

## On Dilation of Local Semigroups of Contractions and some Applications\*

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Presented by Alfonso Montes

Received August 5, 2012

*Abstract:* We show that a multiplicative family of contractions on a separable Hilbert space, with parameter on the interval  $[0, 1)$  of the dyadic rationals, has a unitary dilation with parameter on the dyadic rationals and values on a larger Hilbert space. This result is used to prove a dilation result for strongly continuous local semigroups of contractions. As an application we give results of extension of positive definite functions on the line, generalizing the Kreĭn extension theorem.

*Key words:* Operator semigroup, contraction operator, isometric operator, positive definite.

*AMS Subject Class.* (2010): 47D03, 43A35, 42A82

### 1. BASIC NOTIONS AND DEFINITIONS

Results about dilation and extension of multiplicative families of contractions and local semigroups of contractions are a useful tool in some interpolation and dilation problems, see for example [1, 2, 3, 8]. Multiplicative families of operators, where the domain of the operators depends on the parameter, are considered in the papers [1, 2]. The techniques for obtaining the dilations and extensions are based on special extensions of operators. In the papers [3, 8] the domain of the operators does not depend on the parameter and discretization techniques are used.

In this paper we use discretization techniques to extend, for a separable Hilbert space, some of the dilation results obtained in [1] and we give, as an application, some results about extension of positive definite functions.

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\*All authors were supported in part by the CDCH of the Univ. Central de Venezuela.

Let  $(\Gamma, +)$  be an additive subgroup of the real numbers  $(\mathbb{R}, +)$ , let  $a \in \mathbb{R}$ ,  $a > 0$  or  $a = +\infty$  and let  $I_\Gamma = [0, a) \cap \Gamma$ .

DEFINITION 1. Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  be a Hilbert space. A *multiplicative family of contractions* on  $\mathcal{H}$  with parameter on  $I_\Gamma$  is a family  $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$  such that:

- (i)  $\mathcal{H}(s)$  is a closed subspace of  $\mathcal{H}$ ,  $T(s) : \mathcal{H}(s) \rightarrow \mathcal{H}$  is a contraction operator,  $\mathcal{H}(r) \subset \mathcal{H}(s)$  for  $r, s \in I_\Gamma$ ,  $s < r$  and  $\mathcal{H}(0) = \mathcal{H}$ ,  $T(0) = I_{\mathcal{H}}$ .
- (ii) If  $r, s \in I_\Gamma$  are such that  $r + s < a$  then  $T(s)\mathcal{H}(r + s) \subset \mathcal{H}(r)$  and  $T(r + s)h = T(r)T(s)h$  for all  $h \in \mathcal{H}(r + s)$ .

The multiplicative family is *strongly continuous* if for  $r \in I_\Gamma$  and  $f \in \mathcal{H}(r)$  the function  $s \mapsto T(s)h$  is continuous on  $[0, r] \cap I_\Gamma$ .

If the operators  $T(s)$  are isometries we will say that  $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$  is a *multiplicative family of isometries*.

The following is an extension of a definition given in [1].

DEFINITION 2. Let  $\mathcal{H}$  be a Hilbert space. A *local semigroup of contractions* on  $\mathcal{H}$  with parameter on  $I_\Gamma$  a multiplicative family of contractions  $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$  such that

$$\bigcup_{r \in (x, a) \cap I_\Gamma} \mathcal{H}(r)$$

is dense in  $\mathcal{H}(x)$  for all  $x \in I_\Gamma$ .

If the operators  $T(s)$  are isometries we will say that  $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$  is a *local semigroup of isometries*.

## 2. MULTIPLICATIVE FAMILIES OF CONTRACTIONS WITH PARAMETER ON THE DIADIC RATIONAL NUMBERS

Let  $\Delta$  be the set of the diadic rational numbers, that is

$$\Delta = \left\{ \frac{k}{2^n} : k \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

For each  $m \in \mathbb{N}$  let

$$\Delta_m = \left\{ \frac{k}{2^m} : k \in \mathbb{Z} \right\},$$

$\Delta^+ = \{s \in \Delta : s \geq 0\}$  and  $\Delta_m^+ = \{s \in \Delta_m : s \geq 0\}$ .

