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On Conics in Minkowski Planes

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Abstract: We study basic geometric properties of metric ellipses, hyperbolas, and parabolas in normed (or Minkowski) planes and obtain results on their shapes as well as respective extensions of further statements well known in the Euclidean plane. For the special case of polygonal norms, we prove a theorem on bunches of Minkowskian ellipses and hyperbolas which are pairwise Birkhoff orthogonal.

Key words: Birkhoff orthogonality, conics, ellipses, hyperbolas, Minkowski plane, parabolas, smooth normed plane, strictly convex normed plane.

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1. Introduction

It is surprising that there are not many results about the geometry of conics in arbitrary normed planes; see [4], [3], [6], [1] and [2]. In a natural way we continue here the investigations from [3]. It turns out that various well-known definitions of conics, equivalent in the Euclidean plane, do no longer coincide in normed planes, i.e., they can yield, in general, different types of curves. This is clarified in [3], and we use the most natural analogues of metrical definitions taken from there for our purpose (see below). Based on these definitions we present a collection of results describing conics in general normed planes and, particularly, also in polygonal normed planes. At least for the case of polygonal norms, we are also able to prove a theorem on bunches of Minkowskian ellipses and hyperbolas, which are pairwise Birkhoff orthogonal.

2. Notation and preliminary results

By $M^2(B)$ we denote the Minkowski plane with *unit disc* B, which is a compact, convex set with non-empty interior in $\mathbb{R}^2$, centered at the origin o. The boundary of B is called the *unit circle* of $M^2(B)$ and denoted by S, and $\|\cdot\|$ is the norm induced by B. By $B(m, r)$ we denote a homothet of B with center m and radius r, i.e., a *(Minkowskian) disc*. Its boundary is called a *(Minkowskian) circle*.

If x and y are two different points of the plane, we write $[x, y]$ for the *straight segment* joining x and y, i.e., the convex hull of x and y, and the *line* passing through x and y is denoted by (x, y).

A convex curve C is called *smooth* if through every point of C a unique supporting line of C passes, and it is called *strictly convex* if it does not contain any straight segment. A Minkowski plane $M^2(B)$ as well as the corresponding unit disc B are called *smooth* (*strictly convex*) if S is *smooth* (*strictly convex*).

If for two points x and y we have that $\|x\| \leq \|x + ty\|$ for all $t \in \mathbb{R}$, then x is called *Birkhoff orthogonal* to y and we write $x \perp_B y$. Note that this relation is, in general, not symmetric.

For the definition of metric ellipses and hyperbolas we use similar notation as in [4] and [3].

DEFINITION 1. Let $x, y \in M^2(B)$ and $c \in \mathbb{R}$ be such that $2c > \|x - y\|$. A *metric ellipse* with foci x and y and of size c is defined by

$$E(x, y, c) = \{z \in M^2(B) : \|z - x\| + \|z - y\| = 2c\}.$$

Without loss of generality we can consider the ellipse

$$E(x, c) = \{z \in M^2(B) : \|z - x\| + \|z + x\| = 2c\},$$

where $x \in C$. In this case, the condition for c is reduced to $c > 1$.

Due to [3] we have the following

PROPOSITION 1. *Let $E(x, c)$ be a metric ellipse. Then*

$$E(x, c) = \{z \in M^2(B) : \exists r > 0 \text{ such that } B(z, r) \text{ touches } B(x, 2c) \text{ from inside and contains -}x \text{ in its boundary.}\}$$

The boundary of $B(x, 2c)$ is called the *leading circle* of $E(x, c)$; see Figure 1.

This statement yields the possibility to visualise the shape of metric ellipses in an alternative way. We will use this more geometric definition later, in a proof.

DEFINITION 2. Let $x, y \in M^2(B)$ and $c \in \mathbb{R}$ be such that $0 < 2c < \|x - y\|$. A *metric hyperbola* with foci x and y and of size $2c$ is defined by

$$H(x, y, c) = \{z \in M^2(B) : \big| \|z - x\| - \|z - y\| \big| = 2c.\}$$

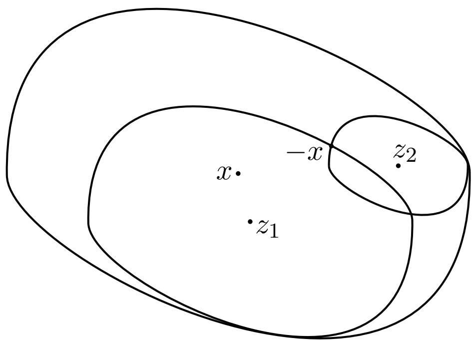

Figure 1: Two leading circles of an ellipse.

