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Some Problems Concerning Basic Sequences in Banach Spaces

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Abstract: In this note we present some open problems concerning the existence of certain sequences in a Banach space, we discuss known techniques and constructions related, and possible ways to attack them.

Key words: unconditional basic sequences, Ramsey methods, non-separable Banach spaces, separable quotient problem.

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1. Introduction and basic facts

The existence of interesting type of sequences in a Banach has been one of the most studied topic in the theory of Banach spaces. For example, it is known from the beginning of the theory that every infinite dimensional Banach space has a basic sequence. It was a long-standing open problem to know if infinite dimensional spaces always have sequences equivalent to the unit bases of some of the classical sequence spaces c_0 or ℓ_p, $p \geq 1$. Tsirelson space T [39] is the first "non-classical" Banach space, as it does not have such sequences. Indeed this space does not have subsymmetric sequences. It took a little more time to find a space without unconditional basic sequences (Gowers-Maurey space [15]). On the other direction there is the Elton's result stating that every non-trivial weakly-null sequence has a near-unconditional subsequence (see [11]), or the result by Ketonen [21] stating that every Banach space X whose density is a ω-Erdös cardinal has a subsymmetric basic-sequence (see Definition 2.23 for full details). From this, it readily follows by Rosenthal ℓ_1-dichotomy that in this case X also has unconditional basic sequences. In the paper [10] it is proved that it is consistently true (relative of the existence of large cardinals) that every Banach space of density $\aleph_\omega$ has an unconditional sequence.

The existence of uncountable sequences, e.g. uncountable biorthogonal sequences, is also known to be interesting, particularly in renorming theory.

In this paper we present several open problems asking for the existence of certain countable or uncountable sequences, from a combinatorial point of view.

We pass now to recall some basic concepts. Along this note a Banach space will be always supposed to be infinite dimensional.

Let κ be a cardinal number and let $\bar{x} = (x_\alpha)_{\alpha<\kappa}$ be a long sequence in a Banach space X indexed in κ. The sequence $\bar{x}$ is called *seminormalized* when

$$0 < \inf_{\alpha<\kappa} \|x_\alpha\| \leq \sup_{\alpha<\kappa} \|x_\alpha\| < \infty\,. \tag{1}$$

The sequence $\bar{x}$ is called a *Schauder basic sequence* if it is seminormalized, and if there is a constant $K \geq 1$ such that

$$\left\|\sum_{\alpha<\beta} a_\alpha x_\alpha\right\| \leq K \left\|\sum_{\alpha<\kappa} a_\alpha x_\alpha\right\| \tag{2}$$

for every sequence $(a_\alpha)_{\alpha<\kappa}$ of scalars and every $\beta < \kappa$. The minimum of those constants K is called the *basic constant* of $\bar{x}$. If in addition the linear span of $\{x_\alpha\}_{\alpha<\kappa}$ is dense in X, then the sequence is called a Schauder basis of X. Equivalently, $\bar{x}$ is a Schauder basis of X if and only if every point $x \in X$ is uniquely written as $x = \sum_{\alpha<\kappa} a_\alpha x_\alpha$. It is well known that there are separable Banach spaces without bases, although every Banach space has an infinite basic sequence. Indeed, it is a result of Mazur that every non-trivial weakly-null sequence has a basic subsequence.

Among the basic sequences, there are the unconditional ones: A basic sequence $\bar{x}$ is called (suppression) *unconditional basic sequence* when there is a constant $K \geq 1$ such that

$$\left\|\sum_{\alpha\in M} a_\alpha x_\alpha\right\| \leq K \left\|\sum_{\alpha<\kappa} a_\alpha x_\alpha\right\| \tag{3}$$

for every sequence $(a_\alpha)_{\alpha<\kappa}$ of scalars and every subset $M \subseteq \kappa$. The minimum of those constants K is called the constant of unconditionality. It is not true that every non-trivial weakly-null sequence has an unconditional basic subsequence[1]. The first example was provided by Maurey and Rosenthal

[1] There are very simple non-weakly-null sequences without unconditional subsequences, for example the summing basis of c_0.

[27]. Many spaces have unconditional bases, as the sequence spaces $\ell_p(\kappa)$, or $c_0(\kappa)$ but there are also without unconditional bases, as for example $C[0,1]$. More difficult to find, but still existing, are spaces without unconditional basic sequences. The first example was given by Gowers and Maurey [15].

Finally, a basic sequence $\bar{x}$ is called subsymmetric if there is a constant $C \geq 1$ such that for every $s, t \in \mathcal{S}$ of the same cardinality the subsequences $(x_\alpha)_{\alpha\in s}$ and $(x_\alpha)_{\alpha\in t}$ are C-equivalent, i.e., denoting by $\theta : s \to t$ the unique order-preserving bijection between s and t, then the linear extension of $x_\alpha \in \langle x_\beta\rangle_{\beta\in s} \mapsto x_{\theta(\alpha)} \in \langle x_\beta\rangle_{\beta\in t}$ is an isomorphism of norm at most C. Examples of subsymmetric bases are the corresponding unit bases of $\ell_p(\kappa)$, or $c_0(\kappa)$. There are also spaces without subsymmetric basic sequences, for example the Tsirelson space [39].

The problems we propose are divided into two categories: existence of countable sequences and existence of uncountable ones.

2. Countable sequences

2.1. Separable spaces. We present here two problems: Elton's near unconditionality constant, and the bounded distortion of the Tsirelson space. Before we state them, we introduce well-known positive results for certain asymptotic properties. They play an important role in the theory of separable Banach spaces, and some of the ideas involved are used also to see how uncountable combinatorial principles force the existence of unconditional basic sequences. It is also interesting that there is no obvious generalization of the separable asymptotic properties to non-separable spaces as we will see in Subsection 2.2 with the notion of asymptotic ℓ_1-spaces.

Recall that the *Schreier family* $\mathcal{S}$ is the collection of all finite subsets s of $\mathbb{N}$ such that $|s| \leq \min s + 1$.

Definition 2.1. We call a basic sequence $(x_n)_{n<\omega}$ *asymptotically unconditional*[2] when there is a constant $C \geq 1$ such that every finite subsequence $(x_n)_{n\in s}$ is C-unconditional for every $s \in \mathcal{S}$. In this case we say that $(x_n)_n$ is asymptotically C-unconditional.

Proposition 2.2. ([30]) *For every non-trivial weakly-null sequence $(x_n)_{n<\omega}$ and every $C > 1$ there is a subsequence $(y_n)_{n<\omega}$ of $(x_n)_{n<\omega}$ which is asymptotically C-unconditional.*

[2] Often also called Schreier unconditional.

For a proof of this fact one can use the following result (see [23] for a proof and more information).

