

Subalgebras of $\mathfrak{gl}(3, \mathbb{R})$

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Abstract: This paper finds all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ up to change of basis in \mathbb{R}^3 . For each such algebra the corresponding matrix Lie subgroup of $GL(3, \mathbb{R})$ obtained by exponentiation is given. An interesting phenomenon is that for algebras of dimension three and higher one frequently encounters inequivalent subalgebras and subgroups, one obtained from the other, by transposing about the anti-diagonal.

Key words: three-dimensional matrix, Lie algebra, Lie group, solvable algebra, non-trivial Levi decomposition.

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1. INTRODUCTION

This paper is concerned with finding all the subalgebras of the 3×3 space of matrices $\mathfrak{gl}(3, \mathbb{R})$. For a number of reasons it is desirable to know all such subalgebras up to change of basis in \mathbb{R}^3 . For example, one question of interest is to find all the representations of the three-dimensional Lie algebras in $\mathfrak{gl}(3, \mathbb{R})$. Another related problem is that of finding minimal dimensional representations of the four and higher dimensional Lie algebras. In a different direction an interesting question is to find the holonomy groups of three-dimensional linear connections.

Rather than giving subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ as a list of 3×3 space of matrices we shall in each case give a matrix *subgroup* of $GL(3, \mathbb{R})$ whose Lie algebras produces the given $\mathfrak{gl}(3, \mathbb{R})$ in the standard way. It is *not* claimed that we are giving all possible Lie subgroups of $GL(3, \mathbb{R})$ and in fact we are not: we shall leave that issue for another day. However, it is more convenient to give the matrix Lie group that is generated by exponentiating the basis elements in the Lie algebra, rather than listing those matrices separately.

In Section 2 we revisit the subalgebras of $\mathfrak{gl}(2, \mathbb{R})$ again working at the group rather than the algebra level. In the succeeding Sections we go through,

each dimension at a time, the subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ considering first of all indecomposable and then decomposable algebras. In Section 12 we summarize our results by listing the corresponding matrix subgroup of $GL(3, \mathbb{R})$. Appendix we list a number of “abstract” low-dimensional *indecomposable* Lie algebras that arise in the classification of the subalgebras of $\mathfrak{gl}(3, \mathbb{R})$.

A striking result is that the standard representation of, for example, the (three-dimensional) Heisenberg algebra and group is not the most general representation in $\mathfrak{gl}(3, \mathbb{R})$ or $GL(3, \mathbb{R})$, see Section 5, the point is that multiples of the identity may be added to two of the generators. Another interesting phenomenon is that in obtaining all possible representations of the three and higher dimensional algebras we obtain “dual” representations, one obtained from the other by, roughly speaking, transposing the group matrix about its anti-diagonal.

As regards references we refer to [1] for basic facts about Lie algebras and to [3] for the Schur-Jacobson theorem that is quoted in Section 5. We mention [2, 4, 5] for information about and lists of the low-dimensional solvable Lie algebras. Finally we refer also to [6] for lists of low-dimensional Lie algebras that have a non-trivial Levi decomposition.

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2. TWO-DIMENSIONAL SUBALGEBRAS OF $\mathfrak{gl}(2, \mathbb{R})$

For the convenience of the reader we list all the subalgebras of $\mathfrak{gl}(2, \mathbb{R})$. In fact we shall not quite do that; instead we give a matrix *subgroup* of $GL(2, \mathbb{R})$ such that its Lie algebra corresponds to one of the possible Lie subalgebras of $\mathfrak{gl}(2, \mathbb{R})$. It is *not* claimed that we are giving all possible Lie subgroups of $GL(2, \mathbb{R})$ and in fact we are not. However, by giving the matrix Lie group that is obtained by exponentiating the generators in the Lie algebra one gets a much better feel for the algebra rather than listing a number of matrices separately. Here are the *subgroups* of $GL(2, \mathbb{R})$ where a denotes an arbitrary real number

$$\begin{aligned} & \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^x & 0 \\ 0 & e^{ax} \end{bmatrix} \begin{bmatrix} e^x & xe^x \\ 0 & e^x \end{bmatrix} \begin{bmatrix} e^{ax} \cos x & e^{ax} \sin x \\ -e^{ax} \sin x & e^{ax} \cos x \end{bmatrix} \\ & \begin{bmatrix} e^x & 0 \\ 0 & e^y \end{bmatrix} \begin{bmatrix} e^{(a+1)x} & y \\ 0 & e^{ax} \end{bmatrix} \begin{bmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{bmatrix} \begin{bmatrix} e^x & z \\ 0 & e^y \end{bmatrix} \end{aligned}$$

to which have to be added $SL(2, \mathbb{R})$ and $GL(2, \mathbb{R})$ itself.

It is instructive to see where the list above comes from. So briefly, the first four on the list arise from putting a 2×2 matrix into Jordan normal form. Then, we find the two-dimensional abelian algebras by taking in turn each of the first four algebras and finding a matrix that commutes with them. The first and the third give rise to the sixth matrix in the special case $a = 1$. For the second matrix on the list if $a \neq 1$ we get the fifth matrix on the list and the fourth gives rise to the seventh. It is easy to say that the second matrix for which $a = 1$ gives no new cases.

Of the eight listed matrices above only the sixth and eighth give non-abelian Lie algebras. Let us consider how to represent the non-abelian two-dimensional algebra $[e_1, e_2] = e_2$ in $\mathfrak{gl}(2, \mathbb{R})$. We seek two 2×2 matrices E_1, E_2 such that $[E_1, E_2] = E_2$. Since E_2 is a commutator it has trace zero and so by Lie's Theorem if there is such a representation then it has generators of the form

$$E_1 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad E_2 = \begin{bmatrix} d & e \\ 0 & -d \end{bmatrix}.$$

Now if we demand that $[E_1, E_2] - E_2 = 0$ we find that $d = 0$ and so $e \neq 0$ and so also $c = a - 1$. We may scale E_2 so as to assume that $e = 1$; thereafter we subtract a multiple of E_2 from E_1 so as to obtain $b = 0$. We obtain thereby the sixth matrix group in the list above on replacing a by $a + 1$.

To obtain three-dimensional algebras we note that the identity commutes with every 2×2 matrix; as such we are reduced to looking for two-dimensional algebras of $\mathfrak{sl}(2, \mathbb{R})$ in addition to $\mathfrak{sl}(2, \mathbb{R})$ itself. It is not difficult to convince oneself that, up to change of basis, the only possibility is the space of upper (or lower) triangular matrices and gives the eighth matrix listed above as the corresponding group matrix.

3. ONE-DIMENSIONAL ALGEBRAS

To obtain the one-dimensional algebras simply take the real Jordan Normal Forms for 3×3 matrices and we obtain the following possibilities:

LEMMA 3.1. *Over \mathbb{R} every 3×3 matrix is equivalent under change of basis to one of the following:*

$$(a) \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad (b) \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad (c) \begin{bmatrix} \alpha & 1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad (d) \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

We have the freedom to scale each such matrix and so we obtain the matrix Lie groups given in Section 11.

4. TWO-DIMENSIONAL ALGEBRAS

4.1. INDECOMPOSABLE ALGEBRAS. We know that up to isomorphism there are just two “abstract” Lie algebras of dimension two, the abelian algebra \mathbb{R}^2 and the one for which $[e_1, e_2] = e_1$. To find the abelian subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ we take a matrix A of the form found in Section 3. Then we find the most general matrix B that commutes with it and then normalize the entries of B .