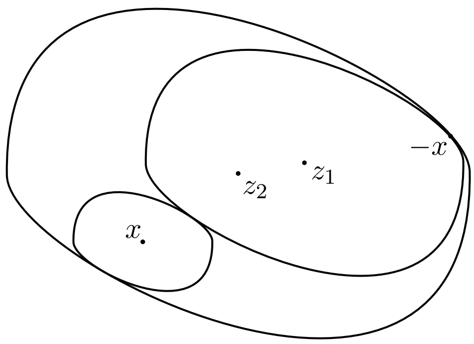

Figure 2: Two leading circles of a hyperbola.

Without loss of generality, we can consider

$$H(x,c) = \{z \in M^2(B) : \big| \|z-x\| - \|z+x\| \big| = 2c\},$$

where $x \in C$. In this case, the condition for c reduces to $0 < c < 1$.

In analogy to metric ellipses we have the following

Proposition 2. *Let $H(x,c)$ be a metric hyperbola. Then*

$$H(x,c) = \{z \in M^2(B) : \exists r > 0 \text{ such that } B(z,r) \text{ touches } B(x,2c) \text{ from outside and contains -}x \text{ in its boundary}\}.$$

The boundary of $B(x,2c)$ is called the *leading circle* of $H(x,c)$; see Figure 2.

In addition, we define metric parabolas.

Definition 3. Let $L \subset M^2(B)$ be a line and $x \in M^2(B) \setminus L$. A *metric parabola* with focus x and *leading line* L is defined by

$$P(x, L) = \{z \in M^2(B) : \|z - x\| = \min\{\|z - y\|, y \in L\}\}.$$

Without loss of generality, we let $x \in S$, and L should support B.

It is easy to see that the following statement holds.

Proposition 3. *Let $P(x, L)$ be a metric parabola. Then*

$$P(x, L) = \{z \in M^2(B) : \exists r > 0 \text{ such that } B(z, r) \text{ touches } L \text{ and contains } x \text{ in its boundary}\}.$$

From now on, and for the sake of simplicity, we sometimes write ellipse, hyperbola and parabola instead of metric ellipse, metric hyperbola and metric parabola.

3. On the shape of conics

In this section we will see that strict convexity (smoothness) of the unit disc is closely related to strict convexity (smoothness) of conic sections.

3.1. On metric ellipses.

In [4] we find basic properties of metric ellipses.

Proposition 4. *Let $M^2(B)$ be a Minkowski plane. Then*

- *$E(x, c)$ is a centrally symmetric, closed convex curve for every $x \in C$ and $c > 1$, and*
- *$E(x, c)$ is strictly convex for every $x \in C$ and $c > 1$ if and only if B is strictly convex.*

Senlin Wu clarifies the second property in some unpublished notes.

Theorem 1. *Let $x \in S$, $c > 1$ and $y_1,\ y_2 \in E(x, c)$. Then the following properties are equivalent:*

i) $[y_1, y_2] \subset E(x, c)$,

ii) $\left[\frac{x-y_1}{\|x-y_1\|}, \frac{x-y_2}{\|x-y_2\|}\right] \subset S$ *and* $\left[\frac{x+y_1}{\|x+y_1\|}, \frac{x+y_2}{\|x+y_2\|}\right] \subset S$.

This means that in general a segment in $E(x,c)$ is determined by two segments in the unit circle S. Senlin Wu also showed that, on the other hand, not any two segments in S yield a segment in $E(x,c)$.

THEOREM 2. *Let* $[a,b] \subset S$ *and* $[c,d] \subset S$, *where* a *and* c *as well as* b *and* d *are linearly independent,* $x \in S$ *and* $c' > 1$. *Then* $[y_1, y_2] \subset E(x,c')$, *where* $a = \frac{x-y_1}{\|x-y_1\|}$, $b = \frac{x-y_2}{\|x-y_2\|}$, $c = \frac{x+y_1}{\|x+y_1\|}$ *and* $d = \frac{x+y_2}{\|x+y_2\|}$, *if and only if* $|m| + |r| = |n| + |s|$, *where* $m, n, r, s > 0$ *satisfy* $2x = ma + rc = nb + sd$.

We introduce the notion of *corner points* and add a theorem that yields information about the connection between smoothness of B and smoothness of ellipses.

DEFINITION 4. A line L is said to *support locally* a closed curve C at $x \in C$, or to be a *locally supporting line* of C at x, if $x \in L$ and there is a neighbourhood $N(x)$ of x such that $N(x) \cap C$ is entirely contained in one of the two half-planes determined by L.

Let now C be a curve and $x \in C$. We call x a *corner point* of C if C is non-smooth in x, i.e., there is no unique locally supporting line of C at x. The *supporting cone* of C at x is defined as the set of all directions of locally supporting lines of C at x, and we call the two directions enclosing the cone *limit directions* of C at x.

Remark. The notion of *locally supporting line* coincides with the common definition for supporting line in the case when C is a convex curve.