LEMMA 2.3. *Let $n < \omega$, let $M \subseteq \omega$ be an infinite set, and let $f : [M]^{\leq n} \to c_0$ be a mapping whose image is a weakly-pre-compact subset of c_0. Then for every $\varepsilon > 0$ there is an infinite $N \subseteq M$ such that*

$$\sum_{n\in N\setminus s} |(f(s))_n| \leq \varepsilon \qquad \text{for every } s \in [N]^{\leq n}.$$

In other words, the support of $f(s)$ is almost included in s for every $s \in [N]^{\leq n}$.

In the next, given $s, t \subseteq \mathbb{N}$ of the same cardinality, we write $\varpi_{s,t} : s \to t$ to denote the unique order-preserving mapping between s and t.

Proof of Proposition 2.2. Fix a space X and a seminormalized weakly-null sequence $(x_i)_i$ on it. Indeed, by normalizing if needed, we assume that $\|x_i\| = 1$ for every $i < \omega$. By the Mazur's theorem, we may assume, by going to a subsequence if needed, that $(x_n)_n$ is 2-basic. Fix now $C > 1$. We prove that for every M and $n < \omega$ there is $N \subseteq M$ such that $(x_i)_{i\in s}$ is C-unconditional for every $s \in [N]^n$. So, we fix such M and $n < \omega$. Since $(x_i)_i$ is normalized and basic,

- there is a finite subset $F_n \subseteq \mathbb{R}^n$ such that for every $s \in [\omega]^n$ and every $(b_i)_{i<n} \in \mathbb{R}^n$ there is $(a_i)_{i<n} \in F_n$ such that

$$\left\|\sum_{i\in s} b_{\varpi_{s,|s|}(i)} x_i - \sum_{i\in s} a_{\varpi_{s,|s|}(i)} x_i\right\| \leq \varepsilon \left\|\sum_{i\in s} a_{\varpi_{s,|s|}(i)} x_i\right\|.$$

Now for each $\bar{b} = (b_i)_{i<n} \in F_n$ and $t \in [M]^{\leq n}$, let $f_{\bar{b},t} \in S_{X^*}$ be such that

$$\left\|\sum_{i\in t} b_{\varpi_{t,|t|}(i)} x_i\right\| = f_{\bar{b},t}\left(\sum_{i\in t} b_{\varpi_{t,|t|}(i)} x_i\right).$$

Then by Lemma 2.3 there is some $N \subseteq M$ such that for every $\bar{b} = (b_i)_{i<n} \in F_n$ and every $t \in [M]^{\leq n}$ we have that

$$\sum_{i\in N\setminus t} \left|f_{\bar{b},t}(x_i)\right| \leq \varepsilon\,, \tag{4}$$

where $\varepsilon > 0$ is such that $(1+\varepsilon)(1-\varepsilon)^2 \leq C$ Now let $s \in [N]^n$, and we prove that $(x_i)_{i\in s}$ is C-unconditional: Let $(a_i)_{i\in s}$ be scalars, and let $t \subseteq s$. Without loss of generality, we assume that $\|\sum_{i\in s} a_i x_i\| = 1$. Find, using the properties of F_n, $\bar{b} = (b_i)_{i<n} \in F_n$ such that

$$\left\|\sum_{i\in t} a_i x_i - \sum_{i\in t} b_{\varpi_{t,|t|}(i)} x_i\right\| \leq \varepsilon \left\|\sum_{i\in t} a_i x_i\right\|.$$

Then, by (4) and setting $f := f_{\bar{b},t}$,

$$\begin{aligned}\left\|\sum_{i\in s} a_i x_i\right\| &\geq \left|f(\sum_{i\in s} a_i x_i)\right| \\ &\geq \left|f(\sum_{i\in t} b_{\varpi_{t,|t|}(i)} x_i)\right| - \left\|\sum_{i\in t}(b_{\varpi_{t,|t|}(i)} - a_i)x_i\right\| - \sum_{i\in s\setminus t} |f(x_i)| \\ &\geq (1-\varepsilon)\left\|\sum_{i\in t} b_{\varpi_{t,|t|}(i)} x_i\right\| - \varepsilon \\ &\geq (1-\varepsilon)^2 \left\|\sum_{i\in t} a_i x_i\right\| - \varepsilon \left\|\sum_{i\in s} a_i x_i\right\|\end{aligned}$$

and we are done. ∎

Definition 2.4. (Elton's near unconditionality) A sequence $(x_n)_{n<\omega}$ is called δ-near unconditional with constant $C \geq 1$ when that for every sequence of scalars $(a_n)_n$ and every $s \subseteq \{n < \omega : |a_n| \geq \delta\}$ one has that

$$\left\|\sum_{n\in s} a_n x_n\right\| \leq C \left\|\sum_{n<\omega} a_n x_n\right\|. \tag{5}$$

Theorem 2.5. (Elton [11]) *Every non-trivial weakly-null sequence has a δ-near unconditional subsequence with constant proportional to* $\log(1/\delta)$.

A proof of this result can be done by using similar combinatorial ideas as the ones exposed in Lemma 2.3. The reader can find more information in [23].

Problem 1. Does there exist a constant $C \geq 1$ such that every normalized weakly null sequence has for every $\delta > 0$ a δ-near unconditional subsequence with constant C?

One of the few facts known about this problem is that the constant is at least 5/4 (see [9]).

The last problem we discuss in this subsection is the distortion of the Tsirelson space. Although this problem is not of the same nature than the rest of the questions we propose, the combinatorial nature of the Tsirelson space invites us to present it here.

Recall that the Tsirelson space $T = \big(T, \|\cdot\|_T\big)$ is defined as the unique norm $\|\cdot\|_T$ such that

$$\|x\|_T = \max\left\{\|x\|_\infty\,,\,\sup\left\{\frac{1}{2}\sum_{i=1}^{n}\|E_i x\|_T \,:\, n \le E_1 < \cdots < E_n\right\}\right\} \tag{6}$$

for every $x \in T$. Tsirelson space T is a reflexive space with an unconditional basis which is asymptotic ℓ_1. It follows that T does not contain subsymmetric sequences.

The distortion of a renorm $\|\cdot\|_0$ of a given Banach space $(X, \|\cdot\|)$ is defined by

$$D\big((X, \|\cdot\|), \|\cdot\|_0\big) = \inf_{Y \subseteq X,\, \dim Y = \infty} \left\{\sup_{x,y \in S_{Y,\|\cdot\|}} \frac{\|y\|_0}{\|x\|_0}\right\}. \tag{7}$$

The space $(X, \|\cdot\|)$ is distortable if

$$D\big(X, \|\cdot\|\big) := \sup_{\|\cdot\|_0 \sim \|\cdot\|} D\big((X, \|\cdot\|), \|\cdot\|_0\big) > 1\,, \tag{8}$$

$(X, \|\cdot\|)$ is arbitrarily distortable when $D\big(X, \|\cdot\|\big) = \infty$, and $(X, \|\cdot\|)$ has bounded distortion when $1 < D\big(X, \|\cdot\|\big) < \infty$.

It is well known that many Banach spaces, including the separable Hilbert space [31], are arbitrarily distortable. It follows from a classical work of James that c_0 and ℓ_1 are not distortable. It is also known that the Tsirelson space is $(2-\varepsilon)$-distortable for every $\varepsilon > 0$, but the following is still open.