Throughout this section  $\mathcal{H}$  will be a separable Hilbert space,  $I_\Delta = [0, 1) \cap \Delta$  and  $(T(s), \mathcal{H}(s))_{s \in I_\Delta}$  will denote a multiplicative family of contractions with parameter on  $I_\Delta$ .

For  $s \in I_\Delta$  let  $V(s) : \mathcal{H} \rightarrow \mathcal{H}$  be the contraction operator defined by

$$V(s) = T(s)P_{\mathcal{H}(s)}^{\mathcal{H}},$$

where  $P_{\mathcal{H}(s)}^{\mathcal{H}}$  denotes the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{H}(s)$ .

LEMMA 3. *If  $m, k \in \mathbb{N}$  and  $k < 2^m$  then*

$$\left( V\left(\frac{1}{2^m}\right) \right)^k \Big|_{\mathcal{H}\left(\frac{k}{2^m}\right)} = T\left(\frac{k}{2^m}\right).$$

*Proof.* We will use induction.

For  $k = 2$ .

Let  $h \in \mathcal{H}\left(\frac{2}{2^m}\right)$ . Then  $h \in \mathcal{H}\left(\frac{1}{2^m}\right)$ , so  $P_{\mathcal{H}\left(\frac{1}{2^m}\right)}^{\mathcal{H}}h = h$ . Since  $\mathcal{H}\left(\frac{2}{2^m}\right) = \mathcal{H}\left(\frac{1}{2^m} + \frac{1}{2^m}\right)$  we have that  $T\left(\frac{1}{2^m}\right)h \in \mathcal{H}\left(\frac{1}{2^m}\right)$ , thus

$$P_{\mathcal{H}\left(\frac{1}{2^m}\right)}^{\mathcal{H}}T\left(\frac{1}{2^m}\right)h = T\left(\frac{1}{2^m}\right)h.$$

Therefore

$$\begin{aligned} \left( V\left(\frac{1}{2^m}\right) \right)^2 h &= \left( T\left(\frac{1}{2^m}\right) P_{\mathcal{H}\left(\frac{1}{2^m}\right)}^{\mathcal{H}} \right) \left( T\left(\frac{1}{2^m}\right) P_{\mathcal{H}\left(\frac{1}{2^m}\right)}^{\mathcal{H}} \right) h \\ &= T\left(\frac{1}{2^m}\right) T\left(\frac{1}{2^m}\right) h \\ &= T\left(\frac{2}{2^m}\right) h \end{aligned}$$

Suppose that  $\left( V\left(\frac{1}{2^m}\right) \right)^k \Big|_{\mathcal{H}\left(\frac{k}{2^m}\right)} = T\left(\frac{k}{2^m}\right)$  and that  $k + 1 < 2^m$ . Let  $h \in \mathcal{H}\left(\frac{k+1}{2^m}\right)$ , then  $h \in \mathcal{H}\left(\frac{1}{2^m}\right)$  and  $V\left(\frac{1}{2^m}\right)h = T\left(\frac{1}{2^m}\right)h$ , so

$$\left( V\left(\frac{1}{2^m}\right) \right)^{k+1} h = \left( V\left(\frac{1}{2^m}\right) \right)^k V\left(\frac{1}{2^m}\right)h = \left( V\left(\frac{1}{2^m}\right) \right)^k T\left(\frac{1}{2^m}\right)h,$$

since  $\mathcal{H}\left(\frac{k+1}{2^m}\right) = \mathcal{H}\left(\frac{1}{2^m} + \frac{k}{2^m}\right)$ , we have that  $T\left(\frac{1}{2^m}\right)h \in \mathcal{H}\left(\frac{k}{2^m}\right)$ . From the induction hypothesis

$$\left( V\left(\frac{1}{2^m}\right) \right)^k T\left(\frac{1}{2^m}\right)h = T\left(\frac{k}{2^m}\right) T\left(\frac{1}{2^m}\right)h = T\left(\frac{k+1}{2^m}\right)h$$

thus

$$\left( V\left(\frac{1}{2^m}\right) \right)^{k+1} h = T\left(\frac{k+1}{2^m}\right)h. \quad \blacksquare$$

