Property (V) Still Fails the Three-Space Property

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Abstract: We construct twisted sums of \( C[0, 1] \) with itself in which the quotient map does not fix any copy of either \( C[0, 1] \) or \( c_0 \). We moreover show that every twisted sum of \( c_0(\Gamma) \) must have Pełczyński’s property (V).

Key words: Banach spaces, property (V), twisted sums.


1. Introduction and preliminaries

In [13], Pełczyński discovered that \( C(K) \) spaces enjoy the property that all operators on them are either weakly compact or an isomorphism on a copy of \( c_0 \); he called this property (V). Thus, a Banach space \( X \) has property (V) if every operator from \( X \) into any other Banach space is either weakly compact or an isomorphism on some copy of \( c_0 \). Pełczyński’s property (V) is not a 3-space property as it was shown in [5] –see also [6]–. The counterexample there presented required a careful inspection of a quite involved example of Ghoussoub and Johnson [8]. In this note we present a rather natural example: there is an exact sequence

\[
0 \rightarrow C[0, 1] \rightarrow \mathfrak{S} \rightarrow C[0, 1] \rightarrow 0 \tag{1}
\]

in which the operator \( q_\mathfrak{S} \) is not an isomorphism on any copy of \( c_0 \), and therefore the space \( \mathfrak{S} \) fails to have Pełczyński’s property (V). In [3] it was shown that for every separable Banach space \( X \) not containing \( \ell_1 \) there exists an exact sequence

\[
0 \rightarrow C[0, 1] \rightarrow \Omega(X) \rightarrow q_X \rightarrow X \rightarrow 0
\]

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in which \( q_X \) is a strictly singular operator. Therefore, the choice \( X = \ell_1 \) yields an exact sequence

\[
0 \longrightarrow C[0,1] \longrightarrow \Omega(c_0) \longrightarrow q_0 \longrightarrow 0
\]

in which the space \( \Omega(c_0) \) fails property (V), providing in this way another counterexample to the 3-space problem for property (V). Moreover, from (2) it follows the existence of an exact sequence

\[
0 \longrightarrow C[0,1] \longrightarrow \Omega(c_0) \oplus C[0,1] \longrightarrow C[0,1] \longrightarrow 0
\]

in which the space \( \Omega(c_0) \oplus C[0,1] \) necessarily fails property (V) although the operator \( Q \) is an isomorphism on a copy of \( C[0,1] \).

An exact sequence 0 → Y → X → Z → 0 of Banach spaces is a diagram formed by Banach spaces and linear continuous operators in which the kernel of each operator coincides with the image of the preceding; the middle space \( X \) is also called a twisted sum of \( Y \) and \( Z \). By the open mapping theorem this means that \( Y \) is a subspace of \( X \) and \( Z \) is the corresponding quotient. An exact sequence is said to split if it is equivalent to the trivial sequence 0 → Y → Y ⊕ Z → Z → 0. There is a correspondence (see [10, 11, 6]) between exact sequences 0 → Y → X → Z → 0 of Banach spaces and the so-called \( z \)-linear maps which are homogeneous maps \( F : Z \rightarrow Y \) (we use this notation to stress the fact that these are not linear maps) with the property that there exists some constant \( C > 0 \) such that for all finite sets \( x_1, \ldots, x_n \in Z \) one has

\[
\|F(\sum_{n=1}^{N} x_n) - \sum_{n=1}^{N} F(x_n)\| \leq C \sum_{n=1}^{N} \|x_n\|.
\]

The infimum of those constants \( C \) is called the \( z \)-linear constant of \( F \) and denoted \( Z(F) \). Given a \( z \)-linear map \( F : Z \rightarrow Y \) the twisted sum space \( Y \oplus_F Z \) is by definition the vector space \( Y \times Z \) endowed with the quasi-norm \( \|(y,z)\|_F = \|y - Fz\| + \|z\| \). The \( z \)-linearity of \( F \) makes this quasi-norm \( Z(F) \)-equivalent to a norm (see [4]). A \( z \)-linear map \( F : X \rightarrow Y \) induces an exact sequence of Banach spaces 0 → Y → Y ⊕_F X → X → 0. Indeed, the map \( y \mapsto (y,0) \)
embeds \( Y \) isometrically into \( Y \oplus_F X \) and \( (y, x) \mapsto x \) extends to a quotient map \( q : Y \oplus_F X \to X \) whose kernel is \( Y \). Two quasi-linear maps \( F, G : X \to Y \) are said to be equivalent—or that one is a version of the other—when the difference \( F - G \) can be written as \( B + L \), where \( B : X \to Y \) is a homogeneous map bounded on the unit ball and \( L : X \to Y \) is linear.

