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On some Curvature Properties of K-contact Manifolds

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Abstract: The object of the present paper is to study projective curvature tensor in K-contact manifolds. Projectively flat and projectively semisymmetric K-contact manifolds are considered. Projectively pseudosymmetric and pseudoprojectively flat K-contact manifolds are also studied. It is shown that in all the cases the K-contact manifold becomes Sasakian.

Key words: K-contact manifolds, projective curvature tensor, projectively flat, projectively semisymmetric, projectively pseudosymmetric, pseudoprojectively flat.

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Introduction

An emerging branch of modern mathematics is the geometry of contact manifolds. The notion of contact geometry has evolved from the mathematical formalism of classical mechanics [11]. Two important classes of contact manifolds are K-contact manifolds and Sasakian manifolds [1], [16]. K-contact and Sasakian manifolds have been studied by several authors, viz, [10], [24], [6], [23], [15], [5] and many others. It is well known that every Sasakian manifold is K-contact, but the converse ia not true, in general. However a three-dimensional K-contact manifold is Sasakian [12]. The nature of a manifold mostly depends on its curvature tensor. Using the tools of conformal transformation geometers have deduced conformal curvature tensor. In the similar way with the help of projective transformation the notion of projective curvature has been defined [21]. Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view.

The object of the present paper is to enquire under what conditions a K-contact manifold will be a Sasakian manifold. In Section 1 we discuss about some preliminaries that will be used in the later sections.

Section 2 of the present paper is devoted to study projectively flat K-contact manifolds. It is known that [25] a Riemannian manifold of dimension greater than three is projectively flat if and only if it is of constant curvature. In Section 2 we consider projectively flat K-contact manifold and prove that a K-contact manifold is projectively flat if and only if it is locally isometric with a unit sphere.

Semisymmetry of a Riemannian manifold was first studied by Cartan [4]. A general study of semisymmetric Riemannian manifolds was made by Szabo [22]. Semisymmetric manifolds have been studied by several authors such as Sekigawa and Tanno [19], Sekigawa and Takagi [17] and Sekigawa [18]. In Section 3 we study projectively semisymmetric K-contact manifolds and prove that a projectively semisymmetric K-contact manifold is Sasakian.

The notion of pseudosymmetric manifolds has been introduced by Deszcz [7], [8]. In Section 4 of the present paper we introduce the notion of projectively pseudosymmetric manifolds and prove that a projectively pseudosymmetric K-contact manifold is Sasakian.

Finally we study pseudoprojectively flat K-contact manifolds and prove that a pseudoprojectively flat K-contact manifold is an Einstein manifold. As a consequence we obtain that a compact pseudoprojectively flat K-contact manifold is Sasakian.

1. Preliminaries

A $(2n+1)$-dimensional Riemannian manifold (M, g) is called an almost contact manifold if the following relations hold [1], [2]:

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{1.1}$$

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \tag{1.4}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \tag{1.5}$$

where ϕ is a $(1,1)$ tensor, η is a 1-form, ξ is the corresponding vector field to the 1-form η and g is the Riemannian metric. An almost contact metric manifold is called contact metric manifold if $d\eta = \Phi = g(X, \phi Y)$. Φ is called the fundamental 2-form of the manifold. If, in addition, ξ is a Killing vector, the manifold is called a K-contact manifold [1], [2], [16]. It is well known that a contact metric manifold is K-contact if and only if $\nabla_X \xi = -\phi X$, for any vector field X on M. Also a contact metric manifold is K-contact if and only if the $(1,1)$ type tensor field h defined by $h = \frac{1}{2}\pounds_\xi \phi$ is equal to zero, where $\pounds$ denotes Lie differentiation. It is known that [1] an almost contact structure is normal if and only if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0.$$

A normal contact metric manifold is known as a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

for any vector fields X, Y on M. Every Sasakian manifold is K-contact but the converse is not true, in general. However a three-dimensional K-contact manifold is Sasakian [12]. It is well known that (M, g) is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all vector fields X and Y on M.