THEOREM 3. *Let* $M^2(B)$ *be a Minkowski plane. Then* $E(x,c)$ *is smooth for every* $x \in C$ *and* $c > 1$ *if and only if* B *is smooth. More precisely,* $z \in E(x,c)$ *is a corner point of* $E(x,c)$ *if and only if* $\frac{z-x}{\|z-x\|}$ *or* $\frac{z+x}{\|z+x\|}$ *is a corner point of* S.

Proof. Let $x \in C$, $c > 1$, and $L = B(x, 2c)$ be the leading circle of $E(x,c)$. According to Proposition 1 we consider a disc $B(z,r)$ that touches L from inside and contains $-x$ in its boundary. When the disc "moves" between L and x, its center draws the ellipse $E(x,c)$. If we want to decide whether a point $z \in E(x,c)$ is a corner point of the ellipse, it suffices to study the movement of the disc (or rather its center) in a neighbourhood of z.

From now on, y denotes the intersection point of the line (x,z) with L and thus a touching point of L with the disc $B(z,r)$. There are three possibilities for the locus of the center z.

- Case 1: Neither $\frac{z-x}{\|z-x\|}$ nor $\frac{z+x}{\|z+x\|}$ is a corner point of S. In this case, the disc moves locally between the unique supporting lines at x and y, and thus we have smoothness.
- Case 2: $\frac{z-x}{\|z-x\|}$ is a corner point of S. First we assume that $\frac{z+x}{\|z+x\|}$ is not a corner point of S. This means that y is a corner point of $B(z,r)$ (as well as of L), but $-x$ is not. So the limit direction of the supporting lines of the disc in y moving into this position is different from the limit direction of the supporting lines at y when leaving this position, whereas the limit directions in $-x$ coincide. As in the first case, for every other position in a neighbourhood small enough, the change in the direction of the movement of z occurs exactly in this position, yielding that $E(x,c)$ is non-smooth.
- Case 3: $\frac{z+x}{\|z+x\|}$ is a corner point of S. Again we assume that $\frac{z-x}{\|z-x\|}$ is not a corner point of S. This means that $-x$ is a corner point of $B(z,r)$, but y is not. With the same argument as in case 2, we get that $E(x,c)$ is non-smooth in z.

It remains to state that if $\frac{z-x}{\|z-x\|}$ and $\frac{z+x}{\|z+x\|}$ are both corner points of S at the same time, then the curve of centers changes its direction two times in one point. Since the ellipse is convex, this results again in a corner point. ∎

Remark. The proof of Theorem 3 does not use the property that B is centrally symmetric. Thus the theorem holds also for *gauges*, i.e., for convex distance functions whose unit discs are not necessarily centrally symmetric.

Now we will have a closer look at normed planes with polygonal unit discs.

Definition 5. A Minkowski plane is called *polygonal* if B is the convex hull of finitely many points.

For information on conics in a special polygonal plane, the rectangular plane, we refer to [5].

Theorem 3 yields the following corollary.

Corollary 1. *Let $M^2(B)$ be a polygonal Minkowski plane. Then for all $x \in S$ and $c > 1$ the ellipse $E(x,c)$ is a polygon. Moreover, the possible directions of the sides of $E(x,c)$ do not depend on x or c, but only on the shape of B. More precisely, let s be one side of the polygon $E(x,c)$. Then there exist two sides of B such that the lines containing these sides intersect in a point u, where $(o,u) \parallel s$.*

Proof. Both statements follow from the main idea of the proof. We use the same notions and consider the disc $B(z,r)$ that moves between $-x$ and L. If B is polygonal, then the supporting lines of $B(z,r)$ at $-x$ and y do not change, unless one of these two points is a corner point. Let these two supporting lines intersect in a point v. Then z moves (locally) on the line (z,v). If on the other hand the two supporting lines are parallel, then z moves on a line parallel to them.

In particular, z moves on a straight line between each two of the corner points of the ellipse, and the direction of this straight line (= side of $E(x,c)$) does not depend on the position of x or the size of the ellipse, but only on the (local) properties of the supporting lines of $B(z,r)$ in $-x$ and y. ∎

3.2. On metric hyperbolas. Due to [3] the following holds.

Proposition 5. *Let $M^2(B)$ be a Minkowsky plane. Its unit disc B is strictly convex if and only if for every $x \in C$ and $0 < c < 1$ the hyperbola $H(x,c)$ consists of two simple curves, called branches, where each of them is intersected by any line parallel to (o,x) in exactly one point.*

We add the following

Proposition 6. *Let $M^2(B)$ be a Minkowski plane that is not strictly convex, and let $[u,v] \subset S$ be a straight segment in the boundary of B. Let $x \in M^2(B)$ be such that (o,x) is not parallel to (u,v). Then there exists a value $0 < c < 1$ such that $H(x,c)$ is not the union of two simple curves.*

Proof. We consider the leading circle $L = B(x, 2c)$ where c is chosen such that $x+v \in [x+u, -x]$; see Figure 3. Let $z \in M^2(B)$ be the intersection point of $(x, x+v)$ and $(-x, -x+u)$. Then every point of the cone $\{z+su+tv,\ s,t \in \mathbb{R}\}$ has equal distance to $-x$ and $x+v$, hence it belongs to $H(x,c)$. ∎

Concerning corner points, we have a result for hyperbolas which is similar like for ellipses.

