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Derivations of Locally Convex $*$-Algebras

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Abstract: This is a survey account on some old and recent results on derivations from a locally convex ($*$-) algebra $A[\tau]$ in a locally convex A-bimodule. Essential differences with known results in the normed setting will be pointed out and natural questions will be put.

Key words: Locally convex algebra, pro-C^*-algebra, GB^*-algebra, locally convex bimodule, derivation, inner derivation.

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This paper is dedicated to the memory of Susanne Dierolf

1. Introduction

Soon after the theory of C^*-algebras was founded by Gel'fand and Naimark in 1943, the field of derivations came to light, and after that, their continuous investigation gave rise to one of the central branches in the theory of Banach ($*$-) algebras. If A is an algebra and X an A-bimodule, a linear map $\delta : A \to X$ is called a *derivation* if it fulfills the Leibnitz formula, i.e. $\delta(ab) = \delta(a)b + a\delta(b)$, for every $a, b \in A$. If there exists an $x \in X$ with $\delta = \delta_x$, where $\delta_x(a) = ax - xa$, for all $a \in A$, then δ is called an *inner derivation*. If $X = A$, we shall speak of a derivation of A. A $*$-*derivation* of a $*$-algebra A is a $*$-preserving derivation. The motivation for the study of derivations comes mainly from the following facts:

(i) Their theory is closely related to the cohomology groups $H^n(A, X)$ of the (topological) algebra A and the (topological) A-module X (see, for example, [21, 28, 29]). In particular, the first cohomology group $H^1(A, X)$ measures how much the space of all (continuous) derivations of A in X differs from the space of all (continuous) inner derivations of A in X. In particular, there are important applications of such results to cohomology theory concerning contractibility and amenability, but also cohomology of higher dimensions (see, for instance, [29, Theorem VII.3.37, Corollary VII.3.38] and [34, p. 9]).

(ii) Derivations of an algebra A with identity give rise to automorphisms of A. The set $\mathrm{Aut}(A)$ of automorphisms of A is a subgroup of the group of all invertible linear operators from A onto A. Therefore, one could say that $\mathrm{Aut}(A)$ shows how rich A is in basic symmetries. Moreover, such automorphisms are essentially used for the proof of the famous Singer-Wermer theorem, which in fact, tells us that among Banach algebras, only the non-commutative ones are rich in derivations (for more details, see [16, Paragraph 18], [29, Section VII, Paragraph 2] and Section 3 of this note).

(iii) The connection of derivations with physics. Derivations are divided into two parts: the bounded and the unbounded ones. The study of bounded derivations started from the mid 1940's and continues up to the present day. As a result a plethora of excellent results have been obtained, which can offer important tools to the investigation of unbounded derivations. The motivation for the study of unbounded derivations was given by the problem of constructing the dynamics in statistical mechanics. Moreover, according to S. Sakai the necessity of studying unbounded derivations came to surface from some observations of Kaplansky [36], in 1958, on two apparently non related papers, one by Shilov [50], in 1947, having to do with differentiation, and another one by Wielandt [57], in 1949, related to quantum mechanics. About all these, the reader is referred to the seminal treatise of S. Sakai [48] "Operator Algebras in Dynamical Systems", where a rich literature on derivations can also be found, together with a very informative preface, interesting comments and historical remarks in all sections. Historical remarks are also contained in the introduction and in the main body of the book of O. Bratteli [18], where the names of the main initiators and contributors to the theory of unbounded derivations are mentioned.

Another connection between derivations and quantum physics is derived by (ii) above, since the symmetries of a quantum system are described by an automorphism group of the algebra involved, which has its self-adjoint elements as the observables of the quantum system.

2. Preliminaries

The algebras we deal with are complex and all topological spaces are assumed to be Hausdorff.

A *topological algebra* is a topological vector space, which is also an algebra such that the multiplication is separately continuous [40]. A topological algebra whose underlying topological vector space is locally convex is called a *locally convex algebra*. If a topological algebra is endowed with an involution $*$, this will always be assumed to be continuous, and the corresponding algebra will be called a *topological $*$-algebra*. The symbol $A[\tau]$ will denote a topological ($*$-) algebra, whose given topology is τ. If $A[\tau]$ is a locally convex ($*$-) algebra, then τ will be induced by an upwards directed defining family of ($*$-) seminorms denoted by (p_λ), $\lambda \in \Lambda$. If $A[\tau]$ is a locally convex ($*$-) algebra, where each p_λ, $\lambda \in \Lambda$, is submultiplicative, i.e. $p_\lambda(ab) \leq p_\lambda(a)p_\lambda(b)$, for all $a, b \in A$, then $A[\tau]$ is called an *m-convex* ($*$-) *algebra*. If $A[\tau]$ is metrizable, the corresponding family of ($*$-) seminorms can be chosen to be countable, and it will be denoted by (p_n), $n \in \mathbb{N}$. A complete metrizable topological ($*$-) algebra will be called a *Fréchet topological* ($*$-) *algebra*.

