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Some Results on Automatic Continuity of Group Representations and Morphisms

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Abstract: In the first part of the paper, some criteria of continuity of representations from a locally compact group in a Banach algebra are given. The common feature of these results is the fact that the continuity of a representation can be deduced from the continuity of its composition with some linear forms on its range. The second part uses the result of the first part to deduce automatic continuity results of Haar-measurable morphism from locally compact groups to infinite dimensional linear or unitary groups.

Key words: Automatic continuity, locally compact groups, linear representations, Glicksberg-De Leeuw decomposition.

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1. Introduction

The papers [4], [8], [7], [20] are, to a large degree, devoted to various but somewhat similar characterization of norm continuity for representations of locally compact group in a Banach algebra and so is the present one. Actually, the roots of these results can be traced back to Phillip's work (see [12]) on one parameter semi-groups. In [8] the following fact was proved:

Theorem 1.1. (Esterle) *Let G be an Abelian locally compact group, A a unital Abelian Banach algebra and θ a locally bounded (norm bounded on compact subsets of G) representation from G in A. The following assertions are equivalent:*

1. *θ is continuous.*
2. $\lim_{g\to e} \rho(\theta(g)-1)=0$, *where ρ denotes the spectral radius in A and e the unit element of G. (This condition is called "spectral continuity" of θ).*

This equivalence is generalized in [7] to some non-Abelian groups. In [4], using the preceding result, the following was obtained:

Theorem 1.2. *Let G, A and θ be as in Theorem* 1.1. *Then θ is continuous if and only if for each character χ of A, the composition $\chi \circ \theta$ is continuous from G to the complex plane $\mathbb{C}$.*

This strategy was already used by Phillips who divided the proof for $G = \mathbb{R}$ into two steps. First, he proved that the continuity of $\chi \circ \theta$ for each character implies the condition 2 of Theorem 1.1 (by an analytic argument) and next, that this last condition implies continuity using the Gelfand-Hille theorem and Riesz-Dunford functional calculus. None of these steps is immediately generalizable to Abelian groups (except perhaps to $\mathbb{R}^n$). In the proof of Theorem 1.1 (see [8]), beside the Gelfand-Hille theorem, Silov's idempotents and the general structure result on locally compact Abelian groups has been used and in [4], the fact that continuity of $\chi \circ \theta$ for each χ implies spectral continuity for θ uses arguments of commutative harmonic analysis (Fourier transforms, Pontryagin duality,...).

In [20] an analogous continuity criterion is proved in the non-Abelian case but only for unitary representations with values in C^*-algebras.

Theorem 1.3. *Let G be a locally compact group (non-Abelian in general), A a C^*-algebra and θ a unitary representation of G in A. The following assertions are equivalent:*

1. *θ is continuous.*
2. *For each state ω on A, the composition $\omega \circ \theta$ is continuous.*

If G is first countable, we can replace "states" by "pure states" in 2.

Recall that states on A are positive functionals of norm one and, if we denote by $W(A)$ the (compact in the weak-$*$ topology of the dual A') set of states and by $\rho_W(a) = \sup_{\omega \in W(A)} |\omega(a)|$, $(a \in A)$, ρ_W is an equivalent norm on A (see [1, p. 34]). Indeed, θ is continuous if $\lim_{g \to e} \rho_W(\theta(g) - 1) = 0$ and the only thing to prove in Theorem 1.3 is that

$$\forall\, \omega \in W(A),\ \omega \circ \theta \text{ continuous} \quad \Rightarrow \lim_{g \to e} \rho_W\big(\theta(g) - 1\big) = 0.$$

In [20] the proof of this fact used tools of unitary representations theory (essentially functions of positive type) that are not available outside the scope of C^*-algebras and unitary representations.

The crucial fact in Theorems 1.2 and 1.3 is, for a weak-$*$ compact subset Ω of the dual A' of the algebra (the set of characters in 1.2, of states in 1.3), to prove that:

$$\forall\, \omega \in \Omega,\ \omega \circ \theta \text{ continuous } \ \Rightarrow\ \sup_{\omega \in \Omega} \bigl|\omega\bigl(\theta(g) - 1\bigr)\bigr| \xrightarrow[g \to e]{} 0.$$

Part 2 of this paper is devoted to a general strategy for obtaining such a result using the Glicksberg-De Leeuw decomposition (see 2.1) in place of group characters or functions of positive type. We give, in this way, another proof of Theorem 1.2 and a generalization of Theorem 1.3 to norm bounded representations of locally compact groups in Banach algebras (not only unitary representations in C^*-algebras).

