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# On the Ergodicity of Banach Spaces with Property $(H)$

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*Abstract*: In this note we discuss the ergodicity of the class of Banach spaces which are characterized by property $(H)$.

*Key words*: weak Hilbert spaces, Borel reducibility.

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In this survey we will discuss some results concerning the complexity of the isomorphism relation between separable Banach spaces via the notion of Borel reducibility.

The general problem regarding the classification of analytic equivalence relations on Polish spaces by Borel reducibility originated from the works of Friedman and Stanley [11] and independently from the works of Harrington, Kechris and Louveau [13]. In the context of Banach spaces, this direction of research has its roots in the work of Bossard [3].

Definition 1. Given Polish spaces $E$ and $F$ carrying equivalence relations $\mathcal{R}$ and $\mathcal{R}'$, one says that $(E,\mathcal{R})$ is *Borel reducible* to $(E',\mathcal{R}')$ if there exists a Borel map $f : E \to E'$ such that

$$x\mathcal{R}y \iff f(x)\mathcal{R}'f(y)\,.$$

In this case we write $(E,\mathcal{R}) \leq_B (E',\mathcal{R}')$, or simply $E \leq_B E'$.

This basically means that assigning invariants to $\mathcal{R}$-classes is at least as difficult as assigning them to $\mathcal{R}'$-classes.

The notion of Borel reducibility is an extremely useful tool in measuring complexity in analysis. An important measure of complexity is the following relation $E_0$ of eventual agreement of countable binary sequences. It is defined on $2^\omega$ by

$$\alpha E_0 \beta \iff \exists\, m\ \forall\, n \geq m\,,\quad \alpha(n) = \beta(n)\,.$$

One can also look at the relation $E_1$ of eventual agreement for countable sequences of real numbers: it is defined on $\mathbb{R}^\omega$ by

$$\alpha E_1 \beta \quad \Longleftrightarrow \quad \exists\, m\ \forall\, n \geq m\,, \quad \alpha(n) = \beta(n)\,.$$

It is easy to see that $E_0 \leq_B E_1$ and it can be shown that $E_1 \not\leq_B E_0$.

In order to investigate the problem concerning the complexity of the isomorphism relation in the context of Banach spaces we need to have an appropriate topological framework.

A Borel space, that is, a space with a distinguished $\sigma$-algebra of sets, is called a standard Borel space if its Borel structure is induced by an underlying Polish topology. For a Banach space $X$ we consider $\mathcal{B}_X$ the set of its closed linear subspaces, equipped with the Effros-Borel structure (see [9]) which turns it into a standard Borel space.

There are two directions of research concerning the complexity of the isomorphism relation between separable Banach spaces via the notion of Borel reducibility. One studies the complexity of the isomorphism relation between subspaces of a *given* separable Banach space, while the second one deals with the complexity of the isomorphism relation between *all* separable Banach spaces; note that in the latter case all separable Banach spaces can be seen as subspaces of an isomerically universal Banach space such as $C([0,1])$ and thus we can work with $\mathcal{B}_X$ where $X = C([0,1])$.

In this note we concentrate on the former question. The positive solution to the homogeneous Banach space problem, which was obtained by combining the Gowers' dichotomy theorem [12] with results of Komorowski and Tomczak-Jaegermann [16], states that if a Banach space $X$ has only one class of isomorphism of infinite dimensional subspaces then $X$ must be isomorphic to $\ell_2$. It is therefore natural to investigate the complexity of the relation of isomorphism between subspaces of a given separable Banach space which is not isomorphic to $\ell_2$.

We begin with the following notion introduced and studied by Ferenczi and Rosendal [10].

Definition 2. A separable Banach space $X$ is said to be *ergodic* if the relation $E_0$ is Borel reducible to isomorphism between subspaces of $X$.

It is clear from the definition that if $X$ has a subspace $Y$ which is ergodic then $X$ itself is ergodic.

There are several known results regarding ergodic spaces.

It was observed in [20] that hereditarily indecomposable Banach spaces are ergodic. Recall that an infinite dimensional Banach space is hereditarily indecomposable (HI) if no subspace can be written as the topological direct sum of two infinite dimensional subspaces. Gowers' dichotomy [12] states that every infinite-dimensional Banach space contains either a subspace with an unconditional basis or an HI subspace. Thus the results of [20] reduces the investigation of ergodicity to spaces with an unconditional basis.