Furthermore, a *pro-C^*-algebra* is a complete topological $*$-algebra $A[\tau]$, whose topology τ is defined by an upwards directed family (p_λ), $\lambda \in \Lambda$, of C^*-seminorms, i.e. each seminorm p_λ fulfils the C^*-property $p_\lambda(x^*x) = p_\lambda(x)^2$, for every $x \in A$. A metrizable pro-C^*-algebra is called a *σ-C^*-algebra* [43, 24]. Every pro-C^*-algebra is represented as an inverse limit of C^*-algebras, denoted by A_λ, $\lambda \in \Lambda$. Each A_λ is, in fact, the quotient A/N_λ, where $N_\lambda = \{a \in A : p_\lambda(a) = 0\}$, $\lambda \in \Lambda$, endowed with the induced C^*-norm $\|\cdot\|_\lambda$, i.e.

$$\|a_\lambda\|_\lambda = p_\lambda(a), \ \forall \ a_\lambda := a + N_\lambda \in A/N_\lambda, \ \lambda \in \Lambda.$$

The quotient A/N_λ is always complete under $\|\cdot\|_\lambda$, $\lambda \in \Lambda$ [24, Subsection 10.2]. Let $A[\tau]$ be a pro-C^*-algebra. Put

$$A_b := \Big\{a \in A : \sup_\lambda p_\lambda(a) < \infty\Big\}.$$

A_b is a C^*-algebra under the C^*-norm $\|a\|_b := \sup_\lambda p_\lambda(a)$, $a \in A_b$; it is dense in $A[\tau]$ and is called *the bounded part* of $A[\tau]$ [24, Theorem 10.23].

Let $A[\tau]$ be a pro-C^*-algebra and $X[\tau']$ a complete locally convex space, which is also an A-bimodule with separately continuous module actions. Then we call $X[\tau']$ a *complete locally convex A-bimodule*. Suppose that τ' is derived by a family $(q_\nu)_{\nu \in \Sigma}$ of seminorms. When the index set Σ is the same as that of the family $(p_\lambda)_{\lambda \in \Lambda}$ defining τ, and the continuity of the module actions of X is described by the inequalities

$$q_\lambda(ax) \leq p_\lambda(a)q_\lambda(x), \ \text{resp.} \ q_\lambda(xa) \leq p_\lambda(a)q_\lambda(x),$$

for all $a \in A$, $x \in X$ and $\lambda \in \Lambda$, then the complete locally convex A-bimodule $X[\tau']$ is called *smooth* [44, p. 75]. Take, for instance, a complete locally convex space E and $X = C_c(\mathbb{R}, E)$, the complete locally convex space of all continuous E-valued-functions on the real line under the compact open topology c. Then X is a smooth $C_c(\mathbb{R})(= C_c(\mathbb{R}, \mathbb{C}))$-bimodule. However, the Arens-algebra $L^\omega[0,1] := \bigcap_{1\leq p<\infty} L^p[0,1]$, endowed with the topology of L^p-norms $\|\cdot\|_p$ [40, p. 12], is not a smooth bimodule over itself. For many more examples and counter-examples of smooth modules, see [44, Subsections 8.1 and 8.2].

Let now $A[\tau]$ be a pro-C^*-algebra and δ a derivation of $A[\tau]$. Then, δ is called *approximately inner* [11] if there is a net $(x_\nu)_{\nu\in\Sigma}$ in A such that

$$\delta(a) = \lim_\nu \delta_{x_\nu}(a), \ \forall\ a \in A.$$

(For the notation δ_{x_ν}, see Introduction).