In Part 3, we apply the results of Part 2 and a recent theorem of Kuznetsova [16] to prove that, for morphisms from locally compact groups into some topological groups, Haar measurability implies continuity. These facts are classical (due to Steinhauss, Weil, Pettis,...) if the range group is Polish but are, to our knowledge, new if this group is non-separable.

2. Continuity of representations seen through composition by some functionals

2.1. Preliminaries. In this section we collect some facts used below that cannot be considered as very classical. First, we will introduce the Glicksberg-De Leeuw decomposition theorem. We will state it in the form most convenient for us, although not the most general one. Let G a locally compact group, X a Banach space, $\mathcal{L}(X)$ the Banach algebra of bounded operators on X. A representation θ from G to $\mathcal{L}(X)$ is said strongly continuous if for each x in X the map $g \mapsto \theta(g)x$ is continuous from G to X (with its norm topology) and weakly continuous if, for each x the same map is continuous into X endowed with its weak topology. θ is said "averaging to zero" if, for each $x \in X$ and each neighborhood V of e, 0 is in the closed convex hull of $\theta(V) \cdot x = \{\theta(g)x : g \in V\}$. Finally, θ is "locally weakly almost periodic" if, for some neighborhood V_0 of e, $\theta(V_0)x$ has weakly compact (weak) closure in X for each x in X.

The following is true (see [6, th. 3.2]):

Theorem 2.1. (Glicksberg-De Leeuw) *Let θ be a locally weakly almost periodic representation of a locally compact group G on a Banach space X such that $g \mapsto \langle x^*, \theta(g)x\rangle$ is locally Haar-measurable for every $x \in X$ and for x^* in a total (in the weak-* topology) subset of the topological dual X' of X. We have a direct decomposition $X = X_c \oplus X_0$, X_c and X_0 being closed*

$\theta(G)$-invariant subspaces such that the subrepresentation induced by θ on X_c is weakly continuous and on X_0 is averaging to zero.

We need also the following result well known for unitary Hilbertian representations.

Remark 2.2. The equivalence between strong and weak continuity is also classical for one parameter semi-groups (see *e.g.* [5] p.15).

THEOREM 2.3. ([6, th. 2.8]) *Every weakly continuous representation of a locally compact group on a Banach space is strongly continuous.*

2.2. CONTINUITY OF REPRESENTATIONS ON SPACES OF CONTINUOUS FUNCTIONS. Let G be a locally compact group and Ω a compact Hausdorff space. We have the following result for representations of $\mathcal{C}(\Omega)$.

PROPOSITION 2.4. *Let $\theta : G \to \mathcal{L}(\mathcal{C}(\Omega))$ be a locally bounded representation. The following assertions are equivalent:*

(i) *For every $f \in \mathcal{C}(\Omega)$, $g \mapsto \theta(g)f$ is continuous from G into $\mathcal{C}(\Omega)$ endowed with the pointwise convergence topology.*

(ii) *θ is strongly continuous.*

Proof. Clearly, only (i) $\Rightarrow$ (ii) needs to be proved. Let V be a compact neighborhood of e in G and $f \in \mathcal{C}(\Omega)$. θ being locally bounded, $\theta(V)f$ is bounded in $\mathcal{C}(\Omega)$ norm and also compact in $\mathcal{C}(\Omega)$ endowed with the pointwise convergence topology by (i). So, by Grothendieck's theorem ([11, th. 5]), $\theta(V)f$ has weakly compact closure in $\mathcal{C}(\Omega)$ and the representation is weakly almost periodic.

Denote by $D_\Omega = \{e_\omega, \omega \in \Omega\}$ the set of Dirac point masses on Ω ($e_\omega(f) = f(\omega)$). D_Ω is a total subset of $(\mathcal{C}(\Omega))'$ and (always by (i)) $g \mapsto \langle e_\omega, \theta(g)f\rangle = (\theta(g)f)(\omega)$ is continuous for each $\omega \in \Omega$ and $f \in \mathcal{C}(\Omega)$. Thus, by Theorem 2.1, we have

$$\mathcal{C}(\Omega) = \big(\mathcal{C}(\Omega)\big)_c \oplus \big(\mathcal{C}(\Omega)\big)_0$$

with the properties of Theorem 2.1.