Theorem 4. *Let $M^2(B)$ be a Minkowski plane. Let $x \in C$ and $0 < c < 1$ be such that $H(x,c)$ consists of two simple curves. If B is smooth, then the branches of $H(x,c)$ are smooth. More precisely, if $z \in H(x,c)$ is a corner point of $H(x,c)$, then $\frac{z-x}{\|z-x\|}$ or $\frac{z+x}{\|z+x\|}$ is a corner point of S.*

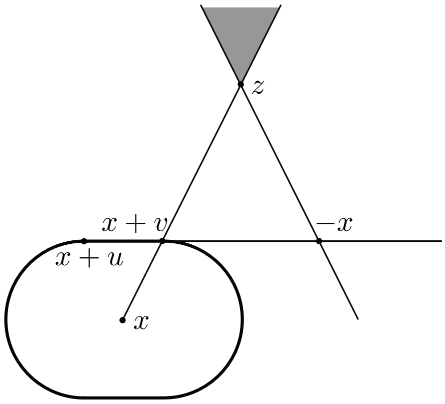

Figure 3: Every point of the shaded cone belongs to the hyperbola.

Proof. The proof works in the same way as that one for ellipses (Theorem 3), where we use local supporting lines instead of supporting lines. With analogous arguments we have that, for every corner point z of $H(x,c)$, $\frac{z-x}{\|z-x\|}$ or $\frac{z+x}{\|z+x\|}$ is a corner point of S. The other direction is, in general, not true. Assume that $\frac{z-x}{\|z-x\|}$ and $\frac{z+x}{\|z+x\|}$ are both corner points of S. In this case we have again two changes in direction. But since the branches of $H(x,c)$ are not necessarily convex, they may cancel out each other; see Figure 4 for an example. ∎

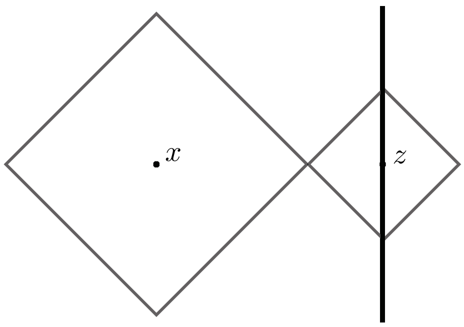

Figure 4: The hyperbola (thick line) has a "double corner point" in z.

3.3. On metric parabolas.

Theorem 5. *Let L be a line supporting B, and let $x \in S \setminus L$. Then $P(x, L)$ is a simple, convex curve.*

Proof. Ghandehari [1] showed that the area bounded by a parabola is convex.

It remains to prove that $P(x, L)$ is a simple curve. For this reason we show that each ray $\{x + tu,\ t > 0\}$ starting in x with direction $u \in S$ intersects $P(x, L)$ in at most one point. Assume that there are different values $s, t > 0$ such that $x + tu = y_1 \in P(x, L)$ and $x + su = y_2 \in P(x, L)$. Obviously, this is not possible for $(o, u) \parallel L$. Let $z = (x, x + u) \cap L$, and let $w_1, w_2 \in L$ with minimal distance to $y_1,\ y_2$, respectively; see Figure 5. Then the similarity of triangles yields

$$\frac{\|y_1 - z\|}{\|y_1 - w_1\|} = \frac{\|y_2 - z\|}{\|y_2 - w_2\|} \iff \frac{\|y_1 - z\|}{\|y_1 - x\|} = \frac{\|y_2 - z\|}{\|y_2 - x\|}$$

$$\iff (\|y_1 - x\| + \|x - z\|)\,\|y_2 - x\| = (\|y_2 - x\| + \|x - z\|)\|y_1 - x\|$$

$$\iff \|x - z\|\,\|y_2 - x\| = \|x - z\|\,\|y_1 - x\| \iff \|x - z\|\,\|y_2 - y_1\| = 0.$$

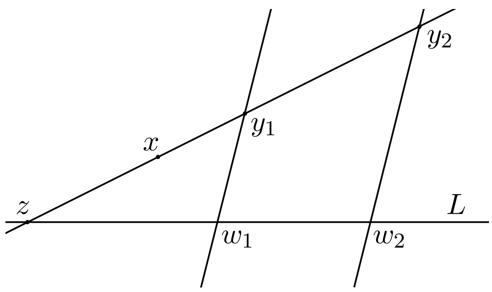

Figure 5: The lines intersecting L are Birkhoff orthogonal to L.