Finally, let $X[\tau], Y[\tau']$ be Fréchet topological vector spaces and $\phi : X[\tau] \to Y[\tau']$ a linear map. Put

$$S(\phi) = \Big\{y \in Y : \exists \text{ a sequence } (x_n) \text{ in } X \text{ with } x_n \underset{\tau}{\to} 0 \text{ and } \phi(x_n) \underset{\tau'}{\to} y\Big\}.$$

Then $S(\phi)$ is called the *separating space* of ϕ (see, for example, [51, p. 3]). By the closed graph theorem for Fréchet topological vector spaces, it is clear that ϕ is continuous if and only if $S(\phi) = \{0\}$.

3. Derivations of Fréchet topological algebras

In this section, we discuss the Singer-Wermer theorem for Banach algebras and its important consequences in the non-normed setting.

Let A be a Banach algebra with identity e. Let a be an arbitrary element in A. Then the series $\sum_{n=0}^{\infty} \frac{a^n}{n!}$, with $a^0 = e$, converges absolutely in A, so that one may consider the exponential map

$$\begin{aligned} \exp : A &\longrightarrow A \\ a &\mapsto \exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!}\,. \end{aligned}$$

For brevity, we shall write e^a instead of $\exp(a)$, $a \in A$. This map has similar properties with the usual exponential map on the real line [16, Paragraph 8]. Suppose now that δ is a bounded derivation of A and let $\mathcal{B}(A)$ be the Banach algebra of all bounded linear operators on A. Then $e^\delta \in \mathcal{B}(A)$ and e^δ is a topological automorphism of A [29, Theorem VII.3.2], its continuous inverse being $e^{-\delta}$. So *each continuous derivation of a Banach algebra A gives rise* *to a topological automorphism of A.* This important result, combined with Gel'fand theory, leads, through Liouville's theorem, to the following famous result (see [16, p. 92, Theorem 16] and/or [29, Theorem VII.2.10]).

THEOREM 3.1. (Singer-Wermer) *Let A be a commutative Banach algebra and δ a continuous derivation of A. Then the image of δ is contained in the (Jacobson) radical of A.*

Note that not every derivation of a commutative Banach algebra is continuous. Take, for instance, an arbitrary Banach space endowed with the trivial multiplication. Then A is a commutative Banach algebra and every linear operator on A is clearly a derivation. But, of course, not every operator on a Banach space is continuous. In 1968, Johnson and Sinclair proved in [35] that *every derivation of a semi-simple Banach algebra is continuous.* This result, combined with the Singer-Wermer theorem, leads to the following

COROLLARY 3.2. ([33]) *The only derivation of a commutative semi-simple Banach algebra is the zero derivation.*

Corollary 3.2 has very interesting consequences in the non-normed setting. These, together with some further information, are included in the remarks that follow.

Remark 3.3. (1) M. P. Thomas showed in 1988 (see [52]) that the continuity of a derivation in the Singer-Wermer theorem is redundant. Therefore, *the image of any derivation of a commutative Banach algebra lies in its radical.*

(2) An immediate consequence of Corollary 3.2 is that several commutative semi-simple, complete m-convex algebras, initiated from distribution theory, are not Banach under their usual topologies. The reason is that these algebras admit non-trivial derivations. Namely, some of these algebras are: (i) The Fréchet m-convex algebra $C^\infty[0,1]$ of all smooth functions on $[0,1]$. (ii) The Fréchet m-convex algebra $S(\mathbb{R})$ of all rapidly decreasing smooth functions on $\mathbb{R}$. (iii) The complete m-convex algebra $\mathcal{D}(\mathbb{R})$ of all smooth functions on $\mathbb{R}$ with compact support. (iv) The Fréchet m-convex algebra $\mathcal{O}(\mathbb{C})$ of all entire functions on the complex plane. For more details, see [24, Theorem 4.26 and its proof].

(3) In 1971, R. L. Carpenter [20] extended the result of Johnson-Sinclair (mentioned after Theorem 3.1) to the class of Fréchet topological algebras. More precisely, he proved that *every derivation of a commutative, semi-simple Fréchet m-convex algebra $A[\tau]$, with an identity element, is continuous*, see

also [26, Subsection 8.2]. The main tools used for the proof of Carpenter's result is Shilov's idempotent theorem for commutative Fréchet m-convex algebras with identity (ibid., p. 132, (6.1.5)), and a result of B. E. Johnson in [33], according to which: *If $A[\tau]$ is a commutative Fréchet m-convex algebra with an identity element, δ a derivation of A and $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of distinct points in the spectrum $\mathcal{M}(A)$ of A* (continuous non-zero complex multiplicative linear functionals of A) *such that the functionals $\varphi_n \circ \delta : A \to \mathbb{C}$ are discontinuous for all $n \in \mathbb{N}$, then there exists $a \in A$ such that the Gel'fand transform of $\delta(a)$, is unbounded on $\{(\varphi_n)_{n\in\mathbb{N}}\}$* [26, Proposition (8.2.2)]. It is now clear from (2) and Carpenter's theorem that *the Singer-Wermer theorem cannot be true for commutative Fréchet algebras, in general.* Hence, one naturally is led to the following

Question. For which class of locally convex algebras is the Singer-Wermer theorem true?