Problem 2. Has the Tsirelson space bounded distortion? In general, is there a Banach space with bounded distortion?

It is known (see [17] for full information) that if X has bounded distortion, then it has an unconditional basic sequence and the space is asymptotic c_0 or asymptotic ℓ_p, $p \ge 1$, and in this last case ℓ_1 must be finitely representable in X.

Perhaps a way to prove that the space T has bounded distortion is to see first a Ramsey result for it, or more likely for its dual T^*. Recall that the dual space T^* is asymptotic c_0, and for c_0 there is a very strong Ramsey result by Gowers [14]: Given any uniformly continuous function $f : S_{c_0} \to \mathbb{R}$ and given $\varepsilon > 0$ there is an infinite dimensional subspace X of c_0 such that the oscillation of f in the unit sphere of X is at most ε.

2.2. Non-separable spaces.

Definition 2.6. Let $\mathcal{K}$ be a class of Banach spaces. Let $\mathfrak{nc}_{\mathcal{K}}$ be the minimal cardinal number with the property that every Banach space in $\mathcal{K}$ has an unconditional basic sequence. For $\mathcal{K}$ the class of all Banach spaces, we simply write $\mathfrak{nc}$.

Gowers and Maurey [15] proved that $\mathfrak{nc}$ is uncountable, and later from the Argyros and Tolias work [3] we know that $\mathfrak{nc}$ is indeed bigger than the continuum. On the other hand, Ketonen result [21] states that $\mathfrak{nc}$ is smaller or equal to the first ω-Erdös cardinal number, and in [10] we prove that it is consistent relative to the existence of large cardinals that $\mathfrak{nc} \leq \aleph_\omega$.

Similarly, we introduce the following cardinal number.

Definition 2.7. Let

$$\mathfrak{nc}_0 := \sup \left\{ \kappa \ : \begin{array}{c} \text{every non-trivial w-null sequence} \\ (x_\alpha)_{\alpha<\kappa} \text{ has an unconditional subsequence} \end{array} \right\}.$$

By the Amir-Lindenstrauss Theorem [1] it follows that $\mathfrak{nc}_{\mathrm{Refl}} \leq \mathfrak{nc}_0$, where Refl is the class of all reflexive spaces. The Schauder basic sequence in [2] gives that $\mathfrak{nc}_0 > \aleph_1$, while a result from [10] says that it is consistent relative to the existence of large cardinals that $\mathfrak{nc}_0 \leq \aleph_0$.

The following combinatorial principle is crucial to find unconditional basis subsequences. Before we need to recall the following standard notation. Given a set X and an integer n, by $[X]^n$ we denote the family of all subsets of X of cardinality n, and by $[X]^{<\omega}$ we denote the set of all finite subsets of X.

Definition 2.8. Given $d \geq 1$, we say that κ is d-polarized, and we write $\mathrm{Pol}_d(\kappa)$ if for every $c : [[\kappa]^d]^{<\omega} \to \omega$ there is an infinite block sequence $(X_i)_i$ of infinite subsets of κ such that $c \restriction \prod_{i<n}[X_i]^d$ is constant for every $n < \omega$.

Theorem 2.9. ([10]) *Suppose that* κ *has the property* $\mathrm{Pol}_2(\kappa)$. *Then* $\mathfrak{nc} \leq \kappa$.

The proof uses the following multidimensional asymptotic unconditional fact (see [10] for a proof):

PROPOSITION 2.10. *Suppose that* $(x_n^{(0)})_{n\in\mathbb{N}},\dots,(x_n^{(k)})_{n\in\mathbb{N}}$ *are non trivial weakly-null sequences. Then for every* $\varepsilon>0$ *there is an infinite subset* M *of* $\mathbb{N}$ *such that* $(x_{m_0}^{(0)},\dots,x_{m_k}^{(k)})$ *is* $(1+\varepsilon)$*-unconditional for every* $m_0<\cdots<m_k$ *in* M.

Proof of Theorem 2.9*. (Sketch)* The proof uses an asymptotic argument. We first fix a separated sequence $(x_\alpha)_{\alpha<\kappa}$ in a given Banach space X. Without loss of generality we assume that ℓ_1 does not embed in X. Now we color with c each finite block sequence $(\{\alpha_0<\beta_0\},\dots,\{\alpha_k<\beta_k\})$ by $l>0$ if $((x_{\alpha_i}-x_{\beta_i}))_{i\le k}$ is $(1+2^{1-l})$-unconditional but it is not $(1+2^{-l})$-unconditional and by 0 if it is 1-unconditional. Since $\mathrm{Pol}_2(\kappa)$ holds, there are infinite subsets $X_0<X_1<X_2<\cdots<$ of κ such that $c\restriction\prod_{i\le k}[X_i]^2$ is constant with value l_k. We claim that $l_k=0$ for every $k\ge 0$, which gives that if we choose arbitrary $\{\alpha_n<\beta_n\}\subseteq X_n$ for every n, then $(x_{\beta_n}-x_{\alpha_n})_n$ is a 1-unconditional basic sequence. So, suppose that there is some $l_k>0$. Since ℓ_1 does not embed in X, we can find, by Rosenthal ℓ_1-Theorem weakly-Cauchy subsequences $(x_{\alpha_n^i})_{n\in\mathbb{N}}$ with $\{\alpha_n^{(i)}\}_{n\in\mathbb{N}}\subseteq X_i$ for every $i\le k$. Then our desired result follows from Proposition 2.10. ∎

Since it is consistent relative to the existence of large cardinals that $\mathrm{Pol}_2(\aleph_\omega)$ holds, then we obtain that $\mathfrak{nc}\le\aleph_\omega$. Moreover, we believe that the following has positive answer:

PROBLEM 3. Is it consistently true that $\mathfrak{nc}=\aleph_2=(2^{\aleph_0})^+$?

Now we pass to discuss some problems concerning the cardinal number $\mathfrak{nc}_0$. We know, by the construction in [2], that $\mathfrak{nc}_0>\aleph_1$. A similar proof that the corresponding of Theorem 2.9 gives the following.

THEOREM 2.11. ([10]) *Suppose that* $\mathrm{Pol}(\kappa)$ *holds. Then* $\mathfrak{nc}_0\le\kappa$.

Interestingly, the next recent result distinguishes the behavior of $\mathfrak{nc}$ and $\mathfrak{nc}_0$, by assuming a positive answer for the Banach Measure Extension Problem.