For  $m \in \mathbb{N}$  we define  $V^{(m)} = (V^{(m)}(s))_{s \in \Delta_m^+}$  by

$$V^{(m)}\left(\frac{k}{2^m}\right) = \left(V_{\frac{1}{2^m}}\right)^k.$$

We have that  $(V^{(m)}(s))_{s \in \Delta_m^+}$  is a semigroup of contractions.

PROPOSITION 4. *The function  $F^{(m)} : \Delta_m \rightarrow L(\mathcal{H})$  defined by*

$$F^{(m)}(s) = \begin{cases} V^{(m)}(s) & \text{if } s \geq 0, \\ (V^{(m)}(-s))^* & \text{if } s < 0, \end{cases}$$

*is positive definite.*

*Proof.* Considering the natural isomorphism between the group  $\Delta_m$  and the integers group  $\mathbb{Z}$  the result is obtained from the Neumark dilation theorem [7], see also Theorem 7.1 of Chapter 1 of [10]. ■

The following result follows from a natural diagonalization procedure.

PROPOSITION 5. *Let  $\Upsilon$  be a topological space and let  $\Lambda$  be a numerable indices set. Let  $\{X_\alpha(n)\}_{n=1}^\infty, \alpha \in \Lambda$  be a family of sequences in  $\Upsilon$  such that for each  $\alpha \in \Lambda$  every subsequence of  $\{X_\alpha(n)\}_{n=1}^\infty$  has a convergent subsequence. Then there exists an increasing sequence  $\{b(n)\}_{n=1}^\infty \subset \mathbb{N}$  such that the sequence  $\{X_\alpha(b(n))\}_{n=1}^\infty$  converges for all  $\alpha \in \Lambda$ .*

LEMMA 6. *There exists an increasing sequence of natural numbers  $\{n_j\}_{j=1}^\infty$  such that for all  $k, m \in \mathbb{N}$  and  $h \in \mathcal{H}$  the sequence*

$$\left\{ V^{(n_j)}\left(\frac{k}{2^m}\right) h \right\}_{j=1}^\infty$$

*converges in the weak topology.*

*Proof.* Since  $\mathcal{H}$  is separable, there is a numerable set  $\{h_i\}_{i=1}^{+\infty} \subset \mathcal{H}$  dense in  $\mathcal{H}$ .

Since the operators  $V^{(n)}\left(\frac{k}{2^m}\right)$  are contractions, we have that for each  $i, k, m \in \mathbb{N}$  the sequence  $\left\{ V^{(n)}\left(\frac{k}{2^m}\right) h_i \right\}_{n \geq m} \subset \mathcal{H}$  is bounded, so it contains a weakly convergent subsequence.

Applying Proposition 5 with the set  $\Lambda = \{(i, k, m) : i, k, m \in \mathbb{N}\}$ ,  $\Upsilon = \mathcal{H}$  with the weak topology and family

$$\left\{ X_\alpha(n) = V^{(n)}\left(\frac{k}{2^m}\right) h_i; \alpha = (i, k, m) \in \Lambda \right\}_{n \geq m}$$

we obtain that there is an increasing sequence of natural numbers  $\{n_j\}_{j=1}^\infty$  such that

$$\left\{ V^{(n_j)} \left( \frac{k}{2^m} \right) h_i \right\}_{j=1}^\infty$$

converges weakly for all  $i, k, m \in \mathbb{N}$ .