1. To begin with let A be a matrix of type (a) in Section 3. We may assume that both matrices are of type (a): if not one of the matrices is of type (b), (c) or (d), each of which will be considered below in turn. Since two commuting diagonalizable matrices are simultaneously diagonalizable we can reduce to

$$(a) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{bmatrix}.$$

2. Now let A be a matrix of type (b) in Section 3. We have to distinguish two cases depending on whether $\lambda \neq \mu$ or $\lambda = \mu$. Suppose the former case holds and that the matrix is called A . Then the most general matrix that commutes with A is of the form

$$B = \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix}.$$

Then considering $\overline{B} := bA - B$ reduces b to zero. If $a \neq 0$ we can reduce a to unity and λ to zero and μ to zero or unity. If, however, $a = 0$ we can assume that $d = 1$ and that $\lambda = 0, 1$. To summarize, in this subcase we have:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} (\mu = 0, 1) \quad \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\lambda = 0, 1).$$

Now suppose that A is a matrix of type (b) in Section 3 and that $\lambda = \mu$. Then the most general matrix that commutes with A is of the form

$$B = \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & e & d \end{bmatrix}.$$

Now if $a \neq d$ then the matrix

$$\begin{bmatrix} 1 & 0 & -\frac{c}{d-a} \\ 0 & 1 & 0 \\ 0 & \frac{e}{d-a} & 1 \end{bmatrix}$$

preserves A and reduces c and e in B to zero. Thus we are reduced to the case of the last paragraph.

The only remaining case is where both matrices are of type (b) in Section 3 with $\lambda = \mu$. In this case we can easily reduce to

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} (\lambda = 0, 1) \quad \begin{bmatrix} a & 0 & c \\ 0 & a & 0 \\ 0 & e & a \end{bmatrix} (a = 0, 1).$$

If we permute the basis vectors e_2 and e_3 then we obtain the matrices in upper triangular form.

3. For a matrix A of type (c) in Section 3 the only possibility for a matrix B that commutes with A is

$$B = \begin{bmatrix} \gamma & \delta & 0 \\ -\delta & \gamma & 0 \\ 0 & 0 & \epsilon \end{bmatrix}.$$

Now we can assume that $\delta = 0$ and then either $\alpha = 0$ and $\gamma = 1$ or $\gamma = 0$, $\epsilon = 1$ and $\beta = 0$.

4. We conclude with a matrix of type (d) in Section 3 that we call A . It has only one Jordan block and so in particular is “non-derogatory”. The only matrices that commute with it are powers of A or equivalently are of the form

$$B = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

If $\lambda \neq 0$ by taking linear combinations and scaling we may assume that $\lambda = 1$ and $a = 0$. Note that if we divide A by λ we may conjugate by a diagonal matrix so as to restore the $(1, 2)$ and $(2, 3)$ entries to unity. We can now scale B by $\frac{1}{b}$ so as to reduce to $b = 1$ if $b \neq 0$. If, however, $b = 0$ then $c \neq 0$ and we can scale c to unity.

If $\lambda = 0$ but $a \neq 0$ we can assume that $b \neq 0$ and then we can reduce the second generator to type (d) in Section 3 for which $\lambda \neq 0$, a case which has just been considered. Finally if $\lambda = a = 0$ or then we can reduce to $b = 0, c = 1$. We can summarize these cases as:

$$(a) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (b) \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} (a = 0, 1) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4.2. TWO-DIMENSIONAL NON-ABELIAN ALGEBRAS. We write the algebra as $[A, B] = A$. Then we take in turn A to be a matrix of type (a), (b), (c) or (d) that appears in Section 3. It turns out that in the case of (a) and (c) there is no solution because of the fact the commutator of an arbitrary matrix and a diagonal matrix has zeroes on the diagonal.

For a matrix of type (b) we find that necessarily $\lambda = \mu = 0$. Then we can subtract a multiple of A from B so as to obtain for A and B

$$(b) \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a & 0 & c \\ 0 & a+1 & 0 \\ 0 & d & e \end{bmatrix}.$$

If $a \neq e$ we can assume that $c = 0$ and if $a - e + 1 \neq 0$ that $d = 0$. Furthermore in these last two cases we may assume by change of basis that $d = 0, 1$ and $c = 0, 1$.

For type (d) It turns out that necessarily $\lambda = 0$. The solution is

$$(d) \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b & 0 & c \\ 0 & b+1 & 0 \\ 0 & 0 & b+2 \end{bmatrix}.$$

However, by change of basis that fixes A , it is possible to reduce c to zero.

5. THREE-DIMENSIONAL ALGEBRAS

5.1. INDECOMPOSABLE ALGEBRAS. Suppose that \mathfrak{g} is a three-dimensional abstract Lie algebra and first of all suppose that \mathfrak{g} is indecomposable.

The list of such algebras can be found in [2, 4]. All but two, $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3)$ are simple. For $\mathfrak{sl}(2, \mathbb{R})$ there are, up to isomorphism, two representations in $\mathfrak{gl}(3, \mathbb{R})$; one coming from the definition of $\mathfrak{sl}(2, \mathbb{R})$ as 2×2 matrices and the second from its adjoint representation. For $\mathfrak{so}(3)$ there is just the standard representation in $\mathfrak{gl}(3, \mathbb{R})$ as skew-symmetric matrices.

The remaining three-dimensional algebras are solvable and only the Heisenberg algebra, which is nilpotent, has a non-trivial center; it has only one non-zero bracket $[e_2, e_3] = e_1$ in a basis $\{e_1, e_2, e_3\}$. The non-zero brackets of a solvable not nilpotent algebra are given by $[e_1, e_3] = ae_1 + be_2$, $[e_2, e_3] = ce_1 + de_2$. Any representation is by Lie's theorem upper triangular over \mathbb{C} .

PROPOSITION 5.1. (SCHUR-JACOBSON [3]) *The maximal commutative subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ is of dimension $1 + [\frac{n^2}{4}]$ where $[\]$ denotes the integer part of a real number. Up to change of basis if n is even the subalgebra consists of the upper left hand block with row entries running from $1 + \frac{n}{2}$ to n and column entries from 1 to $\frac{n}{2}$ together with multiples of the identity; if n is odd the subalgebra consists of the upper left hand blocks with row entries running from either $\frac{n+1}{2}$ to n and column entries from 1 to $\frac{n-1}{2}$ or $\frac{n+3}{2}$ to n and 1 to $\frac{n+1}{2}$, respectively, together with multiples of the identity.*

We note that in the preceding Proposition the cases $n = 2, 3$ are different from $n > 3$ because only in those cases is $n = 1 + [\frac{n^2}{4}]$ so that the dimension of the subalgebras described is the same as the space of diagonal matrices. In the case $n = 3$ an abelian three-dimensional subalgebra consists either of three independent diagonal matrices or the identity and either matrices whose only non-zero entries are in the $(1, 2)$ and $(1, 3)$ or $(1, 3)$ and $(2, 3)$ entries but in all such cases we have the following result: although the result follows from Schur-Jacobson we supply an independent proof.

LEMMA 5.2. *Any abelian three-dimensional subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ contains a multiple of the identity.*

Proof. We refer to Lemma 3.1. In case (d) the space of matrices that commutes with the given matrix is of the form

$$\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

Likewise in case (c) the space of matrices that commutes with the given matrix is of the form

$$\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

and hence the result follows in these two cases. In case (b) if $\lambda \neq \mu$ the centralizer consists of matrices of the form

$$\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

so if there is an abelian three-dimensional algebra it will contain I . In case (b) if $\lambda = \mu$ the centralizer consists of matrices of the form

$$\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{bmatrix}.$$

Now we may assume that $e = a$ or else we would be in a different subcase. Given two such commuting matrices by subtracting multiples of the original matrix we may assume that $b = 0$ and so a linear combination of them will be a multiple of I .

Finally, in case (a) we can assume that all three matrices in the algebra have real eigenvalues of algebraic multiplicity one; otherwise the algebra would fall into case (b), (c) or (d). Thus all three matrices are diagonalizable and since they commute, simultaneously diagonalizable and linearly independent. Hence again the algebra will contain I . ■

COROLLARY 5.3. *A subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ that contains an abelian three-dimensional subalgebra is decomposable as an abstract Lie algebra.*

Let us consider the Heisenberg algebra and suppose that we have a representation of it in $\mathfrak{gl}(3, \mathbb{R})$. We denote the representing matrices corresponding to the basis $\{e_1, e_2, e_3\}$ by E_1, E_2, E_3 . By Lie's theorem we can assume that E_1, E_2, E_3 are upper triangular over \mathbb{C} . One possibility for having E_1 and E_2 commute would be to choose diagonal matrices but then if E_3 were upper triangular $[E_2, E_3]$ would be strictly upper triangular and we could not satisfy the bracket relation. So either by direct verification or else using the result of

Schur-Jacobson [3] we may assume that

$$E_1 = \begin{bmatrix} a & \beta & \alpha \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \quad E_2 = \begin{bmatrix} b & \delta & \gamma \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$$

or

$$E_1 = \begin{bmatrix} a & 0 & \alpha \\ 0 & a & \beta \\ 0 & 0 & a \end{bmatrix}, \quad E_2 = \begin{bmatrix} b & 0 & \gamma \\ 0 & b & \delta \\ 0 & 0 & b \end{bmatrix}$$

where $\alpha\delta - \beta\gamma \neq 0$.