THEOREM 2.12. ([25]) *Suppose that the Lebesgue measure extends to a total countably additive measure. Then every non-trivial weakly-null sequence* *of length $2^{\aleph_0}$ has an unconditional basic subsequence. In particular, every reflexive space of density $2^{\aleph_0}$ has an unconditional basic sequence.*

The proof is based on a relationship between the possibility of extending the Lebesgue measure to a total countably additive one, and the combinatorial principle $\mathrm{Pol}(2^{\aleph_0})$. The reader can find the complete proof in [25]. To finish this part, we take a more direct path, and we explain a measure-theoretical approach to guarantee the existence of an unconditional basic subsequence, that we believe may lead to a further interesting research. So, suppose that $(x_\alpha)_{\alpha<\kappa}$ is a normalized weakly null sequence in some space X. Suppose also that κ carries a countably additive probability measure μ defined on some σ-field of subsets of κ that gives measure zero to all countable subsets of κ. Suppose also that all norm-configurations induced by subsets of the finite power κ^n are measurable relative to the power measure μ^n. Let us see how to find an unconditional basic subsequence of $(x_\alpha)_{\alpha<\kappa}$: For each finite set $s\subseteq\kappa$, choose a countable subset $N_s\subseteq S_{X^*}$ 1-norming $\langle x_\alpha\rangle_{\alpha\in s}$. Let $\theta:[\kappa]^{<\omega}\to[\kappa]^{\le\omega}$ be defined for $s\in[\kappa]^{<\omega}$ by

$$\theta(s):=\left\{\alpha<\kappa\,:\,\text{there is some } f\in N_s \text{ such that } f(x_\alpha)\neq 0\right\}.$$

Let us call a finite subset $s\subseteq\kappa$ *θ-free* if

$$\theta(t)\cap s\subseteq t \qquad \text{for every } t\subseteq s\,.$$

Let $F_n\subseteq[\kappa]^n$ be the set of θ-free sequences of cardinality n. By hypothesis, each F_n is μ^n-measurable. We see now, using a Fubini argument, that indeed $\mu_n(F_n)=1$: Suppose otherwise that $\mu_n([\kappa]^n\setminus F_n)>0$. For each $I\subsetneq n$, and $j\in n\setminus I$, let

$$S_{I,j}:=\left\{s\in[\kappa]^n\,:\,(s)_j\in\theta(s[I])\right\},$$

where $s[I]=\{\varpi(i)\,:\,i\in I\}$ for $\varpi:|s|\to s$ being the unique order-preserving bijection between $|s|$ and s. Let $I_0\subsetneq n$ and $j_0\in n\setminus I_0$ be such that $\mu_n(S_{I_0,j_0})>0$. Set $J=I_0\cup\{j_0\}$, and let $\pi_J:[\kappa]^n\to[\kappa]^J$ be the canonical projection $\pi_J(s)=s[J]$. It follows that $\mu_m(\pi_J(S_J))>0$, where $m=|J|$. By Fubini's Theorem, the set of $t\in[\kappa]^{m-1}$ such that $\mu((\pi_J(S_J))_t)>0$, has μ_{m-1}-positive measure, where $(\pi_J(S_J))_t:=\{\alpha<\kappa\,:\,t\cup\{\alpha\}\in\pi_J(S_J)\}$. But $(\pi_J(S_J))_t\subseteq\theta(t)$, so $(\pi_J(S_J))_t$ is countable, and hence $\mu(\pi_J(S_J))_t=0$, a contradiction.

Observe now that if $s\in[\kappa]^n$ is θ-free then it is 1-unconditional: Let $(a_\alpha)_{\alpha\in s}$ be a sequence of scalars and fix $t\subseteq s$. Let $f\in N_t$ be such that

$$f\left(\sum_{\alpha\in t}a_\alpha x_\alpha\right)=\left\|\sum_{\alpha\in t}a_\alpha x_\alpha\right\|. \tag{9}$$

Since s is θ-free, it follows that $f(t)\cap s \subseteq t$. This means that for every $\alpha \in s\setminus t$ one has that $f(x_\alpha) = 0$. So,

$$\left\|\sum_{\alpha\in s} a_\alpha x_\alpha\right\| \geq f\left(\sum_{\alpha\in s} a_\alpha x_\alpha\right) = f\left(\sum_{\alpha\in t} a_\alpha x_\alpha\right) = \left\|\sum_{\alpha\in t} a_\alpha x_\alpha\right\|. \qquad (10)$$

Since the sets $U_n := \{s \in [\kappa]^n : (x_\alpha)_{\alpha\in s} \text{ is 1-unconditional}\}$ have μ_n-measure 1 for every n, it follows that the set of sequences $(\alpha_n)_{n\in\omega}$ such that $(x_{\alpha_n})_{n\in\omega}$ is 1-unconditional has μ_ω-measure 1, where μ_ω denote the infinite product measure on κ^ω, and we are done.

On the other direction, it is possible to generalize the Maurey-Rosenthal construction [27] of a normalized weakly-null sequence without unconditional subsequences to long weakly-null sequences of length $\aleph_n$, for every $n \in \mathbb{N}$. This gives

Theorem 2.13. ([25]) $\mathfrak{nc}_0 \geq \aleph_\omega$.

Its proof is based on the study of the chromatic number of certain graphs in $\aleph_n$ defined by the method of minimal walks.

Problem 4. Is it consistently true that $\mathfrak{nc}_0 > \aleph_\omega$?

Notice that in [10] it is proved that it is consistent relative to the existence of a measurable cardinal that $\mathfrak{nc}_0 \leq \aleph_\omega$.

2.3. Subsymmetric sequences.

The goal is to understand the following cardinal number.

Definition 2.14. Let

$$\mathfrak{ns} := \sup\left\{\kappa \ : \begin{array}{c}\text{every space of density } \kappa \text{ has a}\\ \text{subsymmetric basic sequence}\end{array}\right\}.$$

We know that $2^{\aleph_0} < \mathfrak{ns} \leq$ first ω-Erdös cardinal (see Definition 2.23), by Odell [29] and Ketonen [21] results, respectively. We start with the following simple asymptotic result.

Proposition 2.15. *Every basic sequence has a subsequence which is asymptotically subsymmetric. Moreover for every $C > 1$ there is a C-asymptotically subsymmetric subsequence.*

Proof. The main idea here is that, given $C > 1$ and $n \in \mathbb{N}$ there is a finite list of finite basic sequences such that any other finite basic sequence of length n is C-equivalent to one of the sequences in that list. ∎

The link between subsymmetric and unconditional sequences is the following.

PROPOSITION 2.16. *Suppose that $(x_n)_{n<\omega}$ is a subsymmetric sequence. Then either $(x_n)_n$ has a subsequence equivalent to the unit basis of ℓ_1 or else there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $(x_{n_{2k+1}} - x_{n_{2k}})_k$ is an unconditional sequence.*

Proof. We apply Rosenthal ℓ_1 dichotomy to the sequence $(x_n)_{n\in\omega}$. The first alternative is to find a subsequence equivalent to the unit basis of ℓ_1. The second one that is that there is a subsequence $(x_{m_k})_{k<\omega}$ which is weakly-Cauchy. It is not true in general that a block subsequence of a subsymmetric sequence is subsymmetric, but obviously the different sequence $(x_{m_{2k+1}} - x_{m_{2k}})_k$ is subsymmetric, and in this case also weakly-null. Now the result readily follows from Proposition 2.2. ∎

Consequently,

COROLLARY 2.17. $\mathfrak{nc}_{\mathrm{refl}} \le \mathfrak{ns}_{\mathrm{refl}}$.