2. Rosenthal's property \((V^*)\) is not a 3-space property

Our starting point is Pełczyński's proof \([14]\) that every subspace of \( C[0, 1] \) containing a copy \( C \) of \( C[0, 1] \) contains a further copy \( C' \) of \( C[0, 1] \) that is complemented in \( C[0, 1] \). An examination of the proof shows that for every \( \varepsilon > 0 \) the copy \( C' \) can be chosen \( (1 + \varepsilon) \)-isomorphic to \( C[0, 1] \) and \( (1 + \varepsilon) \)-complemented. Let thus \( \Gamma \) be the family of all subspaces \( \gamma \) of \( C[0, 1] \) which are 2-isomorphic to \( C[0, 1] \) via some isomorphism \( j_\gamma \) and 2-complemented by some projection \( p_\gamma \). The existence of a nontrivial exact sequence

\[
0 \longrightarrow C[0, 1] \longrightarrow X \longrightarrow C[0, 1] \longrightarrow 0
\]

is well known \([6, 3]\). Let \( \omega \) be a quasi-linear map inducing this sequence. We define the map \( \Omega' : C[0, 1] \sim C[0, 1]^F \) by

\[
\Omega'(x) = (\omega j_\gamma p_\gamma(x))_{\gamma \in \Gamma}.
\]

At each coordinate \( \gamma \) one has \( Z(\omega j_\gamma p_\gamma) \leq Z(\omega) \|j_\gamma\| \|p_\gamma\| \). Therefore

\[
\sup_{\gamma \in \Gamma} \left\| \omega j_\gamma p_\gamma \left( \sum x_i \right) - \sum_i \omega j_\gamma p_\gamma(x_i) \right\| \leq 4Z(\omega) \sum_i \|x_i\|
\]

for finite sums. This is therefore enough to define a \( z \)-linear map \( \Omega : X \sim \ell_\infty(\Gamma, C[0, 1]) \) as follows: take \( P : C[0, 1]^F \to \ell_\infty(\Gamma, C[0, 1]) \) a linear projection and set \( \Omega = P\Omega' \). It is \( z \)-linear with \( Z(\Omega) \leq 4Z(\omega) \) since \( P\Omega'(\sum x_i) - \sum_i P\Omega'(x_i) = \Omega'(\sum x_i) - \sum_i \Omega'(x_i) \). Thus, it defines an exact sequence

\[
0 \longrightarrow \ell_\infty(\Gamma, C[0, 1]) \longrightarrow X \longrightarrow C[0, 1] \longrightarrow 0.
\]

Let us show that \( \Omega \) cannot be trivial on any copy of \( C[0, 1] \). It is enough to show that the restriction of \( \Omega \) to any copy \( \gamma_0 \in \Gamma \) is not trivial. Let \( \pi_0 : \ell_\infty(\Gamma, C[0, 1]) \to C[0, 1] \) be the canonical projection onto the \( \gamma_0 \)-th coordinate. For \( x \in \gamma_0 \) one has

\[
\pi_0 \Omega(x) = \Omega j_0 p_{\gamma_0} x = \Omega j_0(x).
\]
So, $\pi_0 \Omega = \omega j_0$ on $\gamma_0$. If this map is trivial so must be $\omega = \omega j_0 j_0^{-1}$, which is not. Hence $\Omega$ cannot be trivial on $\gamma_0$.

This shows that the quotient map $q$ cannot be an isomorphism on any copy of $C[0, 1]$ (see [7]). Now observe that every $z$-linear map $\Omega$ on a separable space has a version $\Omega_s$ having separable range. Since every separable subspace of $\ell_\infty(\Gamma, C[0, 1])$ is contained in a copy of $C[0, 1]$ inside $\ell_\infty(\Gamma, C[0, 1])$, the image of this version actually lies in $C[0, 1]$. Moreover, all versions of a map enjoy simultaneously the property of being not trivial on any copy of $C[0, 1]$. Thus, this produces an exact sequence

$$0 \to C[0, 1] \to C[0, 1] \oplus \Omega_s C[0, 1] \to Q \to C[0, 1] \to 0$$

whose quotient map $Q$ is not an isomorphism on any copy of $C[0, 1]$ and consequently the space $C[0, 1] \oplus \Omega_s C[0, 1]$ cannot have Rosenthal’s property $(V^*)$. Moreover, Johnson and Zippin showed in [9] that every separable Lindenstrauss space is a quotient of $C[0, 1]$, from which it follows that Lindenstrauss spaces also have Rosenthal’s property $(V^*)$. Thus, the twisted sum space $C[0, 1] \oplus \Omega_s C[0, 1]$ cannot be even a quotient of a Lindenstrauss space.

3. 

We produce now the exact sequence (1) in which the quotient map $q_\mathbb{S}$ is not an isomorphism on any copy of $c_0$.