By taking linear combinations of E_1 and E_2 and renaming a and b we may assume that $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$: note that we are using “gauge” freedom here to simplify the form of the algebra since $[E_1, E_2] = 0$. Assuming now that E_3 is upper triangular we find that in order to have $[E_2, E_3] = E_1$ then $a = 0$ and E_3 must be of the form

$$E_3 = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{bmatrix}$$

or

$$E_3 = \begin{bmatrix} c & 1 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}.$$

In either case we recover the standard representation for the Heisenberg algebra up to the addition of arbitrary multiples of the identity in the generators E_2 and E_3 .

In the case of solvable not nilpotent indecomposable algebras for which $[e_1, e_3] = ae_1 + be_2$, $[e_2, e_3] = ce_1 + de_2$ and $ad - bc \neq 0$ we proceed similarly and reduce to:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that since $ad - bc \neq 0$ each of E_1 and E_2 must be a sum of commutators and so each must have trace zero.

In the first case we may assume that

$$E_3 = \begin{bmatrix} e & 0 & 0 \\ 0 & f & g \\ 0 & h & k \end{bmatrix}$$

so that

$$[E_1, E_3] = (f - e)E_1 + gE_2, \quad [E_2, E_3] = hE_1 + (k - e)E_2.$$

It is true that E_3 here is not upper triangular: we could make it so but then we would have to consider subcases which is in fact unnecessary.

We may solve as $f = a + e$, $g = b$, $h = c$, $k = d + e$ so as to obtain

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} e & 0 & 0 \\ 0 & a + e & b \\ 0 & c & d + e \end{bmatrix}.$$

Similarly in the second case we may reduce to

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} k - d & -b & 0 \\ -c & k - a & 0 \\ 0 & 0 & k \end{bmatrix}.$$

Thus we have found all the three-dimensional solvable indecomposable subalgebras of $\mathfrak{gl}(3, \mathbb{R})$. Up to change of basis in \mathbb{R}^3 there are two forms of these subalgebras; furthermore in the representing matrices there is an ambiguity in that arbitrary multiples of the identity may be added to E_3 . It is apparent that the relationship between the two forms of the subalgebra is that one is obtained from the other by taking the negative and then transposing about the anti-diagonal: the negative is not important since a, b, c, d are arbitrary. We explicate this phenomenon as follows.

PROPOSITION 5.4. *Suppose that the Lie algebra \mathfrak{g} has a representation as a subalgebra of $\mathfrak{gl}(p, \mathbb{R})$. Suppose that $L : \mathfrak{gl}(p, \mathbb{R}) \rightarrow \mathfrak{gl}(p, \mathbb{R})$ is a (linear) involution. Then mapping a representing matrix M to $-LM^tL$ gives a second not necessarily equivalent representation of \mathfrak{g} .*

Proof. Given $M \in \mathfrak{gl}(p, \mathbb{R})$ a representing matrix for \mathfrak{g} , map it to $\phi(M) = -LM^tL$. For a second such matrix N we have

$$\begin{aligned} [\phi(M), \phi(N)] &= [-LM^tL, -LN^tL] = [LM^tLLN^tL, LN^tLLM^tL] \\ &= [LM^tN^tL - LM^tN^tL] = L[N, M]^tL = -\phi([M, N]). \end{aligned}$$

■

COROLLARY 5.5. *Suppose that the Lie algebra \mathfrak{g} has a representation as a subalgebra of $\mathfrak{gl}(p, \mathbb{R})$. Then transposing the representing matrices about the anti-diagonal gives a second inequivalent representation of \mathfrak{g} .*

Proof. Take for L in the Proposition the matrix whose only non-zero entries are 1's down the anti-diagonal. Then the map ϕ is transposition the about the anti-diagonal. ■

5.2. DECOMPOSABLE ALGEBRAS. Now we consider decomposable three-dimensional abstract algebras. For such an algebra there are two possibilities: either it splits as three one-dimensional algebras or it splits as a direct sum of a one and two-dimensional indecomposable algebra. In the first case we can appeal to the result of Schur-Jacobson [3] to conclude that a three-dimensional abelian subalgebra must be spanned by matrices of the form

$$\begin{bmatrix} z & x & y \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} z & 0 & x \\ 0 & z & y \\ 0 & 0 & z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}.$$

In the second case we start with an indecomposable two-dimensional algebra and in each case we look for a matrix that commutes with both generators. Quoting from subsection 3.2 consider a pair A, B of matrices of type (b) with $(a - e)(a - e + 1) \neq 0$. Then a matrix commutes with and is linearly independent of the two generators if and only if it is of the form

$$\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}.$$

If $a = e$ the entry c is not necessarily zero and in this case a matrix commutes with and is linearly independent of the two generators if and only if it is of the form

$$\begin{bmatrix} x & 0 & y \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}.$$

If $a + 1 = e$ the entry d is not necessarily zero and in this case a matrix commutes with and is linearly independent of the two generators if and only if it is of the form

$$\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & y & x \end{bmatrix}.$$

For a pair A, B of matrices of type (d) in subsection 4.2 it is easy to check that the only matrix that commutes with A and B is a multiple of the identity.

6. FOUR-DIMENSIONAL ALGEBRAS

6.1. INDECOMPOSABLE ALGEBRAS. For the list of four-dimensional indecomposable algebras we refer to [4] which discerns twelve classes of algebra all of which are solvable. Hence if they can be represented in $\mathfrak{gl}(3, \mathbb{R})$ they can be represented by upper triangular matrices over \mathbb{C} . The first six have a three-dimensional ideal so by Corollary 5.2 none of them can be represented in $\mathfrak{gl}(3, \mathbb{R})$.

Let us see next which of algebras 4.7-4.11 can actually be represented in $\mathfrak{gl}(3, \mathbb{R})$. According to Section 4 we may assume over \mathbb{C} that

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \quad E_3 = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}, \quad E_4 = \begin{bmatrix} c & 0 & 0 \\ f & d & 0 \\ h & g & e \end{bmatrix}.$$

Here E_1, E_2, E_3 come from Section 5 in the context of the Heisenberg algebra and E_4 is reduced only by subtracting linear combinations of E_1, E_2, E_3 . To continue we note that for algebras 4.7-4.11 the matrix E_1 spans the center of the nilradical and therefore generates an ideal. Hence $[E_1, E_4]$ must be a multiple of E_1 which gives $f = g = h = 0$. Furthermore in order to have a Lie algebra at all (closure) we need that $a = b = 0$. At this point we obtain

$$\begin{aligned} [E_2, E_3] &= E_1, & [E_1, E_4] &= (e - c)E_1, \\ [E_2, E_4] &= (d - c)E_2, & [E_3, E_4] &= (e - d)E_3. \end{aligned}$$

If we consult algebras 4.7-4.11 we see that we cannot obtain 4.7 because $ad(e_4)$ is not semi-simple: however, bearing in mind that we are working over \mathbb{C} we see that 4.8-4.11 are possible. Furthermore we may assume, without loss of generality, that the restriction of the representation to the nilradical, gives the standard representation of the Heisenberg algebra.

A matrix Lie group that engenders algebras 4.8 and 4.9 is given by

$$e^{aw} \begin{bmatrix} e^{(b+1)w} & x & z \\ 0 & e^{bw} & y \\ 0 & 0 & 1 \end{bmatrix}$$

where the coordinates are taken in the order (z, x, y, w) and a is arbitrary.