PROBLEM 5. Is it true that $\mathfrak{nc}_{\mathrm{refl}} < \mathfrak{ns}_{\mathrm{refl}}$?

It is worth to point out that Odell's non-separable space from [29] without subsymmetric sequences is a dual of a separable space, so non-reflexive. In our understanding, there are no constructions of non-separable reflexive spaces without subsymmetric sequences but with unconditional sequences. For separable spaces, the typical example is the Tsirelson space, whose most common presentation is based on the Schreier family. In looking for a natural generalization of this construction to the non-separable context, we introduce the following generalization of the Schreier family:

DEFINITION 2.18. Let κ be a cardinal number. A family $\mathcal{B}$ of finite subsets of κ is called

(a) *hereditary* if $s \subseteq t \in \mathcal{B}$ implies that $s \in \mathcal{B}$;
(b) *compact* if it is compact when $\mathcal{B}$ is identified with the subset of 2^κ consisting on characteristic functions $\mathbf{1}_s$ of sets $s \in \mathcal{B}$;

(c) *large* if $\mathcal{B} \cap [A]^n \neq \emptyset$ for every infinite subset A of κ and every integer n.

So compact and large families are those that they have "asymptotic copies" of every infinite set yet no copy of the infinite set. Keeping this simple idea in mind, one might imagine that such families can be used to define for example spaces not having copies of ℓ_1, yet ℓ_1 will appear asympotically.

A typical example of compact, hereditary and large family is the Schreier family

$$\mathcal{S} := \{s \subseteq \omega \, : \, |s| \leq \min s + 1\} .$$

We are going to see that compact, hereditary and large families exists for many uncountable cardinal numbers (see Proposition 2.24).

DEFINITION 2.19. A long basic sequence $(x_\alpha)_\alpha$ of a Banach space $(X, \|\cdot\|)$ is called asymptotic ℓ_1-space, hereditary and large family $\mathcal{B}$ when there is a constant $C \geq 1$ such that for every block sequence $(y_n)_n$ of $(x_\alpha)_\alpha$ we have that

$$\left\| \sum_{n \in s} y_n \right\| \geq \frac{1}{C} \sum_{n \in s} \|y_n\|$$

for every $s \in \mathcal{S}$. A space is called asymptotic ℓ_1 when it has a long basis which is asymptotic ℓ_1-space.

It is easy to see that the Tsirelson space is asymptotic ℓ_1. Let us see why it does not have subsymmetric sequences.

PROPOSITION 2.20. *Suppose that X is an asymptotic ℓ_1 space without copies of ℓ_1. Then X does not have subsymmetric basic sequences.*

Proof. Let $(e_\gamma)_{\gamma<\kappa}$ be a Schauder basis of X with basic constant $C \geq 1$, and let $K \geq 1$ be a constant witnessing that $(e_\gamma)_{\gamma<\kappa}$ is asymptotic ℓ_1. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a basic sequence in X, and let $D \geq 1$. We use first a gliding-hump argument: Let γ_0 be the first ordinal number $\gamma < \kappa$ such that there is $\varepsilon > 0$ and some infinite subset $M \subseteq \mathbb{N}$ such that

$$\|P_\gamma x_n\| \geq \varepsilon \qquad \text{for every } n \in M . \tag{11}$$

Fix also $\varepsilon_0 > 0$ witnessing that γ_0 has the property in (11). It follows that we can find some infinite subsequence $(x_n)_{n\in N}$ of $(x_n)_{n\in M}$ such that $(P_{\gamma_0}(x_n))_{n\in N}$ is 2-equivalent to a seminormalized block subsequence $(y_n)_{n\in N}$ of $(e_\gamma)_\gamma$ with

$\|y_n\| \geq \varepsilon_0$ for every $n \in N$. Since the space X does not contain ℓ_1, there are scalars $(a_n)_{n\in s}$, $s \subseteq N$, such that

$$\sum_{n\in s} |a_n| = 1 \qquad \text{and} \qquad \Big\|\sum_{n\in s} a_n x_n\Big\| \leq \frac{\varepsilon_0}{3KCD}\,.$$

Since $(e_\gamma)_{\gamma<\kappa}$ is asymptotic ℓ_1, it follows that, fixing $t \in \mathcal{S} \restriction N$ such that $|t| = |s|$, we obtain that

$$\Big\|\sum_{n\in t} a_{\varpi_{t,s}(n)} y_n\Big\| \geq \frac{1}{K}\varepsilon_0 \sum_{n\in s} |a_n| = \frac{\varepsilon_0}{K}\,. \tag{12}$$

Consequently,

$$\Big\|\sum_{n\in t} a_{\varpi_{t,s}(n)} x_n\Big\| \geq \frac{1}{C}\Big\|P_{\gamma_0} \sum_{n\in t} a_{\varpi_{t,s}(n)} x_n\Big\| \geq \frac{\varepsilon_0}{2CK} > D\Big\|\sum_{n\in s} a_n x_n\Big\|\,, \tag{13}$$

so $(x_n)_n$ is not subsymmetric with constant D. Since $D \geq 1$ was arbitrary, we are done. ∎

So in order to prove that $\mathfrak{ns} > \kappa$, one could try to find a reflexive space with an asymptotic ℓ_1 basis of length κ. However this approach does not work.

PROPOSITION 2.21. *Suppose that $(x_\alpha)_{\alpha<\gamma}$ is asymptotic ℓ_1 basic sequence. Then either $\gamma = \omega$ or else $(x_\gamma)_{\omega\leq\alpha<\gamma}$ is equivalent to the corresponding ℓ_1-unit basis. In particular, there are no non-separable reflexive asymptotic ℓ_1-spaces.*

Proof. The proof is very simple: Suppose that $\gamma > \omega$, let $C \geq 1$ be the constant witnessing that $(x_\alpha)_{\alpha<\gamma}$ is asymptotic ℓ_1. Fix a finite subset $\omega \leq \alpha_0 < \cdots < \alpha_n < \gamma$. Let $(y_k)_k$ be the sequence defined as $y_i = x_i$ for $i < n$; $y_{n+i} := x_{\alpha_i}$ for $i \leq n$ and $y_{2n+i+1} := x_{\alpha_n+i+1}$ for every $i \in \omega$. Then $(y_k)_{k\in\omega}$ is a normalized block sequence of $(x_\alpha)_\alpha$. Since this sequence is asymptotic it follows that

$$\Big\|\sum_{i=0}^{n} a_i x_{\alpha_i}\Big\| = \Big\|\sum_{i=n}^{2n} a_i y_i\Big\| \geq \frac{1}{C}\sum_{i=0}^{n} |a_i|\,.$$

∎

Still, formally one can define, given a compact, hereditary and large family on an uncountable cardinal κ, a norm $\|\cdot\|$ on $c_{00}(\kappa)$, as for the Tsirelson space in (6), now with respect to the family $\mathcal{B}$. The corresponding completion has an unconditional basis of length κ, but it is proved in [25] that the space contains a copy of $\ell_1(\kappa)$.