But since $x \notin L$ and $y_1 \neq y_2$, this is a contradiction. ∎

It is not surprising that we have a result on corner points of a parabola, and that the proof works in the same way as for ellipses and hyperbolas.

Theorem 6. *Let L be a line supporting B, and let $x \in S \setminus L$. The unit disc B is smooth if and only if $P(x, L)$ is smooth. More precisely, $z \in P(x, L)$ is a corner point of $P(x, L)$ if and only if $\frac{z-x}{\|z-x\|}$ is a corner point of S.*

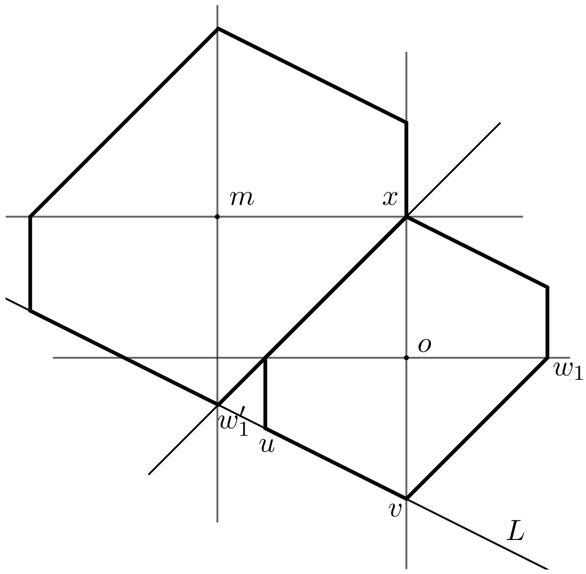

Figure 6: Theorem 7 for the polygonal case.

With the two results above we are able to prove the following theorem that is well known for the Euclidean subcase.

THEOREM 7. *Let L be a line supporting B, and let $x \in S \setminus L$. Let $U = S \cap L$. Then*

i) *the ray $\{x + tu,\ t > 0\}$ does not intersect $P(x, L)$ for all $u \in U$,*

ii) *the ray $\{x - tu,\ t > 0\}$ does intersect $P(x, L)$ for all $u \notin U$.*

Proof. First we assume that $M^2(B)$ is a polygonal Minkowski plane and that $U = [u, v]$, where $u \neq v$. Let $w_1 \in S$ be the corner point of B such that v lies between u and w_1 (with respect to the boundary of B). Let the line passing through x with direction $w_1 - v$ intersect L in w_1'. Then each circle centered at $m = w_1' - rv$, where $r \geq r_0 = \frac{\|w_1' - x\|}{\|w_1 - v\|}$, touches L and contains x in its boundary; see Figure 6. The equality $r = r_0$ yields that $m + rv = w_1'$ and $m + rw_1 = x$. For $r > r_0$, we still have $m + rv = w_1'$. In addition we have $x \in [w_1', m + rw_1]$. Thus, $m \in P(x, L)$ for all $r \geq r_0$.

Analogously we can define w_2 and w_2', and we get that $w_2' - ru \in P(x, L)$ for all $r \geq \frac{\|w_2' - x\|}{\|w_2 - u\|}$. Thus, $x - su - tv \notin P(x, L)$ for all $s, t > 0$. Clearly, every other ray starting in x intersects $P(x, L)$.

For $u = v$, let w_1 and w_2 be the corner points of B such that u lies between w_1 and w_2. Then the same arguments as above yield that the ray $x - tu$ is the only one that does not intersect the parabola.

The non-polygonal case follows by continuity. Let $n \in N$, and let n points be equally distributed in the boundary of a smooth unit disc B. The theorem holds for the convex hull of these points. With $n \to \infty$, the convex hull converges to B. ∎

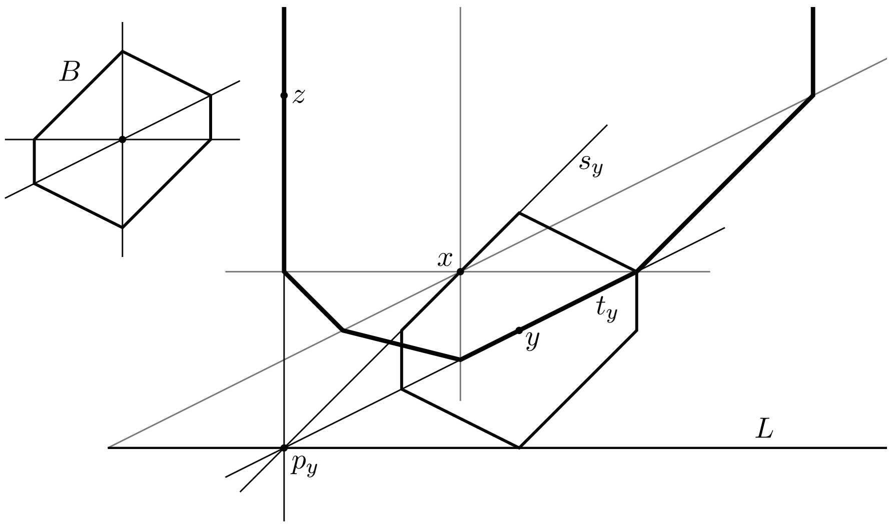

Figure 7: Notation in the proof of Theorem 8.