Likewise a second representation is obtained is given by

$$e^{aw} \begin{bmatrix} 1 & x & z \\ 0 & e^{bw} & y \\ 0 & 0 & e^{(b+1)w} \end{bmatrix}$$

Up to change of basis these two group matrices give the possible representations of algebras 4.8 and 4.9b.

However, it turns out that neither algebra 4.10 nor 4.11a has a faithful representation in $\mathfrak{gl}(3, \mathbb{R})$. For if so, just as in the preceding paragraph, the nilradical could be put into real strictly upper triangular form by a real transformation. Thus without loss of generality we could assume that e_1 , e_2 and e_3 are represented by E_1 , E_2 and E_3 as in equation 1. Of course we have no control over the matrix E_4 that represents e_4 but it does not matter: it is impossible to satisfy either of the brackets $[e_2, e_4] = ae_2 - e_3$, $[e_3, e_4] = e_2 + ae_3$. In fact with E_1 , E_2 and E_3 as in equation 1 and E_4 arbitrary the matrix $[E_2, E_4] - aE_2 + E_3$ has (2, 3)-entry 1. Hence there can be no representation in $\mathfrak{gl}(3, \mathbb{R})$.

The following matrix Lie group gives a three-dimensional representation for algebra 4.12 where a is arbitrary:

$$e^{az} \begin{bmatrix} e^z \cos w & e^z \sin w & x \\ -e^z \sin w & e^z \cos w & y \\ 0 & 0 & 1 \end{bmatrix}.$$

As usual there is a second class of representations given by

$$e^{az} \begin{bmatrix} 1 & x & y \\ 0 & e^z \cos w & e^z \sin w \\ 0 & -e^z \sin w & e^z \cos w \end{bmatrix}$$

but these are the only classes of representations up to change of basis.

6.2. DECOMPOSABLE ALGEBRAS. If an algebra is decomposable it must decompose, as an abstract algebra, as a sum of either \mathbb{R} and a three-dimensional indecomposable, a sum of a two-dimensional indecomposable and two copies of \mathbb{R} , or a sum of four copies of \mathbb{R} , or a sum of two two-dimensional indecomposables. However, these latter two alternatives cannot occur. There

is no four-dimensional abelian subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ according to the Schur-Jacobson result. Also, if we consider a two-dimensional indecomposable subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ we have found all matrices that commute with them in subsection 5.2. However, we can see in each case that when the space of matrices that commute is two-dimensional, it itself is an abelian subalgebra.

Turning now to the first alternative above it is easy to see that the only possibility is to add multiples of the identity to a three-dimensional indecomposable with one exception: indeed for algebras 3.1-3.7 we can appeal to Schur-Jacobson and for 3.9 and the second subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ that is isomorphic to 3.8 we can use Schur's lemma. However, for the first subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ that is isomorphic to 3.8 the space of commuting matrices is of the form

$$\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}.$$

For the second of the alternatives above we refer to subsection 5.2 for the space of matrices that commute with the non-abelian two-dimensional algebras.

7. FIVE-DIMENSIONAL ALGEBRAS

7.1. INDECOMPOSABLE ALGEBRAS. Let us now consider a five-dimensional solvable indecomposable algebra \mathfrak{g} that is a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$. According to [4] the nilradical of \mathfrak{g} can only be one of \mathfrak{g} , \mathbb{R}^4 , $H \oplus \mathbb{R}$, \mathbb{R}^3 or H , where H denotes the Heisenberg algebra. However, in the case of \mathbb{R}^4 , $H \oplus \mathbb{R}$, \mathbb{R}^3 the Lie algebra has a three-dimensional abelian subalgebra and so would be decomposable according to Lemma 5.1. Now if the nilradical of \mathfrak{g} is \mathfrak{g} itself then \mathfrak{g} is nilpotent. The five-dimensional nilpotent indecomposable algebras are listed in [4]: see also the Appendix of the present article. We see in each case that $\{e_1, e_2, e_3\}$ span a three-dimensional abelian subalgebra and so if \mathfrak{g} was a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ then \mathfrak{g} would be decomposable, which is a contradiction. Thus the only possibility is that the nilradical of \mathfrak{g} is isomorphic to H . Of the five-dimensional solvable indecomposable algebras, precisely two, 5.36 and 5.37 [4] have nilradical isomorphic to H .

PROPOSITION 7.1. *The only solvable five-dimensional indecomposable algebras that can be represented faithfully in $\mathfrak{gl}(3, \mathbb{R})$ are 5.36 or 5.37.*

We continue with \mathfrak{g} as at the start of this Section. Since \mathfrak{g} is solvable it has a representation by upper triangular matrices over \mathbb{C} and since the commutator of two upper triangular matrices is a strictly upper triangular matrix we may assume that H has its standard representation in the 3×3 strictly upper triangular matrices. Thus we may write

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If E_4 or E_5 has a complex 2×2 block, instead of being upper triangular over \mathbb{R} , then direct calculation shows that $\{E_1, E_2, E_3, E_4, E_5\}$ do not give a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ so we take E_4 and E_5 in the form

$$E_4 = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \quad E_5 = \begin{bmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & m \end{bmatrix}.$$

Then we find that $\{E_1, E_2, E_3, E_4, E_5\}$ do give a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$. The only possible candidates for the algebra as an abstract algebra are 5.36 or 5.37. If we choose the entries of E_4 and E_5 so as to satisfy the brackets of 5.36 we obtain the solution given by $f = a + 1, m = g, e = 0, d = a + 1, k = 0, j = g - 1, b = 0, h = 0, i = 0$ and it is impossible to obtain 5.37. The representation for 5.36 depends on the three parameters a, c and g , however, by subtracting $c \times E_1$ from E_4 we can assume without loss of generality that $c = 0$.

As regards algebra 5.37 let us note that it is equivalent over \mathbb{C} to 5.36 which means that both are equivalent considered as algebras over \mathbb{C} . To see how make a change of basis according to

$$\bar{e}_1 = -\frac{e_1}{2}, \quad \bar{e}_2 = \frac{e_2 + e_3}{2}, \quad \bar{e}_3 = \frac{i(e_2 - e_3)}{2}, \quad \bar{e}_4 = 2e_4 + e_5, \quad \bar{e}_5 = -ie_5.$$

We obtain the following brackets, which are formally identical to $A_{5,36}$ except for the factor of i which can then be removed by scaling e_1 :

$$[e_2, e_3] = ie_1, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_2, e_5] = -e_2, \quad [e_3, e_5] = e_3.$$

We remark finally that algebra $A_{5,40}$ is the only five-dimensional algebra that has a non-trivial Levi decomposition. It is the Lie algebra of the special affine group and so by its very definition has a representation in $\mathfrak{gl}(3, \mathbb{R})$ but not an upper triangular representation.

7.2. DECOMPOSABLE ALGEBRAS. If we have a five-dimensional decomposable algebra we could, in principle, have decompositions according to the following partitions of the integer five, namely, $(1, 4)$; $(1, 1, 3)$; $(2, 3)$; $(2, 2, 1)$; $(2, 1, 1, 1)$; $(1, 1, 1, 1, 1)$. Of these possibilities the last two are excluded because the nilradical would be four or five-dimensional abelian, respectively. The third and fourth cases are also impossible because, according to subsection 5.2 the space of matrices that commutes with a two-dimensional subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ is at most two-dimensional. For the first of the six partitions we can see that the only matrix that commutes with and is independent of the four generators of a four-dimensional subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ is a multiple of the identity so that case does occur.

Finally in the case of the partition $(1, 1, 3)$ the three-dimensional subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ could not be 3.1-3.7 because the entire algebra would have a four-dimensional abelian ideal; nor can it be 3.9 because only a multiple of the identity commutes with the corresponding subalgebra of $\mathfrak{gl}(3, \mathbb{R})$. The same argument applies to the second representation of 3.8 but the first has a two-dimensional space of commuting diagonal matrices.

8. SIX-DIMENSIONAL ALGEBRAS

8.1. INDECOMPOSABLE ALGEBRAS. In view of Lie's Theorem there can only be one six-dimensional solvable algebra that is isomorphic to a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$, that is, the space of 3×3 upper triangular matrices. This algebra is isomorphic to a direct sum of \mathbb{R} and 5.36. Thus an indecomposable six-dimensional algebra must have a non-trivial Levi decomposition.