Here are some comments for this: In 1992, R. Becker proved that *every derivation of a commutative pro-C^*-algebra is identically zero* [11, Proposition 2 and Corollary 3]. Therefore, the Singer-Wermer theorem is true for commutative pro-C^*-algebras. Allan's GB^*-algebras (GB^* is an abbreviation of "generalized B^*") (see [3, 22], and for some recent survey, [25]), a generalization of C^*-algebras, include pro-C^*-algebras. So *it would be interesting if we could have some information for the Singer-Wermer theorem for commutative GB^*-algebras.*

(4) Concerning the result of Carpenter mentioned in (3), we note that it is not expected that every derivation of a Fréchet topological algebra is continuous. In 2002, C. J. Read [45] proved that the algebra A of all formal power series in countably many variables $x_0, x_1, \ldots$ that are not commuting, equipped with a topology different from the usual one of convergence in coefficients, is a Fréchet m-convex algebra, whose usual derivations are not continuous in the first variable. In fact, what happens is that the separating space $S(\delta)$ of δ coincides with the whole of A.

We close this section with a result relating an arbitrary derivation of $C^\infty[0,1]$ with the usual derivation. Namely, we have the following

Proposition 3.4. *Let δ be a derivation of $C^\infty[0,1]$. Then $\delta(f) = f'\delta(i)$ for all $f \in C^\infty[0,1]$, where f' denotes the first derivative of f and $i = id_{\mathbb{C}} \restriction_{[0,1]}$. In other words, a derivation δ of $C^\infty[0,1]$ is known, if and only if, its value at the element i, is known.*

Proof. Clearly, $i \in C^\infty[0,1]$. Let $\mathbb{C}[t]$ be the algebra of polynomials in one variable $t \in [0,1]$, with complex coefficients. Let $q \in \mathbb{C}[t]$ such that

$$q(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n,\ \alpha_j \in \mathbb{C},\ j = 0,1,\ldots,n,\ t \in [0,1].$$

Then it is easily seen that $q = q \circ i$ and

$$\delta(q) = \delta(q \circ i) = \alpha_1 \delta(i) + 2i\alpha_2 \delta(i) + \cdots + ni^{n-1}\alpha_n \delta(i) = q'\delta(i).$$

But $\mathbb{C}[t]$ is dense in $C^\infty[0,1]$, as it is well known. Therefore, using the continuity of the derivations of $C^\infty[0,1]$ (according to Carpenter's theorem mentioned in Remark 3.3(3)), we are led to the conclusion. ∎

4. Derivations of pro-C^*-algebras

Only a few results exist in the literature about derivations of pro-C^*-algebras, and these are given by R. Becker (1992) in [11], N. C. Phillips (1995) in [43] and Weigt-Zarakas (2010) in [56]. In this section, we shall briefly discuss, these results.

It is well known that *every derivation of a C^*-algebra is continuous* (Sakai); see [48, Theorem 2.3.1], and [48, 2.3.3] for historical notes. Moreover, *every derivation of a commutative C^*-algebra is identically zero* [47, Lemma 4.1.2]. The third important result in this direction is that *every derivation of a simple C^*-algebra with identity and/or of a W^*-algebra is inner* (Sakai); see [47, Theorem 4.1.1], resp. [48, Theorem 2.5.3]. Also see [48, 2.4.14] for historical comments, and Remark 4.4(3) below. There is a great deal of beautiful results on derivations of C^*-algebras, which can be found in [18, 21, 42, 47, 48], together with a very rich literature. In this section, we shall discuss the non-normed analogues of the C^*-algebra results we have just mentioned.