By hypothesis,

$$\forall f \in \mathcal{C}(\Omega),\ \forall\, \omega \in \Omega \quad e_\omega\big(\theta(g)f\big) = \big(\theta(g)f\big)(\omega) \xrightarrow[g\to e]{} f(\omega)$$

and for each $f \in (\mathcal{C}(\Omega))_0$ and any neighborhood V of e, $0 \in e_\omega(Conv(\theta(V)f))$ where $Conv(\theta(V)f)$ is the closed convex hull of $\theta(V)f$. So, for each $\omega \in \Omega$, $0 \in \bigcap_{V \in \mathcal{V}(e)} e_\omega(Conv(\theta(V)f)) = \{f(\omega)\}$ and $(\mathcal{C}(\Omega))_0 = \{0\}$. Thus, $\mathcal{C}(\Omega) = (\mathcal{C}(\Omega))_c$ and θ is weakly and also strongly continuous by Theorem 2.3. ∎

Remark 2.5. One can use the following recent result of H. Pfitzner [18] (answering to a question asked in [10] by G. Godefroy):

THEOREM 2.6. (Pfitzner) *Let X be a real Banach space and Ω a subset of the dual unit ball of X such that*

$$\forall x \in X, \quad \|x\| = \max_{x^* \in \Omega} \langle x^*, x\rangle$$

(the max is actually attained on Ω). Then the weak topology and the $\sigma(X, \Omega)$-topology have the same norm bounded compact sets. (Such sets as Ω are called "James boundary" for X and D_Ω is such a set for $\mathcal{C}(\Omega)$).

Using this result in place of Grothendieck's theorem and with some minor changes (in general the space X considered here is a complex Banach space but its weak topology is the same if we consider the underlying real structure) one can prove:

PROPOSITION 2.7. *Let G be a locally compact group, $\theta : G \to \mathcal{L}(X)$ a locally bounded representation and Ω a subset of the dual unit ball such that*

$$\forall x \in X, \quad \|x\| = \max_{x^* \in \Omega} \Re e\langle x^*, x\rangle$$

($\Re e(\Omega) = \{\Re e(x^*), x^* \in \Omega\}$ is a James boundary for X with its real structure). The following properties are equivalent:

1. *For every $x \in X$, $g \mapsto \theta(g)x$ is continuous from G to X endowed with the $\sigma(X, \Omega)$-topology.*
2. *θ is strongly continuous.*

Only the Proposition 2.4 will be used below.

2.3. APPLICATIONS TO OTHER CONTINUITY CRITERIA. First, we give a proof of Theorem 1.2 different from [4] by applying Proposition 2.4 to an auxiliary representation.

Proof of Theorem 1.2. By Esterle's continuity criterion (Theorem 1.1), it is sufficient to prove that the continuity of $\chi \circ \theta$ for each $\chi \in \hat{A}$ (the character space of A) implies $\lim_{g \to e} \rho(\theta(g) - 1) = 0$. Suppose for every $\chi \in \hat{A}$, $\chi \circ \theta$ continuous. A being a unital Abelian algebra, $\hat{A}$ is a compact Hausdorff space. For $a \in A$, let $\hat{a}$ be its Gelfand transform in $\mathcal{C}(\hat{A})$ ($\hat{a}(\chi) = \chi(a)$ for $\chi \in \hat{A}$). It is well known that

$$\forall a \in A, \quad \|\hat{a}\|_{\mathcal{C}(\hat{A})} = \rho(a) \leq \|a\|.$$

If f is in $\mathcal{C}(\hat{A})$, we denote by M_f the operator of multiplication by f. It is also well known that $\|M_f\|_{\mathcal{L}(\mathcal{C}(\hat{A}))} = \|f\|_{\mathcal{C}(\hat{A})}$. We define $\tilde{\theta} : G \to \mathcal{L}(\mathcal{C}(\hat{A}))$ by $\tilde{\theta}(g) = M_{\widehat{\theta(g)}}$. Clearly $\tilde{\theta}$ is a group representation and

$$\|\tilde{\theta}(g)\| = \|M_{\widehat{\theta(g)}}\| \leq \|\widehat{\theta(g)}\|_{\mathcal{C}(\hat{A})} \leq \|\theta(g)\|.$$

So $\tilde{\theta}$ is locally bounded if θ is.