The semi-simple Lie algebras in dimension six consist of, up to isomorphism, $\mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{so}(3, \mathbb{R})$, $\mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3, 1, \mathbb{R})$. To find representations of these algebras in $\mathfrak{gl}(3, \mathbb{R})$ we could appeal to the entire theory of semi-simple algebras and their representations [1]. Alternatively, we can proceed as follows. It is easy to show using the representation for $\mathfrak{so}(3, \mathbb{R})$ as skew-symmetric matrices that the only 3×3 matrix which commutes with $\mathfrak{so}(3, \mathbb{R})$ is a multiple of the identity. Likewise if we use the representation of $\mathfrak{sl}(2, \mathbb{R})$ coming from the adjoint representation that the only 3×3 matrix which commutes with $\mathfrak{sl}(2, \mathbb{R})$ is a multiple of the identity. Finally, if we use the representation of $\mathfrak{sl}(2, \mathbb{R})$ coming from its definition in $\mathfrak{gl}(2, \mathbb{R})$ and add a zero row and column we find that only that only linearly independent matrices that commute with it are two diagonal matrices. It follows that none of $\mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{so}(3, \mathbb{R})$, $\mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ can appear as subalgebras of $\mathfrak{gl}(3, \mathbb{R})$.

Now let us consider the algebra $\mathfrak{so}(3, 1, \mathbb{R})$. Its Lie brackets appear in the appendix. We note that $\{e_1, e_2, e_3\}$ span a three-dimensional subalgebra that is isomorphic to $\mathfrak{so}(3, \mathbb{R})$. We may then take the standard basis for $\mathfrak{so}(3, \mathbb{R})$. Thereafter we take the remaining three generators as arbitrary matrices. Then brackets such as $[e_4, e_5] = e_1$ impose linear conditions that are easy to solve. However, when all the conditions that are imposed it entails that some of the entries are pure imaginary. Hence there is no representation of $\mathfrak{so}(3, 1, \mathbb{R})$ in $\mathfrak{gl}(3, \mathbb{R})$.

The algebras in dimensions six, seven and eight that have a non-trivial Levi decomposition have been classified by Turkowski [6]. We consider the six-dimensional algebras of which there are four classes.

$L_{6.1}$: Is a semi-direct product of $\mathfrak{so}(3, \mathbb{R})$ and \mathbb{R}^3 . According to Schur-Jacobson [3] the only way to represent the \mathbb{R}^3 factor in $\mathfrak{gl}(3, \mathbb{R})$ is to include multiples of the identity: but then the algebra would have to have a non-trivial center which it does not. We remark that $L_{6.1}$ isomorphic to the Lie algebra of the three-dimensional Euclidean group which is well known to have a representation in $\mathfrak{gl}(4, \mathbb{R})$.

$L_{6.2}$: Is a semi-direct product of $\mathfrak{sl}(2, \mathbb{R})$ and Heisenberg. If it had a representation in $\mathfrak{gl}(3, \mathbb{R})$ then by change of basis we may assume that Heisenberg has a representation as

$$E_4 = \begin{bmatrix} f & 1 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{bmatrix}, \quad E_5 = \begin{bmatrix} g & 0 & 0 \\ 0 & g & 1 \\ 0 & 0 & g \end{bmatrix}, \quad E_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

But this subalgebra, as well as $L_{6.2}$, has a one-dimensional center spanned by E_6 ; as such a matrix that commutes with E_6 can only be of the form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{bmatrix}$$

and so the full algebra could not be six-dimensional. We remark that the algebra does have a representation in $\mathfrak{gl}(4, \mathbb{R})$ as the Lie algebra of the matrix Lie group given by

$$\begin{bmatrix} 1 & q & -p & w \\ 0 & e^x & y & p \\ 0 & z & (yz + 1)e^{-x} & q \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$L_{6.3}$: Is a semi-direct product of $\mathfrak{sl}(2, \mathbb{R})$ and algebra 3.3. As such we can begin with a representation of 3.3 for which generators are E_4, E_5, E_6 and

$$E_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_6 = \begin{bmatrix} a+1 & 0 & 0 \\ 0 & a+1 & 0 \\ 0 & 0 & a \end{bmatrix}.$$

So that $[E_4, E_6] = E_4$ and $[E_5, E_6] = E_5$. To continue we choose E_2 and E_3 as arbitrary 3×3 matrices: then E_1 is defined to be $[E_2, E_3]$. When the remaining brackets are imposed we find that, up to change of basis, in the sense of using linear combinations of $E_1, E_2, E_3, E_4, E_5, E_6$,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding matrix Lie group is given by

$$e^{ar} \begin{bmatrix} x & y & p \\ z & w & q \\ 0 & 0 & e^{-r} \end{bmatrix}$$

where $xw - yz = 1$. For $a = 1$ we obtain the affine group of the plane. If we had started with the alternative representation of 3.3 we would have obtained

$$e^{ar} \begin{bmatrix} e^{-r} & p & q \\ 0 & x & y \\ 0 & z & w \end{bmatrix}$$

where $xw - yz = 1$.

$L_{6.4}$: Is a semi-direct product of $\mathfrak{sl}(2, \mathbb{R})$ and \mathbb{R}^3 . It is the Lie algebra of the tangent bundle of $\mathrm{SL}(2, \mathbb{R})$ and so it is easy to find a representation of it in $\mathfrak{gl}(4, \mathbb{R})$. The same argument as in case $L_{6.1}$ applies to show that no representation of it in $\mathfrak{gl}(3, \mathbb{R})$ exists.

8.2. DECOMPOSABLE ALGEBRAS. As we argued at the beginning of this Section apart from the direct sum of \mathbb{R} and 5.36 any other subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ is not solvable and hence can only be a direct sum of \mathbb{R} and 5.40. It is easy to see that the only possibility is to add multiples of the identity to 5.40.

9. SEVEN-DIMENSIONAL ALGEBRAS

Turkowski lists seven classes of algebra in dimension seven denoted as $L_{7.1} - L_{7.7}$ and they are listed in the Appendix below. All of them has at least a three-dimensional abelian subalgebra and since they are not decomposable have no representation of in $\mathfrak{gl}(3, \mathbb{R})$; the algebras together with their associated abelian subalgebra are, respectively,

$$L_{7.1} \langle e_4, e_5, e_6, \rangle; \quad L_{7.2} \langle e_4, e_5, e_6, e_7 \rangle; \quad L_{7.3} \langle e_4, e_5, e_6, \rangle; \quad L_{7.4} \langle e_2, e_4, e_6 \rangle; \\ L_{7.5} \langle e_4, e_5, e_6 \rangle; \quad L_{7.6} \langle e_4, e_5, e_6, e_7 \rangle; \quad L_{7.7} \langle e_4, e_5, e_6, e_7 \rangle.$$

It follows that any seven-dimensional subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ is decomposable. Since it cannot be solvable the only possibilities are a direct sum of \mathbb{R} and $L_{6.3}$ and a direct sum of a two-dimensional algebra and 5.40. The first case does occur by adding multiples of the identity to $L_{6.3}$, or equivalently adding an arbitrary (3,3)-entry into the corresponding group, but the second does not: the only matrices that commute with the generators of 5.40 in either of its manifestations are multiples of the identity.

10. EIGHT-DIMENSIONAL ALGEBRAS

The only semi-simple algebra in dimension eight is $\mathfrak{sl}(3, \mathbb{R})$ which of course consists of the trace-free matrices in $\mathfrak{gl}(3, \mathbb{R})$: $\mathfrak{su}(3)$ and $\mathfrak{su}(2, 1)$ are also eight-dimensional real Lie algebras but they cannot be found in $\mathfrak{gl}(3, \mathbb{R})$. Any other eight-dimensional subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ must have a non-trivial Levi decomposition. The semi-simple factor can only be $\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{so}(3)$ and so the solvable radical must be five-dimensional. Referring to Section 5 the only possibilities for the solvable radical are the direct sum of \mathbb{R} and 4.8-4.9b or the direct sum of \mathbb{R} and 4.12 or 5.36. The first two cases cannot occur because then the algebra would be a direct sum of a one-dimensional and seven-dimensional algebra. But the only seven-dimensional algebra already contains multiples of the identity and so cannot be further extended.