In 1992, R. Becker studied the preceding results in the context of pro-C^*-algebras and, among others, he proved

Theorem 4.1. ([11, Corollary 3, Proposition 2 and Proposition 6]) *Let $A[\tau]$ be a pro-C^*-algebra with $(p_\lambda)_{\lambda\in\Lambda}$ a defining family of C^*-seminorms for τ. Then the following statements hold:*

(1) *There exists $c_\lambda > 0$ such that $p_\lambda(\delta(a)) \leq c_\lambda p_\lambda(a)$, for every $a \in A$ and $\lambda \in \Lambda$, i.e., every derivation of $A[\tau]$ is continuous.*

(2) *If $A[\tau]$ is commutative, then the trivial derivation is the only derivation of A.*

(3) *If $A[\tau]$ has an identity, every derivation δ of $A[\tau]$ is the infinitesimal generator of a continuous one-parameter group of automorphisms of $A[\tau]$, and this group is locally equicontinuous.*

We briefly discuss (3). To prove this claim, one must show that: There is a family $\{\alpha_t : t \in \mathbb{R}\}$ of automorphisms of $A[\tau]$ such that each one of them is continuous; the map $t \in \mathbb{R} \mapsto \alpha_t(a) \in A[\tau]$ is continuous, for all $a \in A$; and that if δ is a derivation of $A[\tau]$, then

$$p_\lambda\big(\delta(a)\big) = \lim_{t\to 0} \frac{1}{t} p_\lambda\big(\alpha_t(a) - a\big), \ \forall\ a \in A \text{ and } \lambda \in \Lambda.$$

Indeed: From (1), there is a positive number c_λ such that $p_\lambda(\delta(a)) \leq c_\lambda p_\lambda(a)$ for every $a \in A$ and $\lambda \in \Lambda$. So, using completeness of $A[\tau]$, one concludes that

$$p_\lambda\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n(a)\right) \leq e^{c_\lambda |t|} p_\lambda(a), \ \forall\ a \in A \text{ and } \lambda \in \Lambda.$$

Therefore, the series $\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n(a)$ converges. Consequently, the map

$$\begin{array}{rcl} \alpha_t : A[\tau] & \longrightarrow & A[\tau] \\ a & \mapsto & \alpha_t(a) := \displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!}\delta^n(a), \end{array}$$

is a well defined continuous automorphism of $A[\tau]$, for every $t \in \mathbb{R}$. Now an easy calculation shows that

$$p_\lambda\big(\alpha_{t+s}(a) - \alpha_s(a)\big) \leq e^{c_\lambda |s|} e^{(c_\lambda |t| - 1)} p_\lambda(a), \ \forall\ a \in A \ \text{ and } \ \lambda \in \Lambda.$$

Therefore, the maps

$$\begin{array}{rcl} \mathbb{R} & \longrightarrow & A[\tau] \\ t & \mapsto & \alpha_t(a) \end{array}$$

are continuous, for every $a \in A$. The remaining part is easily seen.

• Note that when the derivation δ is inner, then the preceding group $\{\alpha_t : t \in \mathbb{R}\}$ of automorphisms generated by δ can be described in detail (see [11, Proposition 18]). Namely, there is an element $x \in A$ such that

$$\alpha_t(a) = e^{tx} a e^{-tx}, \ \forall\ a \in A \text{ and } t \in \mathbb{R}.$$

The element $x \in A$ is that for which $\delta = \delta_x$.

• A motivation for (3) can be given as follows: Consider the Fréchet m-convex algebra $C^\infty(\mathbb{R})$ (with the C^∞-topology), which of course is not a pro-C^*-algebra (see [24, 2.4(2) and 7.7]). We may define a continuous one-parameter group $\{\alpha_t : t \in \mathbb{R}\}$ of automorphisms of our algebra, in the following way

$$\alpha_t(f)(s) := f(t+s), \ \forall \ f \in C^\infty(\mathbb{R}) \text{ and } t, s \in \mathbb{R}.$$

The "infinitesimal generator" of the group $\{\alpha_t : t \in \mathbb{R}\}$, in a generalized sense, is given by the usual derivation of $C^\infty(\mathbb{R})$

$$\delta : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$

with

$$\delta(f)(s) = \lim_{t\to 0} \frac{f(t+s) - f(s)}{t} = f'(s),$$

for every $f \in C^\infty(\mathbb{R})$ and $s \in \mathbb{R}$.

For the proof of Theorem 4.1(1), Becker used essentially the fact that a derivation δ of $A[\tau]$ induces a derivation δ_λ on each quotient C^*-algebra A_λ, $\lambda \in \Lambda$. Thus in [11, Proposition 12] he proved that *if δ is a derivation of a pro-C^*-algebra $A[\tau]$ such that the induced by δ derivations δ_λ on the quotient C^*-algebras are inner, then δ is approximately inner* (see Section 2 for terminology).