Ferenczi and Rosendal proved in a series of papers [9], [10] that a non-ergodic Banach space $X$ with an unconditional basis must have lots of symmetries, in particular it is isomorphic to its hyperplanes, to its square, and more generally to any direct sum $X \oplus Y$, where $Y$ is generated by a subsequence of the basis of $X$. And Ferenczi proved in [7] that if $X$ is a separable space without minimal subspaces than $X$ must be ergodic.

It was conjectured in [10] that every separable Banach space not isomorphic to $\ell_2$ must be ergodic. If true, this would constitute a substantial improvement on the homogeneous space problem since $E_0$ is an equivalence relation with $2^\omega$ classes.

Among the classical Banach spaces, $c_0$ and $\ell_p$, with $1 \leq p < 2$, are known to be ergodic ([8]). Also, the case of strongly asymptotic $\ell_p$ spaces was resolved in [6].

Let $1 \leq p \leq \infty$. A Banach space $X$ with a basis $\{e_i\}_i$ is strongly asymptotic $\ell_p$ if there exist $C < \infty$ and an increasing function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$, every normalized sequence $\{x_k\}_{k=1}^n$ of disjointly supported vectors from $\overline{\operatorname{span}}\,\{e_i\}_{i=f(n)}^\infty$ is $C$-equivalent to the unit vector basis of $\ell_p^n$:

$$1/C\left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \leq \left\|\sum_{k=1}^n a_k x_k\right\| \leq C\left(\sum_{k=1}^n |a_k|^p\right)^{1/p}, \tag{1}$$

for all choices of scalars $\{a_k\}_{k=1}^n$ (with the obvious convention of taking $\max_{k=1}^n |a_k|$ on both left and right sides when $p = \infty$).

THEOREM 3. ([6]) *Let $1 \leq p \leq \infty$ and let $X$ be a Banach space with a strongly asymptotic basis $\{e_i\}_i$. Then either $\{e_i\}_i$ is equivalent to the unit vector basis in $\ell_p$ (or $c_0$ if $p = \infty$), or $X$ is ergodic.*

In light of previous comments, it follows that when $1 \leq p < 2$ we can conclude that a strongly asymptotic $\ell_p$ space is always ergodic.

In the present note we would like to discuss the ergodicity of a class of Banach spaces which are characterized by the so-called *property* $(H)$, which is related to (1) and involves only linear combinations with equal coefficients.

Definition 4. ([19]) A Banach space $X$ is said to have property $(H)$ if for each $\lambda > 1$ there is a constant $K(\lambda)$ such that for every $n \in \mathbb{N}$, whenever $\{u_1, u_2, \dots, u_n\} \subset X$ is a $\lambda$-unconditional normalized basic sequence, then

$$K(\lambda)^{-1}\sqrt{n} \leq \left\| \sum_{k=1}^{n} u_k \right\| \leq K(\lambda)\sqrt{n}$$

It should be noted that a Banach space with property $(H)$ and with an unconditional basis is saturated with subspaces which are strongly asymptotic $\ell_2$. This is a consequence of the following result of Junge, Kutzarova and Odell [15]. We state here the result from [15] in the form that was observed by Tcaciuc (see the remarks after [22, Theorem 17]).

Theorem 5. ([15]) *Let $X$ be a Banach space with a basis $\{e_i\}_i$. Let $1 \leq p < \infty$ and $K > 0$. Assume that for all $n$, if $\{x_k\}_{k=1}^{n}$ is a normalized sequence of disjointly supported vectors on $\{e_i\}_i$, then $\|\sum_{k=1}^{n} x_k\| \sim n^{1/p}$. Then every infinite dimensional subspace of $X$ contains a strongly asymptotic $\ell_p$ basic sequence.*

Property $(H)$ was introduced by Pisier in [19], where he proved that weak Hilbert spaces have this property. Few years later, Nielsen and Tomczak-Jaegermann [18] showed that the converse is also true in the presence of unconditionality: for a Banach space $X$ with an unconditional basis, property $(H)$ is actually equivalent to the fact that $X$ is a weak Hilbert space.