Let us consider the case where the radical is algebra 5.36. One way to exclude 5.36 is to consult the list of eight-dimensional non-solvable indecomposable algebras given in [6] where on a case by case basis it is shown that the algebra 5.36 does not occur in the Levi decomposition. On the the other hand suppose that we start with the Lie algebra representation of algebra 5.36 determined in Section 7. Then it is easy to check that the only 3×3 matrices that stabilize the representation, in the sense of mapping it into itself

under commutator, are precisely the upper triangular matrices. In particular there cannot be a simple algebra that acts on algebra 5.36 and so no such eight-dimensional algebra can exist.

11. NINE-DIMENSIONAL ALGEBRAS

11.1. DECOMPOSABLE ALGEBRAS: $\mathfrak{gl}(3, \mathbb{R})$ itself.

12. MATRIX SUBGROUPS OF $GL(3, \mathbb{R})$

12.1. DIMENSION 1:

$$(a) \begin{bmatrix} e^{\alpha x} & 0 & 0 \\ 0 & e^{\beta x} & 0 \\ 0 & 0 & e^x \end{bmatrix}; \quad (b) \begin{bmatrix} e^{\lambda x} & xe^{\lambda x} & 0 \\ 0 & e^{\lambda x} & 0 \\ 0 & 0 & e^{\mu x} \end{bmatrix} \quad (\lambda = 0, 1 \text{ or } \mu = 0, 1);$$

$$(c) \begin{bmatrix} e^{\alpha x} \cos(x) & e^{\alpha x} \sin(x) & 0 \\ -e^{\alpha x} \sin(x) & e^{\alpha x} \cos(x) & 0 \\ 0 & 0 & e^{\beta x} \end{bmatrix} \quad (\alpha = 0, 1 \text{ or } \beta = 0, 1);$$

$$(d) \begin{bmatrix} e^{\lambda x} & xe^{\lambda x} & \frac{1}{2}x^2e^{\lambda x} \\ 0 & e^{\lambda x} & xe^{\lambda x} \\ 0 & 0 & e^{\lambda x} \end{bmatrix} \quad (\lambda = 0, 1).$$

12.2. DIMENSION 2, ABELIAN:

$$(a) \begin{bmatrix} e^x & 0 & 0 \\ 0 & e^y & 0 \\ 0 & 0 & e^{ax+by} \end{bmatrix};$$

$$(b) \begin{bmatrix} e^x & ye^x & 0 \\ 0 & e^x & 0 \\ 0 & 0 & e^{bx+ay} \end{bmatrix}, \quad \begin{bmatrix} e^{ax} & xe^{ax} & 0 \\ 0 & e^{ax} & 0 \\ 0 & 0 & e^y \end{bmatrix} \quad (a = 0, 1),$$

$$e^{ax+by} \begin{bmatrix} 1 & cx & y + \frac{cdx^2}{2} \\ 0 & 1 & dx \\ 0 & 0 & 1 \end{bmatrix} \quad (a = 0, 1, b = 0, 1);$$

$$(c) \quad \begin{bmatrix} e^y \cos x & e^y \sin x & 0 \\ -e^y \sin x & e^y \cos x & 0 \\ 0 & 0 & e^{bx+ay} \end{bmatrix}, \quad \begin{bmatrix} e^{ax} \cos x & e^{ax} \sin x & 0 \\ -e^{ax} \sin x & e^{ax} \cos x & 0 \\ 0 & 0 & e^y \end{bmatrix};$$

$$(d) \quad e^x \begin{bmatrix} 1 & x+y & cy + \frac{1}{2}(x+y)^2 \\ 0 & 1 & x+y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^x \begin{bmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

12.3. DIMENSION 2, NON-ABELIAN:

$$(b) \quad \begin{bmatrix} e^{ay} & xe^{(a+1)y} & 0 \\ 0 & e^{(a+1)y} & 0 \\ 0 & 0 & e^{by} \end{bmatrix} \quad ((a-b)(a-b+1) \neq 0),$$

$$e^{ay} \begin{bmatrix} 1 & xe^y & cy \\ 0 & e^y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (c = 0, 1), \quad e^{ay} \begin{bmatrix} 1 & xe^y & 0 \\ 0 & e^y & 0 \\ 0 & dye^y & e^y \end{bmatrix} \quad (d = 0, 1);$$

$$(d) \quad e^{by} \begin{bmatrix} 1 & xe^y & \frac{1}{2}x^2e^{2y} \\ 0 & e^y & xe^{2y} \\ 0 & 0 & e^{2y} \end{bmatrix}.$$

12.4. DIMENSION 3, INDECOMPOSABLE:

3.1 $[e_2, e_3] = e_1$:

$$S = e^{ax+by} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

3.2 $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$:

$$e^{az} \begin{bmatrix} e^z & ze^z & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{az} \begin{bmatrix} 1 & x & y \\ 0 & e^z & ze^z \\ 0 & 0 & e^z \end{bmatrix}.$$

3.3, 3.4, 3.5a ($a \neq \pm 1$) $[e_1, e_3] = e_1, [e_2, e_3] = ae_2$:

$$e^{bz} \begin{bmatrix} e^z & 0 & x \\ 0 & e^{az} & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{bz} \begin{bmatrix} 1 & x & y \\ 0 & e^z & 0 \\ 0 & 0 & e^{az} \end{bmatrix}.$$

3.6 ($a = 0$), 3.7a ($a \neq 0$) $[e_1, e_3] = ae_1 - e_2$, $[e_2, e_3] = e_1 + ae_2$:

$$e^{bz} \begin{bmatrix} e^{az} \cos z & e^{az} \sin z & x \\ -e^{az} \sin z & e^{az} \cos z & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{bz} \begin{bmatrix} 1 & x & y \\ 0 & e^{az} \cos z & e^{az} \sin z \\ 0 & -e^{az} \sin z & e^{az} \cos z \end{bmatrix}.$$

3.8 $[e_1, e_3] = -2e_2$, $[e_1, e_2] = e_1$, $[e_2, e_3] = e_3$:

$$\begin{bmatrix} x & y & 0 \\ z & w & 0 \\ 0 & 0 & 1 \end{bmatrix} (xw - yz = 1), \quad \begin{bmatrix} e^{-y} + 2xz + x^2e^yz^2 & x + x^2e^yz & -x^2e^y \\ 2z + 2xe^yz^2 & 1 + 2xe^yz & -2xe^y \\ -e^yz^2 & -e^yz & e^y \end{bmatrix}.$$

3.9 $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$:

$$\begin{bmatrix} \cos y \cos z & -\cos y \sin z & \sin y \\ \sin x \sin y \cos z + \cos x \sin z & -\sin x \sin y \sin z + \cos x \cos z & -\sin x \cos y \\ -\cos x \sin y \cos z + \sin x \sin z & \cos x \sin y \sin z + \sin x \cos z & \cos x \cos y \end{bmatrix}.$$

12.5. DIMENSION 3, DECOMPOSABLE, ABELIAN:

$$\begin{bmatrix} e^x & 0 & 0 \\ 0 & e^y & 0 \\ 0 & 0 & e^z \end{bmatrix}, \quad \begin{bmatrix} e^z & 0 & x \\ 0 & e^z & y \\ 0 & 0 & e^z \end{bmatrix}, \quad \begin{bmatrix} e^z & x & y \\ 0 & e^z & 0 \\ 0 & 0 & e^z \end{bmatrix}.$$