A complete regular contact metric manifold M^{2n+1} carries a K-contact structure (ϕ, ξ, η, g), defined in terms of the almost Kaehler structure (J, G) of the base manifold M^{2n}. Here the K-contact structure (ϕ, ξ, η, g) is Sasakian if and only if the base manifold (M^{2n}, J, G) is Kaehlerian. If (M^{2n}, J, G) is only almost Kaehler, then (ϕ, ξ, η, g) is only K-contact [1]. In a Sasakian manifold the Ricci operator Q commutes with ϕ, that is, $\phi Q = Q\phi$. Recently in [13] it has been shown that there exists K-contact manifold with $\phi Q = Q\phi$ which are not Sasakian. It is to be noted that a K-contact manifold being intermidiate between a contact metric manifold and a Sasakian manifold.

Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborgood of M and a domain in the Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P defined by [21]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\big(S(Y,Z)X - S(X,Z)Y\big), \tag{1.6}$$

vanishes on the manifold for $X, Y, Z \in TM$, where R is the curvature tensor of type $(1,3)$ and S is the Ricci tensor of type $(0,2)$. In fact, M is projectively flat if and only if it is of constant curvature [25].

For a $(2n+1)$-dimensional K-contact manifold we always have [1], [2]

$$\nabla_X \xi = -\phi X, \tag{1.7}$$

$$S(X,\xi) = 2n\eta(X), \tag{1.8}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \;\; R(\xi,X)\xi = -X + \eta(X)\xi, \tag{1.9}$$

$$\eta\big(R(\xi,X)Y\big) = g(X,Y) - \eta(X)\eta(Y), \tag{1.10}$$

$$(\nabla_X\phi)Y = R(\xi,X)Y. \tag{1.11}$$

2. Projectively flat K-contact manifolds

It is known that [25] a Riemannian manifold of dimension $2n+1$, $(n>1)$ is projectively flat if and only if it is of constant curvature. Hence a K-contact manifold of dimension $2n+1$, $(n>1)$ is projectively flat if and only if the manifold is of constant curvature. Olszak [14] proved the following:

Theorem 2.1. *If a contact metric manifold M^{2n+1}, $(n>1)$ is of constant curvature c, then $c=1$ and the structure is Sasakian.*

Therefore from the above theorem we can state the following:

Corollary 2.1. *A K-contact manifold is projectively flat if and only if it is locally isometric with a unit sphere.*

3. Projectively semisymmetric K-contact manifolds

In view of (1.6) we get the projective curvature tensor of a $(2n+1)$-dimensional K-contact manifold as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\big(S(Y,Z)X - S(X,Z)Y\big).$$

Now from the above equation with the help of (1.2) and (1.9) we get

$$P(\xi,V)\xi = P(V,\xi)\xi = 0, \tag{3.1}$$

for any vector field V. We suppose that a K-contact manifold is projectively semisymmetric, that is, $(R(X,Y)\cdot P)(U,V)W = 0$. Hence

$$\begin{aligned} R(X,Y)P(U,V)W - P\big(R(X,Y)U,V\big)W \\ -P\big(U,R(X,Y)V\big)W - P(U,V)R(X,Y)W = 0. \end{aligned} \tag{3.2}$$

Substituting $Y=U=W=\xi$ in (3.2) and using (3.1) it follows that

$$P\big(R(X,\xi)\xi,V\big)\xi + P(\xi,V)R(X,\xi)\xi = 0.$$

Using (1.9) we obtain from the above equation

$$P(X,V)\xi + P(\xi,V)X = 0. \tag{3.3}$$

From (1.6) and (1.9) it follows that

$$\begin{aligned} R(X,V)\xi - \frac{1}{2n}\big(S(V,\xi)X - S(X,\xi)V\big) + R(\xi,V)X \\ - \frac{1}{2n}\big(S(V,X)\xi - S(\xi,X)V\big) = 0. \end{aligned} \tag{3.4}$$