Now, for any $f \in \mathcal{C}(\hat{A})$, $g \mapsto \tilde{\theta}(g)f$ is continuous with $\mathcal{C}(\hat{A})$ endowed with the pointwise convergence topology. Indeed, for each $\chi \in \hat{A}$,

$$\big(\tilde{\theta}(g)f\big)(\chi) = \widehat{\theta(g)}(\chi)f(\chi) = (\chi \circ \theta)(g)f(\chi)$$

depends continuously of g by hypothesis. So, by Proposition 2.4, $\tilde{\theta}$ is strongly continuous and $\|\tilde{\theta}(g) \cdot \hat{1} - \hat{1}\|_{\mathcal{C}(\hat{A})} \xrightarrow[g \to e]{} 0$ but

$$\|\tilde{\theta}(g)\hat{1} - \hat{1}\|_{\mathcal{C}(\hat{A})} = \big\|\big(\widehat{\theta(g) - 1}\big) \cdot \hat{1}\big\|_{\mathcal{C}(\hat{A})} = \big\|\big(\widehat{\theta(g) - 1}\big)\big\|_{\mathcal{C}(\hat{A})} = \rho\big(\theta(g) - 1\big).$$

∎

We can also with another choice of auxiliary representation to which Proposition 2.4 is applied, prove the following generalization of Theorem 1.3.

Theorem 2.8. *Let G be a locally compact group, A a unital Banach algebra and θ a representation of G in A such that for every $g \in G$, $\|\theta(g)\| = 1$. The following assertions are equivalent:*

1. *θ is continuous.*
2. *For each functional $\omega \in A'$, we have $\omega \circ \theta$ continuous (θ is continuous if A is endowed with its weak topology).*

Proof. We can suppose that A is generated, as a Banach algebra by the range $\theta(G)$. Indeed, if it is not the case, we restrict ourselves to the closed subalgebra B generated by $\theta(G)$. Every $\tilde{\omega} \in B'$ can be extended in $\omega \in A'$ (by Hahn-Banach) and so, if $\omega \circ \theta$ is continuous for each $\omega \in A'$, the same is true for each $\tilde{\omega} \in B'$.

We define an algebra anti-representation L^* (that $L^*_{ab} = L^*_b L^*_a$ from A into $\mathcal{L}(A')$) by

$$(L^*_a \omega)(x) = \omega(ax), \quad \omega \in A',\ x \in A.$$

L^*_a is the transpose of the left multiplication by a. We have

$$\forall \omega \in A',\ \forall x \in A \quad |\omega(ax)| \leq \|\omega\| \|a\| \|x\|.$$

So $\|L^*_a \omega\| \leq \|a\| \|\omega\|$ and $\|L^*_a\| \leq \|a\|$. We can thus define an action of the opposite group of G on the unit ball B of A' by

$$\hat{\theta}(g) = L^*_{\theta(g)} \omega, \quad \omega \in B.$$

Moreover, we are going to show that, for every $\omega \in B$, $g \mapsto \hat{\theta}(g)\omega$ is continuous when B is endowed with the weak-* topology. Let $(g_i)_{i \in I}$ be a net in G such that $g_i \to g$. We have to show that

$$\forall a \in A,\ \big(\hat{\theta}(g_i)\omega\big)(a) = (L^*_{\theta(g_i)}\omega)(a) \to (L^*_{\theta(g)})(a) = \big(\hat{\theta}(g)\omega\big)(a).$$

Recall that A is the closed linear span of $\theta(G)$ and suppose first that $a = \theta(h)$ is in $\theta(G)$. We have

$$(L^*_{\theta(g_i)}\omega)(a) = (L^*_{\theta(g_i)}\omega)\big(\theta(h)\big) = \omega\big(\theta(g_i h)\big) \to \omega\big(\theta(gh)\big) = (L^*_{\theta(g)}\omega)(a).$$

Clearly the same is true for a in the linear span of $\theta(G)$ denoted by $Lin(\theta(G))$. Now, let $a \in A$, we have

$$\forall \epsilon > 0,\ \exists a' \in Lin\big(\theta(G)\big) \text{ such that } \|a - a'\| < \frac{\epsilon}{3}.$$

For such a a', we have

$$\exists i_0 \in I \text{ such that } i \geq i_0 \ \Rightarrow\ \big|(L^*_{\theta(g_i)}\omega)(a') - (L^*_{\theta(g)}\omega)(a')\big| < \frac{\epsilon}{3}.$$

and the fact that for each $h \in G$ (in particular for every g_i and g), $\|L^*_{\theta(h)}\omega\| \leq 1$ gives

$$i \geq i_0 \ \Rightarrow\ \big|(L^*_{\theta(g_i)}\omega)(a) - (L^*_{\theta(g)}\omega)(a)\big| < \epsilon.$$

So, for every $a \in A$, $(L^*_{\theta(g_i)}\omega)(a) \to (L^*_{\theta(g)}\omega)(a)$.