We also have that  $\left\{ V^{(n_j)} \left( \frac{k}{2^m} \right) \right\}_{j=1}^\infty \subset L(\mathcal{H})$  is uniformly bounded and that  $\{h_i\}_{i=1}^\infty$  is dense in  $\mathcal{H}$ , so for all  $k, m \in \mathbb{N}$ , the sequence

$$\left\{ V^{(n_j)} \left( \frac{k}{2^m} \right) h \right\}_{j=1}^\infty$$

converges weakly for all  $h \in \mathcal{H}$ . ■

**THEOREM 7.** *Let  $\mathcal{H}$  be a separable Hilbert space and let  $(T(s), \mathcal{H}(s))_{s \in \Delta^+ \cap [0,1]}$  be a multiplicative family of contractions on  $\mathcal{H}$ , then there exists a Hilbert space  $\mathcal{G}$  containing  $\mathcal{H}$  as a closed subspace and a group of unitary operators  $(W(s))_{s \in \Delta} \subset L(\mathcal{G})$  such that*

$$T(s) = P_{\mathcal{H}}^{\mathcal{G}} W(s)|_{\mathcal{H}(s)}, \text{ for all } s \in \Delta^+ \cap [0,1).$$

*Proof.* For  $s \in \Delta_+$  and  $h \in \mathcal{H}$  we define  $V^{(o)}(s)h$  as the weak limit of the sequence  $\{V^{(n_j)}(s)h\}_{j=1}^\infty$  given in Lemma 6, so we have that  $V^{(o)}(s) : \mathcal{H} \rightarrow \mathcal{H}$  is a linear contraction for all  $s \in \Delta_+$  and

$$\left\langle V^{(o)}(s)h, g \right\rangle_{\mathcal{H}} = \overline{\lim}_{j \rightarrow \infty} \left\langle V^{(n_j)}(s)h, g \right\rangle_{\mathcal{H}}, \tag{2.1}$$

for all  $s \in \Delta_+$  and  $h, g \in \mathcal{H}$ .

Let  $F : \Delta \rightarrow L(\mathcal{H})$  defined by

$$F(s) = \begin{cases} V^{(o)}(s) & \text{if } s \geq 0, \\ \left( V_{-s}^{(o)} \right)^* & \text{if } s < 0. \end{cases}$$

From (2.1) we obtain that

$$\left\langle F(s)h, g \right\rangle_{\mathcal{H}} = \lim_{j \rightarrow \infty} \left\langle F^{(n_j)}(s)h, g \right\rangle_{\mathcal{H}}$$

for all  $s \in \Delta$  and  $h, g \in \mathcal{H}$ , where  $F^{(n_j)}$  is the function defined in the Proposition 4, which is positive definite. Since the weak limit of positive definite functions is positive definite, we have that  $F$  is positive definite.

From the Neumark dilation theorem it follows that there exists a Hilbert space  $\mathcal{G}$  containing  $\mathcal{H}$  as a closed subspace and a unitary group  $(W(s))_{s \in \Delta} \subset L(\mathcal{G})$  such that

$$V^{(o)}(s) = P_{\mathcal{H}}^{\mathcal{G}} W(s)|_{\mathcal{H}}$$

for all  $s \in \Delta^+$ .

From Lemma 3 it follows that

$$V^{(o)}(s)|_{\mathcal{H}(s)} = T(s)$$

for all  $s \in \Delta^+ \cap [0, 1)$ .

Finally, we obtain that

$$T(s) = P_{\mathcal{H}}^{\mathcal{G}} W(s)|_{\mathcal{H}(s)}$$

for all  $s \in \Delta^+ \cap [0, 1)$ . ■

*Remark 8.* This is a new result. It is important to note that it was not necessary to assume the strong continuity of the multiplicative family of contractions.

*Remark 9.* Note that if, in Th. 7, we suppose that  $(T(s), \mathcal{H}(s))_{s \in \Delta^+ \cap [0, 1)}$  is a multiplicative family of isometries on  $\mathcal{H}$ , then we obtain that

$$T(s) = W(s)|_{\mathcal{H}(s)}, \text{ for all } s \in \Delta^+ \cap [0, 1),$$

where  $(W(s))_{s \in \Delta} \subset L(\mathcal{G})$  is a group of unitary operators.