12.6. DIMENSION 3, DECOMPOSABLE, NON-ABELIAN:

$$\begin{bmatrix} e^{ay+cz} & xe^{(a+1)y+cz} & 0 \\ 0 & e^{(a+1)y+cz} & 0 \\ 0 & 0 & e^{by+dz} \end{bmatrix} ((a-b)(a-b+1) \neq 0),$$

$$e^{ay+bz} \begin{bmatrix} 1 & xe^y & cy + dz \\ 0 & e^y & 0 \\ 0 & 0 & 1 \end{bmatrix} (c = 0, 1),$$

$$e^{ay+bz} \begin{bmatrix} 1 & xe^y & 0 \\ 0 & e^y & 0 \\ 0 & (cy + dz)e^y & e^y \end{bmatrix} (d = 0, 1), \quad e^{by+z} \begin{bmatrix} 1 & xe^y & \frac{1}{2}x^2e^{2y} \\ 0 & e^y & xe^{2y} \\ 0 & 0 & e^{2y} \end{bmatrix}.$$

12.7. DIMENSION 4, INDECOMPOSABLE:

4.8, 4.9b:

$$e^{aw} \begin{bmatrix} e^{(b+1)w} & x & z \\ 0 & e^{bw} & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{aw} \begin{bmatrix} 1 & x & z \\ 0 & e^{bw} & y \\ 0 & 0 & e^{(b+1)w} \end{bmatrix}.$$

4.12:

$$e^{az} \begin{bmatrix} e^z \cos w & e^z \sin w & x \\ -e^z \sin w & e^z \cos w & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{az} \begin{bmatrix} 1 & x & y \\ 0 & e^z \cos w & e^z \sin w \\ 0 & -e^z \sin w & e^z \cos w \end{bmatrix}.$$

12.8. DIMENSION 4, DECOMPOSABLE:

3.1 $\oplus \mathbb{R}$:

$$S = e^{ax+by+cw} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

3.2 $\oplus \mathbb{R}$:

$$e^{az+w} \begin{bmatrix} e^z & ze^z & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{az+w} \begin{bmatrix} 1 & x & y \\ 0 & e^z & ze^z \\ 0 & 0 & e^z \end{bmatrix}.$$

3.3 $\oplus \mathbb{R}$, 3.4 $\oplus \mathbb{R}$:

$$e^{bz+w} \begin{bmatrix} e^z & 0 & x \\ 0 & e^{az} & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{bz+w} \begin{bmatrix} 1 & x & y \\ 0 & e^z & 0 \\ 0 & 0 & e^{az} \end{bmatrix}.$$

3.6($a = 0$) $\oplus \mathbb{R}$, 3.7a($a \neq 0$) $\oplus \mathbb{R}$:

$$e^{bz+w} \begin{bmatrix} e^{az} \cos z & e^{az} \sin z & x \\ -e^{az} \sin z & e^{az} \cos z & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{bz+w} \begin{bmatrix} 1 & x & y \\ 0 & e^{az} \cos z & e^{az} \sin z \\ 0 & -e^{az} \sin z & e^{az} \cos z \end{bmatrix}.$$

3.8 $\oplus \mathbb{R}$:

$$\begin{bmatrix} xe^{aq} & ye^{aq} & 0 \\ ze^{aq} & we^{aq} & 0 \\ 0 & 0 & e^{bq} \end{bmatrix} (xw - yw = 1),$$

$$e^q \begin{bmatrix} e^{-y} + 2xz + x^2 e^y z^2 & x + x^2 e^y z & -x^2 e^y \\ 2z + 2x e^y z^2 & 1 + 2x e^y z & -2x e^y \\ -e^y z^2 & -e^y z & e^y \end{bmatrix}.$$

3.9 $\oplus \mathbb{R}$:

$$e^w \begin{bmatrix} \cos y \cos z & -\cos y \sin z & \sin y \\ \sin x \sin y \cos z + \cos x \sin z & -\sin x \sin y \sin z + \cos x \cos z & -\sin x \cos y \\ -\cos x \sin y \cos z + \sin x \sin z & \cos x \sin y \sin z + \sin x \cos z & \cos x \cos y \end{bmatrix}.$$

2.1 $\oplus \mathbb{R}^2$:

$$\begin{bmatrix} e^{ay+z} & x e^{(a+1)y+z} & 0 \\ 0 & e^{(a+1)y+z} & 0 \\ 0 & 0 & e^{by+w} \end{bmatrix} \quad ((a-b)(a-b+1) \neq 0),$$

$$e^{ay+z} \begin{bmatrix} 1 & x e^y & w \\ 0 & e^y & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{ay+z} \begin{bmatrix} 1 & x e^y & 0 \\ 0 & e^y & 0 \\ 0 & w & e^y \end{bmatrix}.$$

12.9. DIMENSION 5, INDECOMPOSABLE:

5.36 $[e_2, e_3] = e_1$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_2, e_5] = -e_2$, $[e_3, e_5] = e_3$,
coordinates (z, x, y, q, w) :

$$e^{aw+bq} \begin{bmatrix} e^w & x e^w & z e^w \\ 0 & e^q & y e^q \\ 0 & 0 & e^{w+q} \end{bmatrix}.$$

5.40 $[e_1, e_2] = 2e_1$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = 2e_3$, $[e_1, e_4] = e_5$, $[e_2, e_4] = e_4$,
 $[e_2, e_5] = -e_5$, $[e_3, e_5] = e_4$, coordinates (x, y, z, p, q) :

$$\begin{bmatrix} 1 & p & q \\ 0 & x & y \\ 0 & z & w \end{bmatrix} \quad (xw - yz = 1), \quad \begin{bmatrix} x & y & p \\ z & w & q \\ 0 & 0 & 1 \end{bmatrix} \quad (xw - yz = 1).$$

12.10. DIMENSION 5, DECOMPOSABLE:

$\mathbb{R} \oplus 4.8, \mathbb{R} \oplus 4.9b:$

$$e^q \begin{bmatrix} e^{(a+b+1)w} & x & z \\ 0 & e^{(a+b)w} & y \\ 0 & 0 & e^{aw} \end{bmatrix}, \quad e^q \begin{bmatrix} e^{aw} & x & z \\ 0 & e^{(a+b)w} & y \\ 0 & 0 & e^{(a+b+1)w} \end{bmatrix}.$$

 $\mathbb{R} \oplus 4.12:$

$$e^{az+q} \begin{bmatrix} e^z \cos w & e^z \sin w & x \\ -e^z \sin w & e^z \cos w & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{az+q} \begin{bmatrix} 1 & x & y \\ 0 & e^z \cos w & e^z \sin w \\ 0 & -e^z \sin w & e^z \cos w \end{bmatrix}.$$

 $\mathbb{R}^2 \oplus 3.8:$

$$\begin{bmatrix} xe^q & ye^q & 0 \\ ze^q & we^q & 0 \\ 0 & 0 & e^r \end{bmatrix} (xw - yz = 1).$$

12.11. DIMENSION 6, INDECOMPOSABLE:

 $L_{6.3}:$

$$e^{ar} \begin{bmatrix} x & y & p \\ z & w & q \\ 0 & 0 & e^{-r} \end{bmatrix} (xw - yz = 1), \quad e^{ar} \begin{bmatrix} e^{-r} & p & q \\ 0 & x & y \\ 0 & z & w \end{bmatrix} (xw - yz = 1).$$

12.12. DIMENSION 6, DECOMPOSABLE:

 $\mathbb{R} \oplus 5.36:$

$$e^p \begin{bmatrix} e^w & x & z \\ 0 & e^q & y \\ 0 & 0 & 1 \end{bmatrix}, \quad e^p \begin{bmatrix} 1 & x & z \\ 0 & e^q & y \\ 0 & 0 & e^w \end{bmatrix}.$$

 $\mathbb{R} \oplus 5.40:$

$$e^r \begin{bmatrix} x & y & p \\ z & w & q \\ 0 & 0 & 1 \end{bmatrix} (xw - yz = 1), \quad e^r \begin{bmatrix} 1 & p & q \\ 0 & x & y \\ 0 & z & w \end{bmatrix} (xw - yz = 1).$$

12.13. DIMENSION 7, INDECOMPOSABLE: None.

12.14. DIMENSION 7, DECOMPOSABLE:

$$\begin{bmatrix} x & y & p \\ z & w & q \\ 0 & 0 & r \end{bmatrix} (r(xw - yz) \neq 0), \quad \begin{bmatrix} p & q & r \\ 0 & x & y \\ 0 & z & w \end{bmatrix} (p(xw - yz) \neq 0).$$

12.15. DIMENSION 8, INDECOMPOSABLE: $\mathfrak{sl}(3, \mathbb{R})$.

12.16. DIMENSION 8, DECOMPOSABLE: None.

12.17. DIMENSION 9, DECOMPOSABLE: $\mathfrak{gl}(3, \mathbb{R})$.

13. APPENDIX: LISTS OF LOW-DIMENSIONAL INDECOMPOSABLE LIE ALGEBRAS

Below a, b denote arbitrary real numbers subject to the inequalities listed.

