2} U_i$ there exists a family $\{V_1, \ldots, V_{n+2}\}$ of open subsets of X such that

$$V_i \subseteq U_i \quad \text{for } i = 1, \ldots, n+2\,, \qquad Q \subseteq \cup_{i=1}^{n+2} V_i \qquad \text{and} \qquad \cap_{i=1}^{n+2} V_i = \emptyset\,.$$

Let $\{U_1, \ldots, U_{n+2}\}$ be a family of open subsets of X with $Q \subseteq \cup_{i=1}^{n+2} U_i$. Since the space Q is normal, there exists a closed cover $\{B_1, \ldots, B_{n+2}\}$ of Q such that $B_i \subseteq U_i \cap Q$ for $i = 1, \ldots, n+2$. We set

$$A_i = X \setminus U_i \qquad \text{for } i = 1, \ldots, n+1\,.$$

The family $\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$ consists of $n+1$ pairs of disjoint subsets of X, where A_i's are closed in X and B_i's are closed in Q. Therefore by hypothesis there exist partitions L_i between A_i and B_i such that $Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset$. That is, there exist open subsets W_i, V_i of X such that:

$$A_i \subseteq W_i\,,\ B_i \subseteq V_i\,, \tag{6}$$

$$W_i \cap V_i = \emptyset\,, \tag{7}$$

$$X \setminus L_i = W_i \cup V_i \qquad \text{for } i = 1, \ldots, n+1\,. \tag{8}$$

We set $V_{n+2} = U_{n+2} \cap \cup_{i=1}^{n+1} W_i$. By the definition of A_i's and (6), (7) we have that $V_i \subseteq U_i$ for $i = 1, 2, \ldots, n+2$. We prove that $Q \subseteq \cup_{i=1}^{n+2} V_i$. We observe that

$$\bigcup_{i=1}^{n+1} W_i \cup \bigcup_{i=1}^{n+1} V_i = \bigcup_{i=1}^{n+1} (W_i \cup V_i) = \bigcup_{i=1}^{n+1} (X \setminus L_i) = X \setminus \bigcap_{i=1}^{n+1} L_i \supseteq Q\,. \tag{9}$$

From (6), (9) and the relation $B_{n+2} \subseteq U_{n+2}$ it follows that

$$\begin{aligned}\bigcup_{i=1}^{n+2} V_i &= \bigcup_{i=1}^{n+1} V_i \cup \left(U_{n+2} \cap \bigcup_{i=1}^{n+1} W_i \right) \\ &= \left(\bigcup_{i=1}^{n+1} V_i \cup U_{n+2} \right) \cap \left(\bigcup_{i=1}^{n+1} V_i \cup \bigcup_{i=1}^{n+1} W_i \right) \\ &\supseteq \bigcup_{i=1}^{n+2} B_i \cap Q \;=\; Q \cap Q \;=\; Q\,.\end{aligned}$$

We prove that $\cap_{i=1}^{n+2} V_i = \emptyset$. From (7) we have

$$\bigcap_{i=1}^{n+2} V_i = \bigcap_{i=1}^{n+1} V_i \cap \left(U_{n+2} \cap \bigcup_{i=1}^{n+1} W_i \right) \subseteq \bigcap_{i=1}^{n+1} V_i \cap \bigcup_{i=1}^{n+1} W_i = \emptyset\,.$$

∎

Remark. It was proved (see Proposition 2.2) that if the subset Q of X is closed, then

$$\dim^*(Q, X) = \text{r-dim}(Q, X)\,.$$

So, by Proposition 2.4, Proposition 3.1, Corollary 3.2, Proposition 3.3, Corollary 3.4, and Proposition 4.2 of [4] we have the following propositions and product theorem for the dimension invariant r-dim.

PROPOSITION 5.2. *Let Q be a closed subspace of a normal space X satisfying* r-dim$(Q, X) \leq n$. *Then, for every family* $\{(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})\}$ *of $n+1$ pairs of disjoint closed subsets of X there exist partitions L_i between A_i and B_i such that $Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset$.*

PROPOSITION 5.3. *For every closed subspace Q of a normal space X we have*

$$\text{r-dim}(Q, X) = \text{r-dim}(Q, \beta X) = \text{r-dim}(\beta Q, \beta X)\,.$$

Proposition 5.4. *Let Q^X be a closed subspace of a compact Hausdorff space X and Q^Y a closed subspace of a compact Hausdorff space Y. Then,*

$$\text{r-dim}\big(Q^X \times Q^Y, X \times Y\big) \leq \text{r-dim}\big(Q^X, X\big) + \text{r-dim}\big(Q^Y, Y\big).$$

6. Questions

Question 1. Is it true the property of universality for dimension r-dim? That is, does there exists a universal element in the class $\mathbb{P}$ of all pairs $\big(Q^X, X\big)$, where Q^X is a subset of a space X such that $\text{r-dim}\big(Q^X, X\big) \leq n$?

Question 2. Is it true the product theorem for r-dim in the realm of all metrizable spaces?

For some other questions on relative covering dimensions see [3] and [4].

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References

[1] R. Engelking, "Theory of Dimensions, Finite and Infinite", Sigma Series in Pure Mathematics, 10, Heldermann Verlag, Lemgo, 1995.

[2] K.P. Hart, Jun-iti Nagata, J.E. Vaughan, "Encyclopedia of General Topology", Elsevier Science Publishers, B.V., Amsterdam, 2004.

[3] D.N. Georgiou, A.C. Megaritis, On the relative dimensions dim and dim* I, *Questions Answers Gen. Topology* (article in press).

[4] D.N. Georgiou, A.C. Megaritis, On the relative dimensions dim and dim* II, *Questions Answers Gen. Topology* (article in press).

[5] Jun-iti Nagata, "Modern Dimension Theory", Sigma Series in Pure Mathematics, 2, Heldermann Verlag, Berlin, 1983.

[6] A.R. Pears, "Dimension Theory of General Spaces", Cambridge University Press, Cambridge, England-New York-Melbourne, 1975.

[7] J. Valuyeva, On relative dimension concepts, *Questions Answers Gen. Topology* **15** (1997), 21–24.