### 3. UNITARY DILATION OF STRONGLY CONTINUOUS LOCAL SEMIGROUPS OF CONTRACTIONS

Let  $(\Gamma, +)$  be an additive subgroup of the real numbers such that  $\Delta \subset \Gamma$ . The following result is an extension, for the case of a separable Hilbert space, of Theorem 1 of [1].

**THEOREM 10.** *Let  $\mathcal{H}$  be a separable Hilbert space and let  $(T(s), \mathcal{H}(s))_{s \in [0, 1) \cap \Gamma}$  be a strongly continuous local semigroup of contractions on  $\mathcal{H}$ , then there exists a Hilbert space  $\mathcal{F}$  containing  $\mathcal{H}$  as a closed subspace and a strongly continuous group of unitary operators  $(U(s))_{s \in \mathbb{R}} \subset L(\mathcal{F})$  such that*

$$T(s) = P_{\mathcal{H}}^{\mathcal{F}} U(s)|_{\mathcal{H}(s)}, \text{ for all } s \in [0, 1) \cap \Gamma.$$

For the proof of the theorem we will need the following result.

LEMMA 11. *Let  $\mathcal{F}$  be a Hilbert space and let  $(V(s))_{s \in \Delta} \subset L(\mathcal{F})$  be a strongly continuous group of unitary operators. Then there exists a strongly continuous group unitary operators  $(U(s))_{s \in \mathbb{R}} \subset L(\mathcal{F})$  which extends  $(V(s))_{s \in \Delta}$ .*

*Proof.* We have that, for each  $h \in \mathcal{F}$  the application  $s \mapsto V(s)h$  is uniformly continuous on  $\Delta$ .

Since  $\Delta$  is dense in  $\mathbb{R}$ , for each  $h \in \mathcal{F}$  there exists a unique continuous map  $s \mapsto U(s)h$  from  $\mathbb{R}$  in  $\mathcal{F}$  such that if  $t \in \Delta$  then

$$U(s)h = V(s)h \quad \text{for each } h \in \mathcal{F}.$$

If  $s \in \mathbb{R}$  and  $\{s_n\}_{n \geq 1} \subset \Delta$  is a sequence such that  $\lim_{n \rightarrow \infty} s_n = s$  it holds that

$$U(s)h = \lim_{n \rightarrow \infty} V(s_n)h \quad \text{for all } h \in \mathcal{F}.$$

Finally, it is easy to check that  $(U(s))_{s \in \mathbb{R}}$  is a strongly continuous group unitary operators. ■

*Proof of Theorem 10.* From Theorem 7 it follows that there is a Hilbert space  $\mathcal{G}$  containing  $\mathcal{H}$  as a closed subspace and a group of unitary operators  $(W(s))_{s \in \Delta} \subset L(\mathcal{G})$  such that

$$T(s) = P_{\mathcal{H}}^{\mathcal{G}} W(s)|_{\mathcal{H}(s)}, \quad \text{for all } s \in \Delta^+ \cap [0, 1).$$

Let  $\mathcal{F}$  be the closed subspace of  $\mathcal{G}$  generated by  $\{W(s)h : s \in \Delta, h \in \mathcal{H}\}$  and let  $V(s)$  be the restriction of  $W(s)$  to  $\mathcal{F}$ , then  $(V(s))_{s \in \Delta} \subset L(\mathcal{F})$  is a unitary group such that

$$T(s) = P_{\mathcal{H}}^{\mathcal{F}} V(s)|_{\mathcal{H}(s)} \tag{3.1}$$

for all  $s \in \Delta^+ \cap [0, 1)$ .

Now we will prove that  $(V(s))_{s \in \Delta}$  is strongly continuous at 0 and therefore strongly continuous on  $\Delta$ .

Since  $\bigcup_{n=1}^{+\infty} \mathcal{H}(\frac{1}{2^n})$  is dense on  $\mathcal{H}$  we have that the set

$$\left\{ V(s)h : s \in \Delta, h \in \bigcup_{n=1}^{+\infty} \mathcal{H}\left(\frac{1}{2^n}\right) \right\}$$

is dense on  $\mathcal{F}$ .