We consider the Banach space $\mathcal{C}(B)$ and the representation $\tilde{\theta}$ from G to $\mathcal{L}(\mathcal{C}(B))$ defined by

$$\big(\tilde{\theta}(g)f\big)(\omega) = f\big(\widehat{\theta}(g)\omega\big) = f(L^*_{\theta(g)}\omega).$$

For each $f \in \mathcal{C}(B)$, the map $g \mapsto \tilde{\theta}(g)f$ is continuous relative to the pointwise convergence topology on $\mathcal{C}(B)$. Indeed, if ω is a functional in B, what we have to show is that if (g_i) is a net in G such that $g_i \to g$, then

$$\big(\tilde{\theta}(g_i)f\big)(\omega) = f(L^*_{\theta(g_i)}\omega) \to f(L^*_{\theta(g)}\omega) = \tilde{\theta}(g)\omega.$$

f being continuous, it is a consequence of the fact that $L^*_{\theta(g_i)}\omega \to L^*_{\theta(g)}\omega$ in B proved above.

We can thus apply Proposition 2.4 and $\tilde{\theta}$ is strongly continuous. If $\hat{1} \in \mathcal{C}(B)$ is defined by $\hat{1}(\omega) = \omega(1)$ we can write

$$\forall \omega \in B,\ \big(\tilde{\theta}(g)\hat{1}\big)(\omega) = \hat{1}(L^*_{\theta(g)}\omega) = (L^*_{\theta(g)}\omega)(1) = \omega\big(\theta(g)\big),$$

and

$$\lim_{g\to e} \|\tilde{\theta}(g)\hat{1} - \hat{1}\|_{\mathcal{C}(B)} = \lim_{g\to e} \big(\sup_{\omega\in B} \big|\omega(\theta(g)-1)\big|\big) = \lim_{g\to e} \|\theta(g) - 1\| = 0$$

by strong continuity of $\tilde{\theta}$ and so θ is norm continuous from G to A. ∎

Corollary 2.9. *Let G be a locally compact group and θ a norm bounded representation of G into a unital Banach algebra A. The following facts are equivalent:*

1. *θ is continuous.*
2. *θ is continuous when A is endowed with its weak topology.*

Proof. Setting $M = \sup_{g\in G} \|\theta(g)\|$. We define, for $a \in A$,

$$|||a||| = \sup_{g\in G} \|\theta(g)\cdot a\|.$$

$|||\cdot|||$ is clearly a norm on A equivalent to the initial one ($\|a\| \le |||a||| \le M\|a\|$). Moreover

$$\forall g \in G,\ \forall a \in A, \quad |||\theta(g)\cdot a||| = |||a|||$$

and

$$|||aa'||| = \sup_{g\in G} \|\theta(g)aa'\| \le \sup_{g\in G} \big(\|\theta(g)a\| \, \|a'\|\big) \le |||a||| \, \|a'\| \le |||a||| \, |||a'|||.$$

Hence $|||\cdot|||$ is a Banach algebra norm on A.

Setting now $\|a\|_G = \sup_{|||a'|||\le 1} |||aa'|||$, for $a \in A$, $\|\cdot\|_G$ is a Banach algebra norm on A equivalent to $|||\cdot|||$ since for each a in A we have $\|a\|_G \le |||a|||$ and, choosing $a' = \frac{1}{M} \in A$, we have $|||a'||| = \frac{1}{M}|||1||| = 1$. So for every $a \in A$, $\|a\|_G \ge \frac{1}{M}|||a|||$. Moreover

$$\forall g \in G,\ \|\theta(g)\|_G = \sup_{|||a'|||\le 1} |||\theta(g)a'||| = \sup_{|||a'|||\le 1} |||a'||| = 1.$$

So renorming A with the equivalent Banach algebra norm $\|\cdot\|_G$, we can apply the preceding theorem. ∎

Now we will show how to deduce Theorem 1.3 from the previous results. Recall that if A is a unital Banach algebra, we call "state" of A a functional ω in A' such that $\|\omega\| = \omega(1) = 1$. For C^*-algebras, this definition coincides with that given by positivity. We have the following known result:

Lemma 2.10. (see [2, p. 100]) *If A is a unital Banach algebra, every continuous functional on A is a linear combination of (at most) four states.*

So the continuity of $\omega \circ \theta$ for each state ω implies the continuity for each linear functional and we obtain:

Corollary 2.11. *Let A be a unital Banach algebra, G a locally compact group and θ a norm bounded representation of G to A, the following facts are equivalent:*

1. *θ is continuous.*
2. *For each state ω of A, $\omega \circ \theta$ is continuous.*

As a special case of this, when A is a C^*-algebra and θ a unitary representation, we obtain Theorem 1.3.