In 1995, N.C. Phillips [43], using very interesting techniques, improved the previous result of Becker in two different ways. First he dropped the assumption of the innerness of the derivations of the quotients C^*-algebras A_λ, $\lambda \in \Lambda$, of $A[\tau]$ (see Theorem 4.2 below) using Theorem 4.1(1), and results on $*$-derivations of separable C^*-algebras (see, for example, [42, 8.6.1 and 8.6.12]). Thus he showed the following

THEOREM 4.2. ([43, Theorem 3]) *Every derivation of a pro-C^*-algebra is approximately inner.*

Secondly, Phillips strengthened the initial result of Becker passing from approximately innerness to innerness, by adding appropriate countability assumptions and using for the proof, the structure of C^*-algebras all of whose derivations are inner (see, for instance, [43, Lemma 5]). Namely, he has shown

THEOREM 4.3. ([43, Theorem 6]) *Let $A[\tau]$ be a separable σ-C^*-algebra such that every derivation of the quotient C^*-algebras A_λ, $\lambda \in \Lambda$, is inner. Then every derivation of $A[\tau]$ is inner.*

Remark 4.4. (1) Theorem 4.2 improves also Propositions 13 and 14 of [11] and, at the same time, answers a question stated after Remark 16 of the same reference.

(2) Regarding Theorem 4.3, N. C. Phillips gives an example of a separable pro-C^*-algebra $A[\tau]$ with a derivation δ, which is not inner, but the derivations it induces on the C^*-quotients A_λ, $\lambda \in \Lambda$, of $A[\tau]$, are all inner (see [43, Example 7]). This shows the importance of his assumption for the derivations of the A_λ's in Theorem 4.3.

(3) It is not expected that every derivation, even of a σ-C^*-algebra, is inner. Indeed, let H be a Hilbert space, $\mathcal{B}(H)$ the C^*-algebra of all bounded operators on H and $\mathcal{K}(H)$ the C^*-subalgebra of all compact operators in $\mathcal{B}(H)$. Then, for all $T \in \mathcal{B}(H)$, the map $\delta_T(S) = TS - ST$, $S \in \mathcal{K}(H)$ is a derivation of $\mathcal{K}(H)$. For $S \in \mathcal{K}(H)$, one has that $\delta_T = \delta_S$ if and only if $T - S \in \mathcal{K}(H)' = \mathbb{C}I$, where $\mathcal{K}(H)'$ is the commutant of $\mathcal{K}(H)$ in $\mathcal{B}(H)$, and I the identity operator on H (see [29, Example (VII.2.14)] and [41, Remark 4.4.2]). Hence, $\delta_T = \delta_S$, if and only if there exists $\mu \in \mathbb{C}$ such that $T - S = \mu I$. Therefore, if we choose $T \notin \mathcal{K}(H) + \mathbb{C}I$, then δ_T is not an inner derivation of $\mathcal{K}(H)$. If $H = l^2(\mathbb{N})$, such a T could be the shift operator.

To finish this section, we give a short account on some recent results on derivations from pro-C^*-algebras into locally convex bimodules due to the second and third author of this paper. These results mainly offer extensions of a result of J. R. Ringrose [46], according to which every derivation of a C^*-algebra A into a Banach A-bimodule is continuous. For notation and terminology, see Section 2. The proof of the following result is more or less scheduled along the lines of the proof of [51, Remark 12.3 and Corollary 12.5].

Theorem 4.5. ([56, Theorem 3.9]) *Let $A[\tau]$ be a σ-C^*-algebra with identity, $X[\tau']$ a Fréchet locally convex A-bimodule and $\delta : A[\tau] \to X[\tau']$ a derivation. Then δ is continuous if and only if the two-sided ideal $I = \{a \in A : aS(\delta) = S(\delta)a = \{0\}\}$ has finite codimension in A.*

Theorem 4.6. ([56, Proposition 3.5 and Corollary 3.2]) *Let $A[\tau]$ be a pro-C^*-algebra and $X[\|\cdot\|]$ a Banach A-bimodule. Then the following statements hold:*

(1) *Every derivation $\delta : A[\tau] \to X[\|\cdot\|]$ is continuous.*

(2) *If $A[\tau]$ is commutative and metrizable (i.e., a commutative σ-C^*-algebra) and $X[\|\cdot\|]$ is commutative and involutive (the latter term means that*

X has a vector space involution such that $(ax)^* = x^*a^* = (xa)^*$, for all $a \in A$ and $x \in X$), then every derivation $\delta : A[\tau] \to X[\|\cdot\|]$ is inner.