Therefore it is enough to show that

$$\lim_{\substack{s \rightarrow 0 \\ s \in \Delta}} \|V(s)h - h\|_{\mathcal{H}} = 0, \quad \text{for all } h \in \bigcup_{n=1}^{+\infty} \mathcal{H}\left(\frac{1}{2^n}\right).$$

Let  $h \in \bigcup_{n=1}^{+\infty} \mathcal{H}\left(\frac{1}{2^n}\right)$ , then there exists a natural number  $n_o$  such that  $h \in \mathcal{H}\left(\frac{1}{2^{n_o}}\right)$ .

Let  $s \in \Delta^+ \cap [0, 1)$  such that  $s < \frac{1}{2^{n_o}}$ , then  $h \in \mathcal{H}(s)$  and we have that

$$\begin{aligned} \|V(s)h - h\|_{\mathcal{H}}^2 &= \langle V(s)h - h, V(s)h - h \rangle_{\mathcal{H}} \\ &= \langle V(s)h, V(s)h \rangle_{\mathcal{H}} - 2\operatorname{Re}(\langle V(s)h, h \rangle_{\mathcal{H}}) + \langle h, h \rangle_{\mathcal{H}} \\ &= 2\langle h, h \rangle_{\mathcal{H}} - 2\operatorname{Re}(\langle P_{\mathcal{H}}^{\mathcal{F}}V(s)h, h \rangle_{\mathcal{H}}) \\ &= 2\langle h, h \rangle_{\mathcal{H}} - 2\operatorname{Re}(\langle T(s)h, h \rangle_{\mathcal{H}}). \end{aligned}$$

So the result follows from the strong continuity of  $(T(s), \mathcal{H}(s))_{s \in [0, 1) \cap \Gamma}$ .

From Lemma 11 it follows that there exists a strongly continuous group of unitary operators  $(U(s))_{s \in \mathbb{R}} \subset L(\mathcal{F})$  such that

$$U(s) = V(s) \tag{3.2}$$

for all  $s \in \Delta$ .

Let us show that  $T(s) = P_{\mathcal{H}}^{\mathcal{F}}U(s)|_{\mathcal{H}(s)}$  for all  $s \in [0, 1) \cap \Gamma$ .

Let  $s \in [0, 1) \cap \Gamma$  and  $h \in \mathcal{H}(s)$ . Since  $\Delta$  is dense on  $\mathbb{R}$  there exists an increasing sequence  $\{s_n\}_{n \geq 1}$  in  $\Delta^+ \cap [0, 1)$  such that  $\lim_{n \rightarrow \infty} s_n = s$ . We have that  $\mathcal{H}(s) \subset \mathcal{H}(s_n)$  for all  $n$ .

From the strong continuity of the local semigroup and the group  $(U(s))$  we have

$$\begin{aligned} T(s)h &= \lim_{n \rightarrow \infty} T(s_n)h = \lim_{n \rightarrow \infty} P_{\mathcal{H}}^{\mathcal{F}}V(s_n)h \\ &= \lim_{n \rightarrow \infty} P_{\mathcal{H}}^{\mathcal{F}}U(s_n)h \\ &= P_{\mathcal{H}}^{\mathcal{F}}U(s)h. \end{aligned}$$

■

*Remark 12.* Note that if  $(T(s), \mathcal{H}(s))_{s \in \Delta^+ \cap [0, 1)}$  is a local semigroup of isometries on  $\mathcal{H}$ , then

$$T(s) = U(s)|_{\mathcal{H}(s)}, \quad \text{for all } s \in \Delta^+ \cap [0, 1).$$

*Remark 13.* If a  $\Delta \subset \Gamma$ , with a natural change of variables this theorem can be extended for an interval of the form  $[0, a)$  for  $a \in \mathbb{R}$  and  $a > 0$ , instead of  $[0, 1)$ .



4. EXTENSION OF POSITIVE DEFINITE FUNCTIONS

As in Section 1, let  $(\Gamma, +)$  be an additive subgroup of the real numbers  $(\mathbb{R}, +)$ .