If we denote by $S(A)$ the set of states of A, $S(A)$ endowed with the weak-* topology is a compact subset of A', and so it is the closed convex hull of the set $PS(A)$ of his extreme points, called "pure states". As in [20], an application of the Choquet-Bishop-De Leeuw integral representation theorem allows us to obtain:

Corollary 2.12. *If θ is a norm bounded representation of a locally compact first countable group in a Banach algebra A, the following assertions are equivalent:*

1. *θ is continuous.*
2. *For each pure state ω of A, $\omega \circ \theta$ is continuous.*

Proof. The proof proceeds exactly as in [20] but we repeat it for self-completeness.

$S(A)$ is a convex compact in the weak-* topology of A'. Hence, by the Choquet-Bishop-De Leeuw integral representation theorem (see [19]), for each $\omega \in S(A)$, there exists a Baire measure μ_ω on $PS(A)$ such that for each weak-* continuous functional φ on A', we have

$$\varphi(\omega) = \int_{PS(A)} \varphi d\mu_\omega.$$

Applying this representation to the functional $\widehat{\theta(g)}$ on A' (evaluation of ω at $\theta(g)$), we have for any $g \in G$,

$$(\omega \circ \theta)(g) = \widehat{\theta(g)}(\omega) = \int_{PS(A)} \widehat{\theta(g)} d\mu_\omega.$$

By Corollary 2.11 above, one only have to prove the continuity for $\omega \circ \theta$ for each $\omega \in S(A)$ to obtain that θ is. G being first countable, we have to show that if (g_n) is a sequence in G such that $g_n \to g$ we have $\omega \circ \theta(g_n) \to \omega \circ \theta(g)$. By hypothesis, for each $\tau \in PS(A)$,

$$\tau \circ \theta(g_n) = \widehat{\theta(g_n)}(\tau) \to \widehat{\theta(g)}(\tau) = \tau \circ \theta(g).$$

Moreover, for every $h \in G$ and every $\tau \in PS(A)$, $|\widehat{\theta(h)}\tau| \leq 1$ so, by the dominated convergence theorem,

$$\omega \circ \theta(g_n) = \int_{PS(A)} \widehat{\theta(g_n)} d\mu_\omega \to \int_{PS(A)} \widehat{\theta(g)} d\mu_\omega = \omega \circ \theta(g)$$

and we are done. ∎

To conclude this part, we will state and prove a proposition on unitary representation on C^*-algebras which is implicit in [20] but will be used in the third part below.

PROPOSITION 2.13. *Let G be a locally compact group, A be a unital C^*-algebra and θ be a unitary representation of G in A. The following facts are equivalent:*

1. *θ is continuous.*
2. *For each $\omega \in S(A)$, $\pi_\omega \circ \theta$ is strongly continuous where $\pi_\omega : A \to \mathcal{L}(H_\omega)$ is the Gelfand-Naimark-Segal representation associated to ω.*

Proof. Recall that for each $\omega \in S(A)$, we can construct (see [17]) a Hilbert space H_ω and a Hilbertian cyclic representation π_ω (unique up to equivalence) with cyclic vector $x_\omega \in H_\omega$ such that

$$\forall a \in A, \quad \omega(a) = \big\langle \pi_\omega(a)x_\omega \ , \ x_\omega \big\rangle.$$

(1) $\Rightarrow$ (2) is clear since π_ω is continuous.
Conversely, if $\pi_\omega \circ \theta$ strongly continuous, we have

$$g \mapsto (\omega \circ \theta)(g) = \big\langle (\pi_\omega \circ \theta)(g)x_\omega \ , \ x_\omega \big\rangle$$

continuous and by the results above, θ is continuous. ∎

Remarks 2.14.

1. If G and A are Polish, the fact that the weak continuity implies continuity is a consequence of the Suslin graph theorem ([3]) because, in this case, the graph of the representation being closed in a Polish space, is a Suslin set. But in the general case, there is no "closed graph theorem" for group morphisms as it is known by the classical example of the regular (i.e., by translation) representation of $\mathbb{R}$ on $\mathbb{L}^2(\mathbb{R})$ which, being strongly continuous, has closed graph but is not norm continuous.
2. We do not claim that the proofs of Theorems 1.2 and 1.3, as consequences of the Proposition 2.4 by using "ad hoc" auxiliary representations, are simpler than those of [4] or [20]. Indeed, the use of group characters, Fourier transforms, functions of positive type or GNS representations is replaced by Grothendieck's weak compactness criterion and Glicksberg-De Leeuw decomposition which are highly nontrivial results. Nevertheless, this different approach can have its advantages like the fact that we are able to go further the unitary case for non-Abelian groups.