Another interesting statement is given by

THEOREM 8. *Let L be a line supporting B, and let $x \in S \setminus L$. Let $y, z \in P(x, L)$ such that $x \in (y, z)$, where neither y nor z is a corner point of $P(x, L)$. Then the tangent lines of $P(x, L)$ in y and z intersect in L.*

Proof. Again we discuss the case where $M^2(B)$ is a polygonal plane and conclude the general situation by continuity arguments.

As y is not a corner point of $P(x, L)$, we have that x is not a corner point of the disc $B(y, \|y - x\|)$. Let s_y be the (unique) line that supports this disc at x. From the former theorem we know that s_y intersects L. If it were parallel to L, then the line through y and x would not intersect the parabola for a second time. We call the intersection point p_y.

Consider now the disc that moves between L and x. Its midpoint "draws" the parabola. The tangent line t_y of $P(x, L)$ at y is determined by the movement of the center of the disc in a neighbourhood of y. As the disc moves locally between L and s_y, every point of B moves towards the intersection point of s_y and L; see Figure 7. In particular, $p_y \in t_y$.

On the other hand, with the same arguments there is a unique line s_z that supports $B(z, \|z - x\|)$ at x. Again we have that s_z intersects L, and every point of the disc, that moves between x and L in a neighbourhood of z, moves towards this intersection point. We call it p_z. In particular, the tangent line t_z of $P(x, L)$ at z contains p_z.

But since x lies on the line through y and z, we have that y and z lie on opposite and therefore parallel sides of the (corresponding) leading circle. Thus $s_y \parallel s_z$. Since both of them contain x, they are equal. It follows that $p_y = p_z$, hence t_y and t_z intersect in L. ∎

4. Intersecting ellipses and hyperbolas

For the Euclidean plane, the following statement is well known.

Theorem. *Let $E(x, c)$, $c > 1$, be an ellipse and $H(x, d)$, $0 < d < 1$, a hyperbola with arbitrary foci x and $-x$ that are identical for both conics. Let t_e and t_h be lines tangent at a common point of E and H. Then $t_e \perp t_h$.*

A similar, but slightly weaker theorem holds for polygonal Minkowski planes.

Theorem 9. *Let $M^2(B)$ be a polygonal Minkowski plane, and $x \in S$ be arbitrary. Let $E(x, c)$, $c > 1$, be an ellipse and $H(x, d)$, $0 < d < 1$, a hyperbola that consists of two simple curves. Let t_e and t_h be tangent lines of $E(x, c)$ and $H(x, c)$, respectively, at a common point of E and H. If c is large enough, then $t_h \perp_B t_e$.*

Later on we will see what "large enough" means in this context, and what happens in case that c is small.

To prove this theorem, we need some additional notation. If $M^2(B)$ is a polygonal plane, then there are only finitely many lines each passing through the origin and a corner point of B. We call the lines parallel to them and passing through x and $-x$ *corner lines*. These corner lines divide the plane into (bounded and unbounded) cells; see Figure 8. Any corner point of any metric conic with foci x and $-x$ lies on a corner line; within the cells there are only straight segments. We will have now a closer look at the unbounded cells.

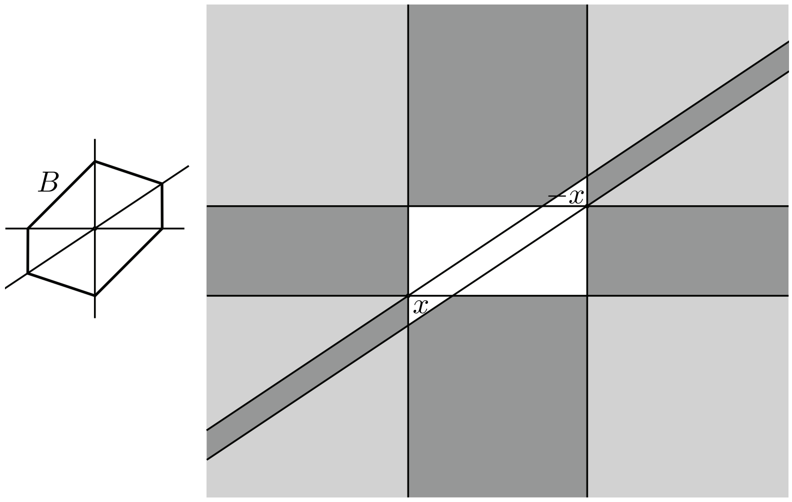

Figure 8: Cones (light grey) and strips (dark grey).