For the proof of (1), notice that from the continuity of the module actions of X, there is $\lambda \in \Lambda$, with respect to which X becomes an A_λ-bimodule and moreover, δ induces a well-defined derivation δ_λ from A_λ in X, which is continuous by Ringrose's result mentioned above. The assertion then follows.

For the proof of (2), you just note that X becomes a (commutative) Banach A_b-bimodule, so that if δ_b is the restriction of δ on A_b, then since every commutative C^*-algebra is amenable, by a result of B. E. Johnson [34, Proposition 8.2], it follows that δ_b is inner, therefore continuous. The assumptions for the vector space involution on X imply now that δ is continuous [56, Proposition 3.1] and since A_b is dense in $A[\tau]$, it follows that δ is inner.

Remark 4.7. (1) Theorem 4.6(i) remains true if the Banach A-bimodule $X[\|\cdot\|]$ is replaced by a smooth complete locally convex bimodule $X[\tau']$ (see [56, Theorem 3.6] and Section 2).

(2) Ringrose's result has been proved by the third author in [58] for every pro-C^*-algebra $A[\tau]$ and any "Hilbert pro-C^*-bimodule" X.

(3) Some structural properties of inner derivations of a "locally W^*-algebra" (inverse limit of W^*-algebras) can be found in [23].

5. Derivations of unbounded operator algebras

Our physical world mainly consists of unbounded operators, like the momentum and position operators denoted by p and q respectively, the annihilation operators a^* and a such that $a^*a - aa^* = I$, or the Hamiltonian H of a harmonic oscillator, where $H = \frac{1}{2}(p^2 + q^2) = a^*a + \frac{1}{2}I$, I the identity operator, see [4, 15, 18, 48, 49]. The algebras of unbounded operators accept non-normed topologies and are of particular importance since they are connected with mathematical physics and quantum field theory questions (see, for instance, [4, 49]). The GB^*-algebras of G. R. Allan [3] are algebras of unbounded operators, as P. G. Dixon has shown in [22]. We would stress here that the annihilation operators considered above, show that there are unbounded operators a_1, a_2 that fulfil the commutation relation $a_1a_2 - a_2a_1 = I$ (and, of course, there are unbounded operators a_1, a_2 that satisfy the canonical commutation relation $a_1a_2 - a_2a_1 = -i\hbar I$, where i is the imaginary unit and $\hbar$ the Planck constant, see e.g., [48, p. 17, 2.2] and [15, p. 10]), but in the bounded case, no operators a_1, a_2 exist such that $a_1a_2 - a_2a_1 = I$ (see, for instance, [29, p. 104, Proposition (II.1.21)]).

Furthermore, pro-C^*-algebras are related to the so-called O^*-algebras introduced by G. Lassner [37] (cf. also [4, 30, 49]), which are $*$-subalgebras of unbounded operators. More precisely, if H is a Hilbert space and D a dense subspace in H, let $\mathcal{L}(D)$ denote the set of all linear operators from D in D. Let

$$\mathcal{L}^{\dagger}(\mathcal{D}) := \{T \in \mathcal{L}(D) : \mathcal{D}(T^*) \supset D \text{ and } T^*D \subset D\},$$

where T^* is the adjoint of T and $\mathcal{D}(T^*)$ the domain of T^*. Then $\mathcal{L}^{\dagger}(D)$ is a $*$-algebra under the usual algebraic operations and involution $T^{\dagger} = T^* \restriction_D$, $T \in \mathcal{L}^{\dagger}(D)$. A $*$-subalgebra of $\mathcal{L}^{\dagger}(D)$ containing the identity operator is called an *O^*-algebra* and is denoted by $\mathcal{A}(D)$. An O^*-algebra $\mathcal{A}(D)$ is said to be of type R [19] if D is represented as an algebraic direct sum of Hilbert spaces, H_j, $j \in J$, that are left invariant by all operators in $\mathcal{A}(D)$. An O^*-algebra $\mathcal{A}(D)$ comes equipped with a topology τ_0 determined by the seminorms

$$p_j(T) := \sup\{\|Tx\| : x \in H_j,\ \|x\| \leq 1\},\ j \in J,\ T \in \mathcal{A}(D).$$

The first paper on derivations of unbounded operator algebras is due to C. Brödel and G. Lassner (1975) [19]. Further results in this direction can be found in [1, 2, 5, 6, 7, 9, 10, 12, 13, 14, 31, 32, 54, 55].