3. Measurability and continuity for group morphisms

Well known results due to Steinhauss, A. Weil, and Pettis show that if H is a Polish group and if G is a Baire group (resp. a locally compact group), a group morphism θ from G into H having the Baire property (resp. being Haar measurable) is in fact continuous (cf. [13], [21]). But, as it is often the case for infinite dimensional representation, when the range group is not separable, these results do not generalize. If G is locally compact and H is the group of invertible elements of a unital Banach algebra or the group of unitaries of a C^*-algebra, the preceding results combined in the non-Abelian case with a recent theorem of Kuznetsova give analogous automatic continuous criteria.

3.1. The commutative case. We will first prove a lemma on the boundedness of regular subadditive or submultiplicative maps from a topological group into the real field.

Lemma 3.1. *Let G a Baire group (resp. a locally compact group) and φ a subadditive map from G to $\mathbb{R}^+$ with the Baire property (resp. Haar-measurable). Then φ is locally bounded on G (bounded on compact subsets).*

Proof. We shall treat the case of Baire Property, the measurable case goes exactly the same way. We can write

$$G = \bigcup_{n\in\mathbb{N}} V_n, \text{ where } V_n = \{g \in G : \varphi(g) \le n,\ \varphi(g^{-1}) \le n\}.$$

φ having the Baire property, all the sets V_n have the Baire property, and at least one of them, V_{n_0}, is non meager. By a well-known result (see [13, th. 9.9]), there is a neighborhood V of e in G such that $V \subset V_{n_0} \cdot V_{n_0}^{-1}$ so, for each $h \in V$, $h = gg'^{-1}$ with g, g' in V_{n_0} and $\varphi(h) \le 2n_0$. Hence, φ is bounded on V and by translation, for each $g \in G$, φ is bounded on the neighborhood $g \cdot V$ of g (by $\varphi(g) + 2n_0$). Now, an easy compactness argument shows that φ is bounded on any compact subset of G. ∎

Corollary 3.2. *Let $\varphi : G \to]0, +\infty[$ where G is a Baire group (resp. a locally compact group) such that $\varphi(gg') \le \varphi(g) \cdot \varphi(g')$. If φ has the Baire property (resp. is Haar measurable), then φ is locally bounded on G.*

Proof. This is a consequence of the lemma above using the logarithm map applied to $\psi(s) = \sup(1, \varphi(s))$. ∎

We are now ready to prove:

THEOREM 3.3. *Let G be a locally compact Abelian group and θ be a morphism from G into the group A^* of invertible elements in a unital Banach algebra. If θ is Haar measurable or θ has the Baire property, then θ is continuous.*

Proof. Let $\theta : G \to A$ and $\hat{A}$ the spectrum of A. If θ is measurable or has the Baire property, the same is true for $\chi \circ \theta$ for each $\theta \in \hat{A}$. Since $\chi \circ \theta$ is a morphism with values in $\mathbb{C}$, we can deduce that for each $\chi \in \hat{A}$, $\chi \circ \theta$ is continuous. Moreover, by Corollary 3.2, $g \mapsto \|\theta(g)\|$ being submultiplicative, θ is locally bounded. We can, thus, apply Theorem 1.2 to obtain continuity of θ as a representation or equivalently as a group morphism to A^*. ∎

3.2. THE NON-ABELIAN UNITARY CASE. We cannot deduce directly a result similar to Theorem 2.8 or its corollaries because $\omega \circ \theta$ for ω a linear functional are not morphisms and, to our knowledge, there is no automatic continuity available in this case. Nevertheless, the continuity of measurable morphisms from a locally compact group to a unitary group can be obtained combining Proposition 2.13 and the following recent continuity criterion of J. Kuznetsova.

THEOREM 3.4. ([16]) *Let G be a locally compact group, H a Hilbert space and $\theta : G \to \mathcal{L}(H)$ a unitary representation. If θ is Haar-measurable when $\mathcal{L}(H)$ is endowed with the weak operator topology, then θ is strongly continuous.*

Remarks 3.5.

1. In the case where H is separable, it is known that weak measurability of θ (i.e., $\forall(u,v) \in H^2$, $g \mapsto \langle\theta(g)u, v\rangle$ measurable) implies the strong continuity of θ (see [9]) but the following example (given in [16]) shows that this cannot be extended to the nonseparable case.