We finish the discussion on subsymmetric sequences with the sequence version of $\mathfrak{ns}$.

Definition 2.22. Let $\mathfrak{ns}_0$ be the minimal cardinal number κ such that every non trivial weakly-null sequence $(x_\alpha)_{\alpha<\kappa}$ has a subsymmetric basic subsequence.

It is proved in [2] that $\mathfrak{ns}_0 > \aleph_1$. In contrast with the number $\mathfrak{ns}$, the situation for $\mathfrak{ns}_0$ is completely understood.

Definition 2.23. A cardinal number κ is called ω-Erdös when

$$\kappa \to (\omega)_2^{<\omega},$$

i.e., whenever $c : [\kappa]^{<\omega} \to 2$ there is an infinite c-homogeneous subset $A \subseteq \kappa$, i.e., such that $c \restriction [A]^n$ is constant for every $n < \omega$.

It is not difficult to see that the definition above is equivalent to $\kappa \to (\omega)_{\mathfrak{c}}^{<\omega}$.

Proposition 2.24. ([25]) *Let κ be an infinite cardinal. The following are equivalent:*

(1) *κ is ω-Erdös;*

(2) *$\mathfrak{ns}_0 \le \kappa$, i.e., every non-trivial weakly-null sequence $(x_\alpha)_{\alpha<\kappa}$ has a subsymmetric basic subsequence;*

(3) *there are no large compact and hereditary families on κ.*

Proof. (Sketch) (1) implies (2) was proved first by Ketonen: Let C be a set of finite basic sequences in some Banach space (for example $C[0,1]$) such that every finite linearly independent sequence in a Banach space is 1-equivalent to some sequence in C, and of cardinality $|C| = \mathfrak{c}$. Let $c : [\kappa]^{<\omega} \to C$ be defined for $s \in [\kappa]^{<\omega}$ by $c(s) \in C$ such that $(x_\alpha)_{\alpha\in s}$ is 1-equivalent to $c(s)$. Then if $A \subseteq \kappa$ is c-homogeneous, then $(x_\alpha)_{\alpha\in A}$ is subsymmetric. Now, by Mazur's result, there is some $B \subseteq A$ such that $(x_\alpha)_{\alpha\in B}$ is in addition basic.

(2) implies (3): Let $\mathcal{B}$ be a large compact and hereditary family on κ. Define on $c_{00}(\kappa)$ the following norm:

$$\|x\|_{\mathcal{B}} := \sup_{s\in\mathcal{B}}\langle x, \mathbf{1}_s\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $c_{00}(\kappa)$. Then the unit Hamel basis $(u_\gamma)_{\gamma<\kappa}$ is a 1-unconditional Schauder basis of the completion of $(c_{00}(\kappa), \|\cdot\|_{\mathcal{B}})$. In addition, it follows from Ptak's Lemma that $(u_\gamma)_{\gamma<\kappa}$ is weakly-null, and by the fact that $\mathcal{B}$ is large and again Ptak's Lemma, it follows that $(u_\gamma)_{\gamma<\kappa}$ does not have subsymmetric basic sequences.

(3) implies (1): Suppose that κ is not ω-Erdös, and let $c : [\kappa]^{<\omega} \to 2$ witnessing that. Let $\mathcal{B}$ be the family of c-homogeneous subsets of κ. Then this is a compact and hereditary family, since c does not have infinite homogeneous sets. In addition, it follows from the finite Ramsey Theorem that $\mathcal{B}$ is large. ∎

3. Uncountable special sequences

The last section is devoted to the existence of certain uncountable sequences. To motivate this search, we recall the following natural variation of the well-known separable quotient problem.

Problem 6. Let X be an infinite dimensional Banach space. Does there exists a quotient of X with a Schauder basis of length the density of X?

This is true if X is separable (Johnson-Rosenthal [20]). This result was later extended by Todorcevic [37] who proved that assuming a standard Baire category principle, the previous problem had a positive answer for spaces of densities at most $\aleph_1$.

Related to this are the existence of the following sequences.

Definition 3.1. A sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa}$ of pairs $(x_\alpha, f_\alpha) \in X \times X^*$ is called an *ε-biorthogonal system* $(0 \le \varepsilon < 1)$ if $f_\alpha(x_\alpha) = 1$ for every $\alpha < \kappa$ and $|f_\alpha(x_\beta)| \le \varepsilon$ for every $\beta \ne \alpha$. The sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa}$ is called an *almost-biorthogonal system* if it is ε-biorthogonal for some $0 \le \varepsilon < 1$, and it is called *biorthogonal* when it is 0-biorthogonal.

Notice that if X has a quotient with a Schauder basis of length κ, then X has automatically a biorthogonal system of length κ. As it is well known, assuming certain set-theoretical axioms, there are examples of non-separable

Banach spaces without uncountable biorthogonal systems: One such a space was given by Kunen (see [28]), who proved that under the Continuum Hypothesis (CH) there is a non-metrizable scattered compacta K which is hereditarily separable in all its powers. This was later improved by Todorcevic [36, Chapter 1] who constructed a compactum with the same properties using the weaker assumption $\mathfrak{b} = \aleph_1$, which today appears to be optimal. It follows that the corresponding space of continuous functions $C(K)$ is a non-separable c_0-saturated Banach space without uncountable biorthogonal systems, indeed $(C(K); \text{weak})^n$ is Hereditarily Lindelöf (HL) in all its finite powers. Slightly before Kunen's construction (see the remark of Johnson in [37]) Shelah in [35], assuming the Diamond principle ($\diamondsuit$), proved that there is a non-separable Banach space S which is Gurarij and such that $(S; \text{weak})^n$ is again (HL) for all $n \in \mathbb{N}$. The densities of these two examples are $\aleph_1$. Brech and Koszmider [7] gave the first example of a Banach space, indeed a $C(K)$ space, of density $\aleph_2$ without uncountable biorthogonal systems. Their proof uses a generic construction.

With the purpose of understanding a general approach to build this kind of spaces, we pass now to explain the general methodology introduced in [24] to give non-separable generic spaces. The construction in [24] is done using forcing but, with the permission of authors, we reproduce here a direct construction (which is, in fact, fairly easy given the crucial notions given below) from a paper in preparation [26] of a non-separable space X with the following properties:

(a) X is an increasing union $X = \bigcup_{\alpha<\omega_1} X_\alpha$ of separable spaces X_α isomorphic to c_0.

(b) X does not have uncountable biorthogonal systems (indeed $(X, \text{weak})^n$ is (HL) for all $n \in \mathbb{N}$).

Definition 3.2. An *homogeneous* family is a family $\mathcal{F}$ consisting on finite subsets of ω_1 with the following properties:

(1) $\mathcal{F} \upharpoonright \lambda$ is $\subseteq$-cofinal in λ and $\subseteq$-directed for every $\lambda < \omega_1$ limit.