It is easy to see that the unbounded cells are either strips, bounded by two parallel corner lines, or cones, bounded by two corner lines corresponding to two adjacent corner points of B. In this sense, each strip is corresponding to a corner point of B, and each cone is corresponding to one side of the polygon B.

The following proposition yields information about the behaviour of ellipses and hyperbolas inside the cones.

PROPOSITION 7. *Let p_1 and p_2 be two points that lie in the boundary of one of the cones, but on different corner lines, such that the line (p_1, p_2) is parallel to the corresponding side of B; see Figure 9. Denote by q the vertex of the cone. Then we have*

i) $\|p - x\| + \|p + x\| = \|p_1 - x\| + \|p_1 + x\|$ *for all* $p \in [p_1, p_2]$, *and*

ii) $|\,\|p - x\| - \|p + x\|\,| = |\,\|q - x\| - \|q + x\|\,|$ *for all* $p \in [p_1, p_2]$.

In other words, if one point of the segment $[p_1, p_2]$ belongs to an ellipse *with foci x and $-x$, then any point of the line does. If one point of the cone belongs to a hyperbola with foci x and $-x$, then any point of the cone does.*

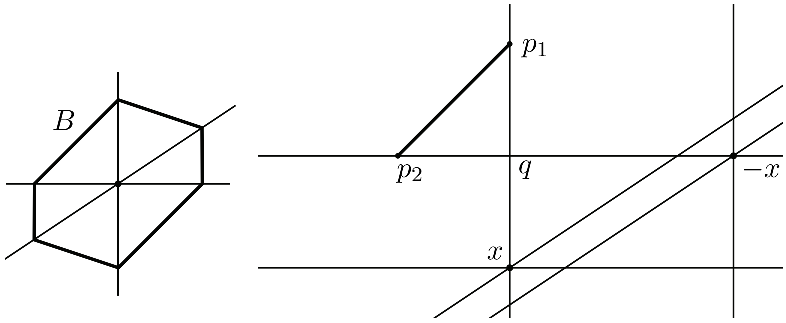

Figure 9: (p_1, p_2) is parallel to the corresponding side of B.

Proof. Obviously, $[p_1, p_2] \subset B(x, \|p_1 - x\|) \cap B(-x, \|p_2 + x\|)$. Thus $\|p - x\| = \|p_1 - x\|$ and $\|p + x\| = \|p_2 + x\| = \|p_1 + x\|$. In particular, i) holds.

In addition, $[p_1, p_2] \subset B(q, \|p_1 - q\|)$, i.e., $\|p - q\| = \|p_1 - q\|$ for all $p \in [p_1, p_2]$. Thus we have

$$\begin{aligned} \big| \|p - x\| - \|p + x\| \big| &= \big| \|p_1 - x\| - \|p_2 + x\| \big| \\ &= \big|(\|p_1 - q\| + \|q - x\|) - (\|p_2 - q\| + \|q + x\|)\big| \\ &= \big| \|q - x\| - \|q + x\| \big|, \end{aligned}$$

proving ii). ∎

Now we study the behaviour of hyperbolas within the strips.

PROPOSITION 8. *Let $u \in S$ be a corner point of B. There are two strips with sides parallel to the line (o, u). We consider the strip for which, with every point p, also the ray $R = \{p + tu,\ t > 0\}$ lies within that strip. If $p \in H(x, c)$ for any $0 < c < 1$, then $R \subseteq H(x, c)$. In other words: Whenever a hyperbola enters a strip, it stays inside.*

Proof. Let $p \in H(x, c)$ be any point of the hyperbola that lies within the strip, and q be any point of the ray R. We define a and b as the points that lie on the cornerline that passes through x, such that $\|x - p\| = \|x - a\|$

and $\|x-q\| = \|x-b\|$. Analogously, c and d are defined as the points lying on the corner line passing through $-x$, such that $\|-x-p\| = \|-x-c\|$ and $\|-x-q\| = \|-x-d\|$; see Figure 10. We note that $\|p-q\| = \|a-b\| = \|c-d\|$. Then we have

$$\begin{aligned}\|x-p\| - \|-x-p\| &= \|x-a\| - \|-x-c\| \\ &= \|x-a\| + \|a-b\| - \|-x-c\| - \|c-d\| \\ &= \|x-b\| - \|-x-d\| \\ &= \|x-q\| - \|-x-q\| .\end{aligned}$$

In particular, the absolute values of the first and of the last difference are equal, and thus $q \in H(x,c)$. ∎

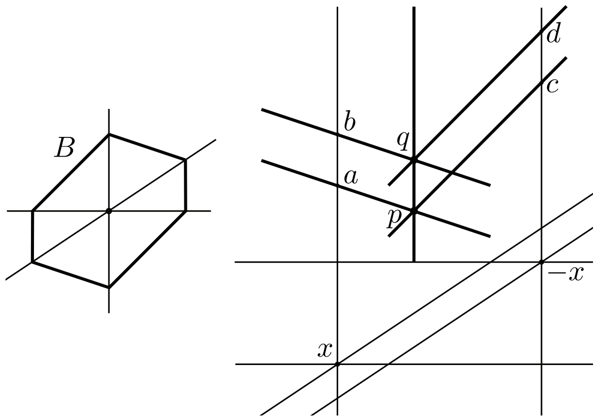

Figure 10: $p \in H(x,c) \iff q \in H(x,c)$.