   EXAMPLE 3.6. Let G be a non-discrete locally compact group, $\ell^2(G)$ be the set of square summable families on G (i.e., $L^2(G_d)$, where G_d is the discrete version of G) and θ the left or right regular representation (by translations) from G on $\ell^2(G)$. θ is weakly (in the above sense) measurable but not strongly continuous.

2. Actually, considering for u and v in H and V an open set in $\mathbb{C}$

$$A_{u,v,V} = \big\{T \in \mathcal{L}(H)\ ,\ \langle Tu, v\rangle \in V\big\},$$

the weak measurability (in the sense of (1)) for θ expresses the measurability of sets in G of the form $\theta^{-1}(A_{u,v,V})$ and the Haar weak operator measurability of the theorem of Kuznetsova expresses the measurability of sets of the form $\theta^{-1}(U)$ with U in the topology generated by the sets $A_{u,v,V}$.

Now we can prove the following theorem:

THEOREM 3.7. *Let G be a locally compact group, H an Hilbert space, $U(H)$ the unitary group of H and θ an Haar-measurable morphism from G to $U(H)$. Then, θ is continuous.*

Proof. θ induces a unitary representation on the C^*-algebra $\mathcal{L}(H)$. For each state ω of $\mathcal{L}(H)$, denote by π_ω the GNS representation associated to ω. If θ is measurable, $\pi_\omega \circ \theta$ is measurable from G into $\mathcal{L}(H_\omega)$ since π_ω is continuous from $\mathcal{L}(H)$ into $\mathcal{L}(H_\omega)$ (where H_ω is the space of the representation π_ω) and so $\pi_\omega \circ \theta$ is also Haar to weakly operator measurable from G into $\mathcal{L}(H_\omega)$. By Kuznetsova's theorem, $\pi_\omega \circ \theta$ is strongly continuous from G into $\mathcal{L}(H_\omega)$ and by Proposition 2.13, θ is continuous from G into $\mathcal{L}(H)$ (or $U(H)$). ∎

Remark 3.8. Kuznetsova's theorem can also be used to give a quick proof of the following theorem due to A. Kleppner.

THEOREM 3.9. (see [15]) *Let G, H be a locally compact groups (no separability hypothesis) and $\varphi : G \to H$ a group morphism. Then φ is continuous if and only if it is Haar-measurable.*

As does Kleppner's proof, one use the regular representation through the following, certainly well known, lemma which is proved in [14].

LEMMA 3.10. *If G is a locally compact group, $\rho : G \to \mathcal{L}(\mathbb{L}^2(G))$ the left regular representation (i.e., $(\rho(g)f)(h) = f(g^{-1}h)$), ρ is a homeomorphism onto is range endowed with the strong topology of $\mathcal{L}(\mathbb{L}^2(G))$*

Proof of Kleppner's theorem. Let G, H be locally compact groups and $\varphi : G \to H$ an Haar measurable morphism. One can consider $\rho_H : H \to \mathcal{L}(\mathbb{L}^2(H))$ the left regular representation of H and $\theta = \rho_H \circ \varphi$ from G to

$\mathcal{L}(\mathbb{L}^2(H))$. ρ_H being continuous (for the strong topology of $\mathcal{L}(\mathbb{L}^2(H))$) and φ being measurable from G into H, θ is Haar to strong-operator (and so also Haar to weak operator) measurable and, by Kuznetsova's theorem, continuous. Now, let V be an open set in H, $\varphi^{-1}(V) = \theta^{-1}(\rho_H(V))$ since ρ_H is injective. By the preceding lemma, $\rho_H(V)$ is a relative strong open set in $\rho_H(H)$ so there is a strong open set U in $\mathcal{L}(\mathbb{L}^2(H))$ such that

$$\varphi^{-1}(V) = \theta^{-1}\big(\rho_H(V)\big) = \theta^{-1}\big(U \cap \rho(H)\big) = \theta^{-1}(U)$$

since $\theta(G) \subset \rho(H)$. So, by the continuity of θ, $\varphi^{-1}(V) = \theta^{-1}(U)$ is open in G and φ is continuous. ∎

Remark 3.11. This proof of the fact that, for a morphism from a locally compact group G to a topological group H, measurability implies continuity can be used in all cases where H admits a unitary representation that is an homeomorphism onto its range endowed with the strong topology. Nevertheless, we do not know if, beside the locally compact groups (by the regular representation) and the unitary group itself, there are classes of groups admitting such representations.

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