Definition 6. A Banach space $X$ is a *weak Hilbert space* provided there exist $\delta > 0$ and $K \geq 1$ such that every finite-dimensional subspace $E \subseteq X$ contains a further subspace $F$ with $\dim F \geq \delta \dim E$ which is $K$-Euclidean and $K$-complemented in $X$.

The original definition of a weak Hilbert space is different and the characterization mentioned above is chosen out of many equivalent properties proved by Pisier. Weak Hilbert spaces are known to behave very nicely, in particular they are reflexive and satisfy some very strong types of approximation properties, for example they admit a finite-dimensional Schauder decomposition (Maurey-Pisier, see [17]).

One of the main problems in the theory of weak Hilbert spaces is the scarcity of known examples. The known weak Hilbert spaces are basically variations of the Tsirelson's construction and this raises several important questions which remain still unanswered.

Question 1. ([4], for example) Does every weak Hilbert space have a Schauder basis?

Question 2. (folklore) Does there exist a weak Hilbert space which is HI?

Related to the latter problem, there is a recent construction of Argyros, Beanland and Raikoftsalis [2] which exhibits a weak Hilbert space such that for every block subspace $Y$ every bounded linear operator on $Y$ is of the form $D+S$, where $D$ is a diagonal operator and $S$ is stricly singular.

Going back to the discussion regarding the ergodicity of a Banach space with property $(H)$, we refer now to another important result from the above mentioned paper of Nielsen and Tomczak-Jaegermann [18]. They studied the structure of unconditional sequences in Banach spaces with property $(H)$ and gave strong estimates of the tail behaviour of such sequences. The estimates have the same order of magnitude as those obtained for the basis of a particular Banach space with property $(H)$, namely the unit vector basis of $T^{(2)}$ (or its dual), where $T^{(2)}$ is the 2-convexification of Tsirelson's space (see [5]). In terms of the complexity of the relation of isomorphism between subspaces, not only that $T^{(2)}$ is ergodic (Theorem 3) but we also know that the relation $E_1$ is Borel reducible to isomorphism between subspaces of $T^{(2)}$, as it was proved by Rosendal in [21].

This suggests that one might be able to show that the relation $E_1$ is Borel reducible to isomorphism between subspaces of $X$, for $X$ a non-Hilbertian Banach space with property $(H)$. We don't know the answer to this question.

If we want to investigate the ergodicity of a Banach space with property $(H)$ we need to look at some other kinds of symmetries that appear in such spaces. The following result of Johnson, which was published in [19], does just this.

Definition 7. Given integers $n \geq 0$, $m \geq 1$ and a constant $K$, we say that a Banach space $X$ satisfies $H(n, m, K)$ provided there is an $n$-codimensional subspace $Y$ of $X$ such that every subspace $E$ of $Y$ with $\dim E \leq m$ is $K$-isomorphic to $\ell_2^{\dim E}$. A Banach space $X$ is said to be *asymptotically Hilbertian* provided there is a constant $K$ so that for every $m$ there exists $n$ such that $X$ satisfies $H(n, m, K)$.

Theorem 8. ([19]) *A Banach space with property $(H)$ is asymptotically Hilbertian.*

Theorem 8 provides now the key in proving that a non-Hilbertian Banach spaces with property $(H)$ is ergodic. The following result was obtained in [1].

Theorem 9. ([1]) *Let $X$ be an asymptotically Hilbertian space which is not isomorphic to $\ell_2$. Then $X$ is ergodic.*

The construction from [1] uses the fact that $X$ satisfies not only $H(n,m,K)$, but also a closely related property. Namely, given integers $n' \geq 0$, $m' \geq 1$ and a constant $K'$, we say that a Banach space $X$ satisfies $C(n',m',K')$ provided there is an $n'$-codimensional subspace $Y$ of $X$ such that every subspace $E$ of $Y$ with $\dim E \leq m'$ is the range of a projection $P$ from $X$ with $\|P\| \leq K'$. It was proved by Johnson [14] that the notions $H(n,m,K)$ and $C(n',m',K')$ are in duality, and this plays an essential part in the proof of Theorem 9.

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