(2) For every $s, t \in \mathcal{F}$, if $|s| = |t|$, then the unique order-preserving bijection $\theta : s \to t$ between s and t is an $\subseteq$-isomorphism between $\mathcal{F} \upharpoonright s$ and $\mathcal{F} \upharpoonright t$.

(3) For every $s \in \mathcal{F}$ the set $(\mathcal{F} \upharpoonright s)^{(-)}$ of immediate $\subseteq$-predecessors of s in $\mathcal{F}$ forms a Δ-system of finite sets of the same cardinality.

Given a set $s \in \mathcal{F}$, we say that $s = s_0 \cup \cdots \cup s_{n(s)}$ is the canonical decomposition of s when $\{s_i\}_{i \leq n(s)}$ is the set of immediate predecessors of s ordered by $s_i < s_j$ iff $\max s_i < \max s_j$.

We pass now to define the normed space. We use $x \vee y$ to denote the supremum of the two vectors $x, y \in c_{00}(\omega_1)$ defined by $(x \vee y)_\gamma := \max{(x)_\gamma, (y)_\gamma}$. Given two finite subsets $s, t \subseteq \omega_1$ of the same cardinality, we write $\theta_{s,t} : s \to t$ to denote the unique order-preserving bijection between s and t, and we use also $\theta_{s,t}$ as the isomorphism between $c_{00}(s)$ and $c_{00}(t)$ linearly defined by $\theta_{s,t}(u_\gamma) = u_{\theta_{s,t}(\gamma)}$.

Definition 3.3. Let $\mathcal{F}$ be an homogeneous family on ω_1. We define inductively on $s \in \mathcal{F}$ a subset $\{h_\gamma^{(s)}\}_{\gamma \in s} \subseteq c_{00}(s)$ as follows:

(a) Suppose that $s = \{\gamma\}$. Then let $h_\gamma^{(s)} := u_\gamma$.

(b) Suppose that $|s| > 1$. Let $s = s_0 \cup \cdots \cup s_n$ be the canonical decomposition of s. Given $\gamma \in s$, we define

$$h_\gamma^{(s)} := \begin{cases} \bigvee_{i \leq n} h^{(s_i)}_{\theta_{s_0,s_i}(\gamma)} & \text{if } \gamma \in s_0\,, \\ h_\gamma^{(s_i)} & \text{if } \gamma \in s_i \setminus s_0, \text{ for some } 1 \leq i \leq n\,. \end{cases}$$

It follows that

(c) if $t \subseteq s$ are both in $\mathcal{F}$, then $h_\gamma^{(t)} \restriction c_{00}(s) = h_\gamma^{(s)}$;

(d) $(h_\gamma^{(s)})_\delta \in \{0,1\}$, $(h_\gamma^{(s)})_\gamma = 1$ while $(h_\gamma^{(s)})_\eta = 0$ for every δ, and $\eta < \gamma \in s \in \mathcal{F}$;

(e) $\|x\|_{\mathcal{F}} = \max_{\gamma \in s}\{|\langle h_\gamma^s, x\rangle\}$ for every $x \in c_{00}(s)$ and $s \in \mathcal{F}$.

(f) if $s, t \in \mathcal{F}$ are such that $|s| = |t|$, then

$$\theta_{s,t} h_\gamma^s = h^{(t)}_{\theta_{s,t}(\gamma)} \qquad \text{for every } \gamma \in s\,.$$

We can now define $h_\gamma := \bigvee_{\gamma \in s \in \mathcal{F}} h_\gamma^{(s)}$ for every $\gamma < \omega_1$. This is well defined because the family $\mathcal{F}$ is cofinal, and (c) above. Now for $x \in c_{00}(\omega_1)$ we can define

$$\|x\|_{\mathcal{F}} := \sup\big\{|\langle x, h_\gamma\rangle| \,:\, \gamma < \omega_1\big\}.$$

It follows from (d) that $\|\cdot\|_{\mathcal{F}}$ is a norm on $c_{00}(\omega_1)$. Let $X_{\mathcal{F}}$ be the completion of $(c_{00}(\omega_1), \|\cdot\|_{\mathcal{F}})$. Since for $\eta < \gamma < \omega_1$ we have that $\|u_\gamma - u_\eta\| \geq \langle h_\gamma, u_\gamma - u_\eta\rangle =$

1, it also follows that $X_{\mathcal{F}}$ is non-separable. For each λ limit, let $X_{\mathcal{F}}^{\gamma}$ be the closed linear span of $\{u_\gamma\}_{\gamma<\lambda}$. Then the space $X_{\mathcal{F}}^{\lambda}$ is isomorphic to c_0: Since $\mathcal{F} \upharpoonright \lambda$ is cofinal in λ, we can find first an strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathcal{F} \upharpoonright \lambda$ such that $\bigcup_{n\in\mathbb{N}} s_n = \lambda$. Now by filling the gaps if needed, and changing the enumeration we may assume that s_n is an immediate predecessor of s_{n+1} for every n. Then for each $n \in \mathbb{N}$ one can find a 1-normalized basis $(x_{\gamma_i})_{i<|s_n|}$, $\{\gamma_i\}_{i<|s_n|} = s_n$, of $(c_{00}(s_n), \|\cdot\|_{\mathcal{F}})$ such that $\{h_\gamma^{(s_n)}\}_{\gamma\in s_n} \subseteq \{f_\gamma, f_\gamma - f_\eta\}_{\gamma,\eta\in s_n}$, where $(f_\gamma)_{\gamma\in s_n}$ is the biorthogonal to $(x_\alpha)_{\alpha\in s_n}$. It follows then that

$$\frac{1}{2}\max_{\gamma\in s_n}|a_\gamma| \le \left|\left\langle h_\gamma^{(s_n)}, \sum_{\gamma\in s_n} a_\gamma x_\gamma \right\rangle\right| \le 2\max_{\gamma\in s_n}|a_\gamma|\,,$$

the first inequality because $(x_{\gamma_n})_{n<|s_n|}$ is a 1-basis.

The following property of the family is crucial for the space $X_{\mathcal{F}}$ to be (KS).

Definition 3.4. We say that a regular family $\mathcal{F}$ on ω_1 is *capturing* if for every uncountable sequence $(t_\alpha)_{\alpha<\omega_1}$ of subsets of ω_1 and every $n \in \mathbb{N}$ there is $s \in \mathcal{F}$ and $\alpha_0 < \cdots < \alpha_{n-1}$ such that

(1) $n \le n(s)$,

(2) $t_{\alpha_i} \subseteq s_i$ for every $i < n$;

(3) $\theta_{s_i,s_j}(t_{\alpha_i}) = t_{\alpha_j}$ for every $i \le j < n$.