With the help of (1.8), (3.4) becomes

$$R(X,V)\xi + R(\xi,V)X - \eta(V)X + 2\eta(X)V - \frac{1}{2n}S(V,X)\xi = 0. \tag{3.5}$$

Interchanging X and V in (3.5) we obtain

$$R(V,X)\xi + R(\xi,X)V - \eta(X)V + 2\eta(V)X - \frac{1}{2n}S(X,V)\xi = 0. \tag{3.6}$$

Subtracting (3.6) from (3.5) we obtain

$$R(X,V)\xi + R(\xi,V)X - R(V,X)\xi - R(\xi,X)V + 3\eta(X)V - 3\eta(V)X = 0.$$

Using Bianchi identity we get from the above equation

$$3R(X,V)\xi = 3\eta(V)X - 3\eta(X)V,$$

or,

$$R(X,V)\xi = \eta(V)\xi - \eta(X)V.$$

Hence the manifold is a Sasakian manifold. Now we are in a position to state the following:

THEOREM 3.1. *A projectively semisymmetric K-contact manifold is Sasakian.*

A Riemannian manifold is said to be projectively recurrent if $\nabla P = A \otimes P$, where A is a non-zero 1-form [21].

Shaikh and Baishya [20] proved that a projectively recurrent Riemannian manifold is projectively semisymmetric.

Hence by virtue of Theorem 3.1 w have the following:

COROLLARY 3.1. *A projectively recurrent K-contact manifold is Sasakian.*

4. Projectively pseudosymmetric K-contact manifolds

A Riemannian manifold M is said to be pseudosymmetric [9] if at every point of the manifold the following relation holds

$$\big(R(X,Y)\cdot R\big)(U,V)W = L_R\big((X\wedge Y)\cdot R(U,V)W\big) \tag{4.1}$$

for all $X,Y,U,V,W \in TM$, where L_R is some function on M. The endomorphism $X\wedge Y$ is defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{4.2}$$

A Riemannian manifold is said to be projectively pseudosymmetric if it satisfies the condition

$$\big(R(X,Y)\cdot P\big)(U,V)W = L_p\Big(\big((X\wedge Y)\cdot P\big)(U,V)W\Big), \tag{4.3}$$

where $L_p(\neq 1)$ is some function on M.

Let us suppose that a K-contact manifold satisfies the condition

$$\big(R(X,Y)\cdot P\big)(U,V)W = L_p\Big(\big((X\wedge Y)\cdot P\big)(U,V)W\Big). \tag{4.4}$$

From (4.4) we obtain

$$\big(R(X,\xi)\cdot P\big)(\xi,V)\xi = L_p\Big(\big((X\wedge \xi)\cdot P\big)(\xi,V)\xi\Big). \tag{4.5}$$

Using (1.9) in (4.5) we have

$$-P(X,V)\xi - P(\xi,V)X = L_p\Big(\big((X\wedge \xi)\cdot P\big)(\xi,V)\xi\Big). \tag{4.6}$$

Now

$$\begin{aligned} L_p\Big(\big((X\wedge \xi)\cdot P\big)(\xi,V)\xi\Big) = L_p\Big(&(X\wedge\xi)P(\xi,V)\xi - P\big((X\wedge\xi)\xi,V\big)\xi \\ &- P\big(\xi,(X\wedge\xi)V\big)\xi - P(\xi,V)(X\wedge\xi)\xi\Big). \end{aligned} \tag{4.7}$$

With the help of (1.9) and (1.6) we get the following:

$$(X\wedge\xi)P(\xi,V)\xi = 0, \tag{4.8}$$

$$P\big((X\wedge\xi)\xi,V\big)\xi = P(X,V)\xi, \tag{4.9}$$

$$P\big(\xi, (X \wedge \xi)V\big)\xi = 0, \tag{4.10}$$

$$P(\xi, V)(X \wedge \xi)\xi = P(\xi, V)(X - \eta(X)\xi) = P(\xi, V)X. \tag{4.11}$$