13.1. DIMENSION 2: 2.1 $[e_1, e_2] = e_1$.

13.2. DIMENSION 3: [4]

3.1 $[e_2, e_3] = e_1$.

3.2 $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$.

3.3 $[e_1, e_3] = e_1, [e_2, e_3] = e_2$.

3.4 $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$.

3.5a ($a \neq \pm 1$) $[e_1, e_3] = e_1, [e_2, e_3] = ae_2$.

3.6 $[e_1, e_3] = -e_2, [e_2, e_3] = e_1$.

3.7a ($a \neq 0$) $[e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2$.

3.8 (semi-simple $\mathfrak{sl}(2, \mathbb{R})$) $[e_1, e_3] = -2e_2, [e_1, e_2] = e_1, [e_2, e_3] = e_3$.

3.9 (semi-simple $\mathfrak{so}(3)$) $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$.

13.3. DIMENSION 4: [4]

4.1 $[e_2, e_4] = e_1, [e_3, e_4] = e_2.$

4.2a ($a \neq 0$) $[e_1, e_4] = ae_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3.$

4.3 $[e_1, e_4] = e_1, [e_3, e_4] = e_2.$

4.4 $[e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3.$

4.5ab ($ab \neq 0, -1 \leq a \leq b \leq 1$) $[e_1, e_4] = e_1, [e_2, e_4] = ae_2, [e_3, e_4] = be_3.$

4.6ab ($a \neq 0, b \geq 0$) $[e_1, e_4] = ae_1, [e_2, e_4] = be_2 - e_3, [e_3, e_4] = e_2 + be_3.$

4.7 $[e_2, e_3] = e_1, [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3.$

4.8 $[e_2, e_3] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = -e_3.$

4.9b ($-1 < b \leq 1$) $[e_2, e_3] = e_1, [e_1, e_4] = (b+1)e_1, [e_2, e_4] = e_2, [e_3, e_4] = be_3.$

4.10 $[e_2, e_3] = e_1, [e_2, e_4] = -e_3, [e_3, e_4] = e_2.$

4.11a ($a > 0$) $[e_2, e_3] = e_1, [e_1, e_4] = 2ae_1, [e_2, e_4] = ae_2 - e_3, [e_3, e_4] = e_2 + ae_3.$

4.12 $[e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1.$

13.4. DIMENSION 5, NILPOTENT: [4]

5.1 $[e_3, e_5] = e_1, [e_4, e_5] = e_2.$

5.2 $[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3.$

5.3 $[e_3, e_4] = e_2, [e_3, e_5] = e_1, [e_4, e_5] = e_3.$

5.4 $[e_2, e_4] = e_1, [e_3, e_5] = e_1.$

5.5 $[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2.$

5.6 $[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3.$

13.5. DIMENSION 5, SOLVABLE: [4]

$$5.36 \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_2, e_5] = -e_2, \\ [e_3, e_5] = e_3.$$

13.6. DIMENSION 5, ALGEBRA HAVING A NON-TRIVIAL LEVI DECOMPOSITION: [4, 6]

$$5.40 \quad [e_1, e_2] = 2e_1, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = 2e_3, \quad [e_1, e_4] = e_5, \\ [e_2, e_4] = e_4, \quad [e_2, e_5] = -e_5, \quad [e_3, e_5] = e_4.$$

13.7. DIMENSION 6, SIMPLE ALGEBRA:

$$\mathfrak{so}(3, 1, \mathbb{R}) \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_1, e_4] = e_5 \\ [e_4, e_5] = -e_1, \quad [e_5, e_1] = e_4, \quad [e_2, e_4] = e_6, \quad [e_4, e_6] = -e_2 \\ [e_6, e_2] = e_4, \quad [e_3, e_5] = e_6, \quad [e_5, e_6] = -e_3, \quad [e_6, e_3] = e_5.$$

13.8. DIMENSION 6, ALGEBRAS HAVING A NON-TRIVIAL LEVI DECOMPOSITION: [6]

$$L_{6.1} \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_1, e_5] = e_6, \\ [e_1, e_6] = -e_5, \quad [e_2, e_4] = -e_6, \quad [e_2, e_6] = e_4, \quad [e_3, e_4] = e_5, \\ [e_3, e_5] = -e_4.$$

$$L_{6.2} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4 \\ [e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5, \quad [e_4, e_5] = e_6.$$

$$L_{6.3} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4 \\ [e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5, \quad [e_4, e_6] = e_4 \\ [e_5, e_6] = e_5.$$

$$L_{6.4} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_4 \\ [e_1, e_6] = -2e_6, \quad [e_2, e_5] = e_4, \quad [e_2, e_6] = e_5, \quad [e_3, e_4] = e_5 \\ [e_3, e_5] = 2e_6.$$

13.9. DIMENSION 7, ALGEBRAS HAVING A NON-TRIVIAL LEVI DECOMPOSITION: [6]

$$L_{7.1} \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_1, e_5] = e_6, \\ [e_1, e_6] = -e_5, \quad [e_2, e_4] = -e_6, \quad [e_2, e_6] = e_4, \quad [e_3, e_4] = e_5, \\ [e_3, e_5] = -e_4, \quad [e_4, e_7] = e_4, \quad [e_5, e_7] = e_5, \quad [e_6, e_7] = e_6.$$

$$L_{7.2} \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_1, e_4] = \frac{1}{2}e_7, \\ [e_1, e_5] = \frac{1}{2}e_6, \quad [e_1, e_6] = -\frac{1}{2}e_5, \quad [e_1, e_7] = -\frac{1}{2}e_4, \quad [e_2, e_4] = \frac{1}{2}e_5, \\ [e_2, e_5] = -\frac{1}{2}e_4, \quad [e_2, e_6] = \frac{1}{2}e_7, \quad [e_2, e_7] = -\frac{1}{2}e_6, \quad [e_3, e_4] = \frac{1}{2}e_6, \\ [e_3, e_5] = \frac{1}{2}e_6, \quad [e_3, e_6] = -\frac{1}{2}e_7, \quad [e_3, e_7] = \frac{1}{2}e_5.$$

$$L_{7.3} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4, \\ [e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5, \quad [e_4, e_7] = e_4, \\ [e_5, e_7] = e_5, \quad [e_6, e_7] = ae_6.$$

$$L_{7.4} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4, \\ [e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5, \quad [e_4, e_5] = e_6, \\ [e_4, e_7] = e_4, \quad [e_5, e_7] = e_5, \quad [e_6, e_7] = 2e_6.$$

$$L_{7.5} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_4, \\ [e_1, e_6] = -2e_6, \quad [e_2, e_5] = e_4, \quad [e_2, e_6] = e_5, \quad [e_3, e_4] = e_5, \\ [e_3, e_5] = 2e_5, \quad [e_4, e_7] = e_4, \quad [e_5, e_7] = e_5, \quad [e_6, e_7] = e_6.$$

$$L_{7.6} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = 3e_4, \\ [e_1, e_5] = e_5, \quad [e_1, e_6] = -e_6, \quad [e_1, e_7] = -3e_7, \quad [e_2, e_5] = 3e_4, \\ [e_2, e_6] = 2e_5, \quad [e_2, e_7] = e_6, \quad [e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \\ [e_3, e_6] = 3e_7.$$

$$L_{7.7} \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = 3e_4, \\ [e_1, e_5] = -e_5, \quad [e_1, e_6] = e_6, \quad [e_1, e_7] = -e_7, \quad [e_2, e_5] = e_4, \\ [e_2, e_7] = e_6, \quad [e_3, e_4] = e_5, \quad [e_3, e_6] = e_7.$$

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