The existence of capturing families follows from the diamond principle (see [26] for more details). Then if $\mathcal{F}$ is capturing, we obtain that $(X_{\mathcal{F}}, \mathrm{weak})^n$ is (HL) in all its finite powers: We only sketch how to prove that $X_{\mathcal{F}}$ does not have uncountable biorthogonal sequences. And to do this we prove the stronger fact that for every uncountable normalized sequence $(x_\alpha)_{\alpha<\omega_1}$ of vectors in $c_{00}(\omega_1)$ and every $n \in \mathbb{N}$ there is $\alpha_0 < \cdots < \alpha_n$ such that

$$\left\| x_{\alpha_0} - \frac{1}{n}\sum_{i=1}^{n} x_{\alpha_i} \right\| \le \frac{1}{n}\,. \tag{14}$$

Let $s_\alpha := \operatorname{supp} x_\alpha$ for every $\alpha < \omega_1$. By passing to an uncountable subsequence if needed we may assume that $|s_\alpha| = |s_\beta|$ and that $\theta_{s_\alpha,s_\beta}(x_\alpha) = x_\beta$ for every $\alpha, \beta < \omega_1$. Now we use that $\mathcal{F}$ is capturing to find $\alpha_0 < \cdots < \alpha_n < \omega_1$, and $s \in \mathcal{F}$ such that (1), (2) and (3) above hold. Then, $x_{\alpha_j} = \theta_{s_i,s_j}(x_{\alpha_i})$ for every

$i, j \leq n$. We check that (14) holds. Notice that $x_{\alpha_i} \in c_{00}(s)$, so by (e), it suffices to prove that

$$\left| \left\langle h_\gamma^s \,,\, x_{\alpha_0} - \frac{1}{n} \sum_{i=1}^n x_{\alpha_i} \right\rangle \right| \leq \frac{1}{n}\,.$$

Let $\gamma \in s$. Suppose first that $\gamma \in s$. Then

$$\begin{aligned}\left\langle h_\gamma^s \,,\, x_{\alpha_0} - \frac{1}{n} \sum_{i=1}^n x_{\alpha_i} \right\rangle &= \langle h_\gamma^{s_0} \,,\, x_{\alpha_0} \rangle - \frac{1}{n} \sum_{i=1}^n \left\langle h^{(s_i)}_{\theta_{s_0,s_i}(\gamma)} \,,\, x_{\alpha_i} \right\rangle \\ &= \langle h_\gamma^{s_0} \,,\, x_{\alpha_0} \rangle - \frac{1}{n} \sum_{i=1}^n \left\langle \theta_{s_0,s_i}(h_\gamma^{(s_0)}) \,,\, \theta_{s_0,s_i}(x_{\alpha_0}) \right\rangle \\ &= \langle h_\gamma^{s_0} \,,\, x_{\alpha_0} \rangle - \frac{1}{n} \sum_{i=1}^n \left\langle h_\gamma^{(s_0)} \,,\, x_{\alpha_0} \right\rangle \;=\; 0\,.\end{aligned}$$

Suppose now that $\gamma \in s_{i_0} \setminus s_0$. Then

$$\begin{aligned}\left| \left\langle h_\gamma^s \,,\, x_{\alpha_0} - \frac{1}{n} \sum_{i=1}^n x_{\alpha_i} \right\rangle \right| &= \left| \langle h_\gamma^{s_{i_0}} \,,\, x_{\alpha_0} \rangle - \frac{1}{n} \sum_{i=1}^n \left\langle h^{(s_i)}_{\theta_{s_0,s_i}(\gamma)} \,,\, x_{\alpha_i} \right\rangle \right| \\ &= \left| \frac{1}{n} \langle h_\gamma^{s_{i_0}} \,,\, x_{\alpha_{i_0}} \rangle \right| \;\leq\; \frac{1}{n}\,.\end{aligned}$$

Notice that all known examples of non-separable (HS) spaces contain a copy of c_0. So, it is natural to ask the following:

PROBLEM 7. Does there exists a non-separable space without uncountable biorthogonal systems and copies of c_0? Even more, does there exist a non-separable space X such that $(X, \mathrm{weak})^n$ is (HL) in all its finite powers and without copies of c_0?

Some other interesting questions concerning uncountable biorthogonal sequences:

PROBLEM 8. Is it true that if X has an uncountable Markushevich system then it has an uncountable Schauder basic sequence?

Recall that a Markushevich system is a double sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa}$ in $X \times X^*$ which is biorthogonal and such that $\langle x_\alpha \rangle_\alpha$ is dense in X, and such that $(f_\alpha)_\alpha$ separates points of X. It seems that the techniques introduced in [24] does not help to solve this problem.

DEFINITION 3.5. Recall that a sequence $(x_\alpha)_{\alpha<\kappa}$ in a Banach space X is called ω-independent if for any given subsequence $(x_{\alpha_n})_{n<\omega}$ of $(x_\alpha)_{\alpha<\kappa}$, the equation $\sum_n a_n x_{\alpha_n} = 0$ implies $a_n = 0$ for every $n < \omega$.

The simpler example of an ω-independent sequence is obviously a biorthogonal sequence. Sersouri [34] proved that separable Banach spaces do not have uncountable ω-independent sequences.

PROBLEM 9. Is it true that if X has an uncountable ω-independent family, then X has an uncountable biorthogonal sequence?

Recall that in [24] there is the construction of a space without uncountable biorthogonal systems but with uncountable ε-biorthogonal systems for all $\varepsilon > 0$. This space is related to the Problem 9 since it was proved in [16, Proposition 32, p. 108] that if $(x_\alpha)_{\alpha<\omega_1}$ is ω-independent, then for every $\varepsilon > 0$ there is an uncountable subset $\Gamma_\varepsilon \subseteq \omega_1$ and functionals $(f_\alpha^{(\varepsilon)})_{\alpha\in\Gamma_\varepsilon}$ such that $(x_\alpha, f_\alpha^{(\varepsilon)})_{\alpha\in\Gamma_\varepsilon}$ is an ε-biorthogonal sequence.

PROBLEM 10. Does there exist a non-separable Gurarij space with a long Schauder basis?

We finish this note with a problem concerning elastic spaces. Recall now the following problem

PROBLEM 11. Suppose that X is a Banach space such that

$$d_{\mathrm{BM}}(X, Y) < \infty\,, \tag{15}$$

where Y is an isomorphic copy of X and $d_{\mathrm{BM}}(X, Y)$ is the Banach-Mazur distance between X and Y. Is then X finite dimensional?

Johnson and Odell recently proved [19] that the answer is affirmative, assuming that X is separable. Later on, Godefroy [13] extended their result to the case of spaces with an uncountable ω-independent sequence. The key for the Godefroy's result is that from the existence of uncountable ε-biorthogonal sequences for every $\varepsilon > 0$ one can naturally define for each n a renorming $X_n = (X, \|\cdot\|_n)$ such that $d_{\mathrm{BM}}(X, x_n) \geq n$. Perhaps the existence of a single uncountable ε-biorthogonal sequence would give the same conclusion. In relation with that, a good space to test this is the generic space provided in [24, Theorem 4.5. (II)] that does not have uncountable ω-independent families but it has an uncountable ε-biorthogonal sequence.

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