In view of (4.8), (4.9), (4.10), and (4.11), from (4.6) we obtain

$$-P(X, V)\xi - P(\xi, V)X = L_p\big(- P(X, V)\xi - P(\xi, V)\xi\big). \tag{4.12}$$

The above equation yields

$$(L_p - 1)\big(P(X, V)\xi + P(\xi, V)X\big) = 0.$$

By assumption $L_p \neq 1$ and hence

$$P(X, V)\xi + P(\xi, V)X = 0.$$

Now computing in the same way as in Theorem 3.1 we finally obtain

$$R(X, V)\xi = \eta(V)X - \eta(X)V.$$

Hence we can state the following:

THEOREM 4.1. *A projectively pseudosymmetric K-contact manifold is Sasakian.*

5. PSEUDOPROJECTIVELY FLAT K-CONTACT MANIFOLDS

A K-contact manifold is said to be pseudoprojectively flat if

$$g\big(P(\phi X, Y)Z, \phi W\big) = 0. \tag{5.1}$$

From (1.6) we have

$$\begin{aligned} g\big(P(\phi X, Y)Z, \phi W\big) &= R(\phi X, Y, Z, \phi W) \\ &\quad - \frac{1}{2n}\big(S(Y, Z)g(\phi X, \phi W) - S(\phi X, Z)g(Y, \phi W)\big), \end{aligned} \tag{5.2}$$

for all $X, Y, Z \in TM$. Let us take an orthonormal basis $\{e_1, e_2, \dots, e_{2n}, \xi\}$ in M^{2n+1}. Then from (5.2) we get

$$\begin{aligned} \sum_{i=1}^{2n} g\big(P(\phi e_i, Y)Z, \phi e_i\big) &= \sum_{i=1}^{2n} R(\phi e_i, Y, Z, \phi e_i) \\ &\quad - \frac{1}{2n}\sum_{i=1}^{2n}\big(S(Y, Z)g(\phi e_i, \phi e_i) - S(\phi e_i, Z)g(Y, \phi e_i)\big). \end{aligned} \tag{5.3}$$

In a $(2n+1)$-dimensional almost contact metric manifold, if $\{e_1, e_2, \dots, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in M, then $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \tag{5.4}$$

$$\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) - S(Y, \xi)\eta(Z), \tag{5.5}$$

for $X, Y \in TM$.

In a K-contact manifold we also have

$$R(\xi, Y, Z, \xi) = g(\phi Y, \phi Z), \tag{5.6}$$

for $X, Y \in TM$. Consequently,

$$\sum_{i=1}^{2n} R(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} R(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) - g(\phi Y, \phi Z). \tag{5.7}$$

Using the equations (5.4) to (5.7), in (5.3) we get

$$\begin{aligned}\sum_{i=1}^{2n} g\big(P(\phi e_i, Y)Z, \phi e_i\big) &= S(Y, Z) - g(\phi Y, \phi Z) \\ &\quad - S(Y, Z) + \frac{1}{2n}\big(S(Y, Z) - S(Y, \xi)\eta(Z)\big).\end{aligned} \tag{5.8}$$

If M^{2n+1} satisfies (5.1), then from (5.8) we get

$$S(Y, Z) - 2n\eta(Y)\eta(Z) - 2ng(Y, Z) + 2n\eta(Y)\eta(Z) = 0,$$

or,

$$S(Y, Z) = 2ng(Y, Z).$$

Hence we obtain the following:

PROPOSITION 5.1. *A pseudoprojectively flat K-contact manifold is an Einstein manifold.*

It is known that [3] a compact K-contact Einstein manifold is Sasakian. Thus we get the following:

THEOREM 5.1. *A compact pseudoprojectively flat K-contact manifold is Sasakian.*

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