Since ellipses are convex curves, we have that the slope of the ellipse within a strip is somewhere between the slopes in the adjacent cones, and thus somewhere between the slopes of the two sides of B that are adjacent to u.

Finally, we are able to prove Theorem 9.

Proof of Theorem 9. Let $c > 1$ be such that every intersection point of $E(x,c)$ with $H(x,d)$ lies within one of the unbounded cells. As $H(x,d)$ does not contain a cone, all these points have to lie within strips. Thus we have that $t_h \parallel (o,u)$. By definition, (o,u) is Birkhoff orthogonal to the two sides of B adjacent to u as well as to any line that has a slope that lies between the slopes of these two sides. Thus, $t_h \perp_B t_e$. ∎

Theorem 9 does not hold for every $c > 1$ in arbitrary Minkowski planes; see the counterexample in Figure 11.

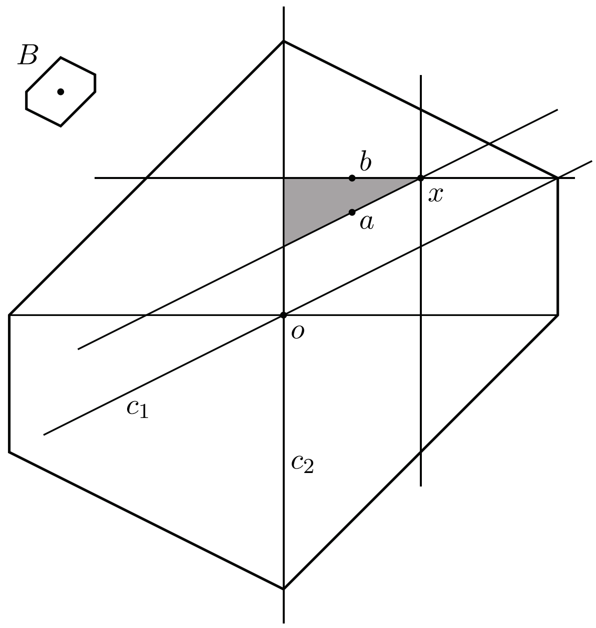

Figure 11: Counterexample if c is not large enough.

Within the shaded cell, the boundaries of the ellipse are parallel to the cornerline c_1, thus only lines parallel to the vertical cornerline c_2 are Birkhoff orthogonal to c_1. However, we have that $\|x - a\| = \|x - b\| = 2$, but $\|a\| = 4$ and $\|b\| = 5$. Thus, the hyperbola that passes through a does not move parallel to c_2 within the shaded cell. Hence, there the ellipse and the hyperbola do not intersect Birkhoff orthogonally.

On the other hand, it is possible that the statement holds for every $c > 1$ in certain normed planes; see Figure 12 for an example.

Again, the boundaries of the ellipse are parallel to cornerlines within the shaded cells, namely parallel to c_1 in the brighter cells and parallel to c_2 in the darker ones. But since the plane is a Radon plane, the corner lines are parallel to the sides of B, and thus the statement of Theorem 9 holds.

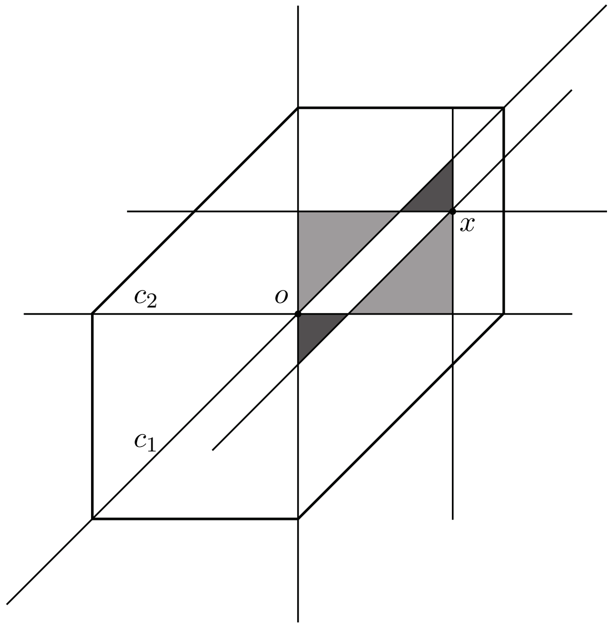

Figure 12: Also for arbitrary $c > 1$ Theorem 9 can be true.

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