We exhibit now two derivation results on O^*-algebras, the first one due to Brödel and Lassner and the second one to Becker.

THEOREM 5.1. ([19, Theorem 3]) *Every pro-C^*-algebra $A[\tau]$ with identity is algebraically and topologically $*$-isomorphic to a complete O^*-algebra $\mathcal{A}(D)$ of type R, all of whose derivations are spatial.*

Note that a derivation δ of an O^*-algebra $\mathcal{A}(D)$ is *spatial* if there is an element $S \in \mathcal{L}^{\dagger}(D)$ such that $\delta(T) = ST - TS$, for every $T \in \mathcal{A}(D)$.

THEOREM 5.2. ([11, Proposition 10]) *Let $\mathcal{A}(D)[\tau_0]$ be a complete O^*-algebra of type R and δ a $*$-derivation of $\mathcal{A}(D)[\tau_0]$. The group of automorphisms $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ (see discussion after Theorem 4.1) generated by δ is given as*

$$\alpha_t(T) = e^{itS}Te^{-itS},\ \forall\ t \in \mathbb{R} \text{ and } T \in \mathcal{A}(D)[\tau_0],$$

where S is the operator $S \in \mathcal{L}^{\dagger}(D)$ that makes δ spatial according to Theorem 5.1.

The investigation of the structure of derivations acting on unbounded operator algebras and, in particular, on GB^*-algebras and locally convex quasi C^*-algebras introduced in [8] (2008) would be interesting. All pro-C^*-algebras are GB^*-algebras. The Arens algebra $L^\omega[0,1] := \bigcap_{1\leq p<\infty} L^p[0,1]$ [40] is a GB^*-algebra, but not a pro-C^*-algebra. Locally convex quasi $*$-algebras were introduced by G. Lassner [38, 39] for facing problems of quantum statistics and quantum dynamics that could not be solved within the algebraic formulation of quantum theories initiated by Haag and Kastler in [27]. Both of the classes of unbounded operator algebras work through a C^*-subalgebra directly related with their structure, see [3, 8, 22]. In the case of the GB^*-algebra $L^\omega[0,1]$, the corresponding C^*-subalgebra is $L^\infty[0,1]$, while the Banach space $L^p[0,1]$, $1 \leq p < \infty$ is a Banach quasi C^*-algebra with corresponding C^*-subalgebra again $L^\infty[0,1]$. The main characteristic of locally convex quasi C^*-algebras is that they belong to the class of partial $*$-algebras [4], whose multiplication is not everywhere defined. The second and third author of the present paper are working in the spirit of the preceding discussion.

The theory of unbounded derivations on C^*-algebras is mainly concerned with closability, generators and transformation groups (see [17, Chapter 3], [48, Chapter 3], [53] together with the relevant literature therein).

A natural question is to see how far a similar project on unbounded derivations in the context of the aforementioned unbounded operator algebras could reach.

We finish by giving the definition and an example of an unbounded derivation in a GB^*-algebra.

Definition 5.3. Let $A[\tau]$ be a GB^*-algebra. An unbounded derivation in $A[\tau]$ is a linear map δ from a dense subalgebra $\mathcal{D}(\delta)$ of $A[\tau]$ in $A[\tau]$, that satisfies the Leibnitz rule. If $\mathcal{D}(\delta)$ is a dense $*$-subalgebra of $A[\tau]$ and $\delta(a^*) = \delta(a)^*$ for every $a \in \mathcal{D}(\delta)$, then δ is called an unbounded $*$-derivation of $A[\tau]$.

Let $A[\tau]$ be the σ-C^*-algebra (hence a GB^*-algebra) $C_c(\mathbb{R})$ of all $\mathbb{C}$-valued continuous functions on $\mathbb{R}$, under the compact open topology c, and

$$\mathcal{D}(\delta) = C^1(\mathbb{R}) := \{f \in C_c(\mathbb{R}) \text{ with continuous } 1^{\text{st}} \text{ derivative } \}.$$

An unbounded derivation δ of $C_c(\mathbb{R})$ is then defined as follows:

$$\begin{array}{rcl} \delta : \mathcal{D}(\delta) & \longrightarrow & C_c(\mathbb{R}) \\ f & \mapsto & f', \end{array}$$

where f' is the first derivative of f.

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