We work as before taking as starting point Sobczyk’s theorem—every copy of $c_0$ is 2-complemented in any separable superspace—and the well-known distortion result for $c_0$ [12]: every Banach space isomorphic to $c_0$ contains for every $\varepsilon > 0$ a $(1 + \varepsilon)$-isometric copy of $c_0$. Let thus $\Gamma$ be the family of all subspaces $\gamma$ of $C[0, 1]$ which are 2-isomorphic to $c_0$ via some isomorphism $j_\gamma$ and 2-complemented by some projection $p_\gamma$.

The existence of a nontrivial exact sequence

$$0 \to C[0, 1] \to \Omega(c_0) \to q_0 \to c_0 \to 0$$

is well-known [2, 6, 15]. Let $\omega$ be a quasi-linear map inducing this sequence. We define the map $\Omega' : C[0, 1] \to C[0, 1]^\Gamma$ by

$$\Omega'(x) = (\omega j_\gamma p_\gamma(x))_{\gamma \in \Gamma}.$$ 

That this is enough to define a $z$-linear map $\Omega : C[0, 1] \to \ell_\infty(\Gamma, C[0, 1])$ as before. Let us show that $\Omega$ cannot be trivial on any copy of $c_0$; equivalently,
the restriction of Ω to any copy γ₀ ∈ Γ is not trivial. Let π₀ : ℓ∞(Γ, C[0, 1]) → C[0, 1] be the canonical projection. For x ∈ γ₀ one has

\[ \pi₀(Ω(x)) = Ω(j₀p₀x) = Ωj₀(x). \]

So, π₀Ω = ω₀ on γ₀. If this map is trivial so must be ω = ω₀j₀−1, which is not. Hence Ω cannot be trivial on γ₀. The reduction of the range to become C[0, 1] is as before.

4. Final remarks and conclusions

The previous device can be applied in other more general situations. In particular, we have proved that given Banach spaces X, Y such that:

1. There is some C such that that every copy of Y inside X contains a further C-isomorphic copy of Y that is C-complemented in X.

2. For some compact space S there is a nontrivial exact sequence 0 → C(S) → ☢ → Y → 0.

Then there is a C(K) space with the same density character as X and an exact sequence

\[ 0 \to C(K) \to ☢ \to X \to 0 \]

such that q is not an isomorphism on any copy of Y. In particular, taking as X a Weakly Compactly Generated space containing c₀ and Y = c₀ produces an exact sequence 0 → C(K) → ☢ → X → 0 in which ☢ cannot have Pełczyński’s property (V). Further counterexamples can be deduced from [3], where it was shown the existence of an exact sequence

\[ 0 \to C(ω) \to Ω_ω \xrightarrow{q₀} c₀ \to 0 \]

with q₀ strictly singular; thus Ωω does not have either property (V).

It is an open problem whether every twisted sum of c₀(Γ) must be isomorphic to a C(K)-space. Related to this is the question of whether every twisted sum of c₀(Γ) must have Pełczyński’s property (V).

**Proposition.** Every twisted sum of c₀(Γ) and a space with property (V) has property (V).
Proof. Let
\[ 0 \rightarrow c_0(\Gamma) \rightarrow X \rightarrow^q Z \rightarrow 0 \]
be an exact sequence in which \( Z \) has property \((V)\) and let \( \phi : X \rightarrow Y \) be an operator. If the restriction \( \phi_0 = \phi|_{c_0(\Gamma)} \) is an isomorphism on some copy of \( c_0 \) then the same does \( \phi \). Otherwise it is weakly compact and admits a weakly compact extension \( \phi_0^{**} : \ell_\infty(\Gamma) \rightarrow Y \). Let \( p : X^{**} \rightarrow \ell_\infty(\Gamma) \) be a linear continuous projection. This gives a weakly compact extension \( \phi_0^{**} p|_X : X \rightarrow Y \). Moreover, there is an operator \( \psi : Z \rightarrow Y \) such that \( \phi_0 - \phi_0^{**} p|_X = \psi q \). If \( \psi \) is weakly compact then so is \( \phi = \phi_0^{**} p|_X + \psi q \). Otherwise, there is a subspace \( c_0' \) of \( Z \) isomorphic to \( c_0 \) on which \( \psi \) is an isomorphism. By Sobczyk’s theorem, there is a linear continuous projection \( P : \ell_\infty(c_0') \rightarrow \ell_\infty(c_0(\Gamma)) \) and \( q \) is thus an isomorphism on \( ker P \), which is isomorphic to \( c_0' \). Finally, since \( p|_{ker P} = 0 \) one gets
\[ \phi|_{ker P} = \phi_0^{**} p|_{ker P} + \psi q|_{ker P} = \psi|_{c_0'} \]
is an isomorphism. \( \square \)

References