Let  $a \in \mathbb{R}$  such that  $a > 0$  and let  $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$  be a Hilbert space, recall that a function  $f : (-a, a) \cap \Gamma \rightarrow L(\mathcal{E})$  is said to be *positive definite* if

$$\sum_{x,y \in [0,a) \cap \Gamma} \langle f(x-y)h(x), h(y) \rangle_{\mathcal{E}} \geq 0$$

for all functions  $h : [0, a) \cap \Gamma \rightarrow \mathcal{E}$  of finite support. We will suppose that  $f(0) = I_{\mathcal{E}}$ , the identity operator.

M. G. Kreĭn [6] proved that a continuous scalar valued positive definite function, on an interval of the real line, can be extended to a continuous positive definite function on the whole line. This result was extended for strongly continuous operator valued functions by M. L. Gorbachuck [4].

A multiplicative family of isometric operators can be associated, in a natural way, to a positive definite function, see for example [1, 2, 5, 9] for more details. This correspondence is established as follows.

Let

$$\mathcal{D} = \{h : [0, a) \cap \Gamma \rightarrow \mathcal{E} \mid \text{support of } h \text{ is finite} \}.$$

Then  $\mathcal{D}$  is a linear space, for  $h, h' \in \mathcal{D}$  we define

$$\langle h, h' \rangle_{\mathcal{D}} = \sum_{x,y \in [0,a) \cap \Gamma} \langle f(x-y)h(x), h'(y) \rangle_{\mathcal{E}},$$

then  $\langle \cdot, \cdot \rangle_{\mathcal{D}}$  is a, possibly degenerate, positive sesquilinear form on  $\mathcal{D}$ . Let  $\mathcal{H}$  be the completion of  $\mathcal{D}$ , after the natural quotient.

For  $r \in [0, a) \cap \Gamma$  we define

$$\mathcal{D}(r) = \{h \in \mathcal{D} \mid \text{support}(h) \subset [0, a-r) \cap \Gamma\}$$

and  $S(r) : \mathcal{D}(r) \rightarrow \mathcal{D}$  by

$$S(r)h(x) = \begin{cases} h(x-r) & \text{if } x \in [r, a), \\ 0 & \text{if } x \in [0, r). \end{cases}$$

It holds that  $S(r)$  is an isometric operator and, if  $\mathcal{H}(r)$  is the closure of  $\mathcal{D}(r)$  in  $\mathcal{H}$ , then  $S(r)$  can be extended to an isometric operator from  $\mathcal{H}(r)$  to  $\mathcal{H}$ , denoting this extension by  $T(r)$ , we have that  $(T(r), \mathcal{H}(r))_{r \in [0,a) \cap \Gamma}$  is a

multiplicative family of contractions. It is easy to check that, if  $f$  is strongly continuous, then  $(T(r), \mathcal{H}(r))_{r \in [0, a) \cap \Gamma}$  is a strongly continuous local semigroup of isometries.

Since  $f(0) = I_{\mathcal{E}}$  there exists a natural immersion of  $\mathcal{E}$  into  $\mathcal{H}$  and it holds that

$$\langle f(x)h, h' \rangle_{\mathcal{E}} = \langle T(x)h, h' \rangle_{\mathcal{H}}$$

for  $h, h' \in \mathcal{E}$  and  $x \in [0, a) \cap \Gamma$ . If the multiplicative family of isometries  $(T(r), \mathcal{H}(r))_{r \in [0, a) \cap \Gamma}$  could be extended to an unitary group  $(U(r))_{r \in \mathbb{R}}$  on a larger Hilbert space  $\mathcal{F}$  we would have

$$f(x) = P_{\mathcal{E}}^{\mathcal{F}} U(x)|_{\mathcal{E}} \quad \text{for } x \in (-a, a) \cap \Gamma.$$

So the extension to an unitary group of the family  $(T(r), \mathcal{H}(r))_{r \in [0, a) \cap \Gamma}$  is a sufficient condition for the extension of  $f$  to a positive definite function in the whole line. If the space  $\mathcal{E}$  is separable, then the corresponding space  $\mathcal{H}$  is also separable. Therefore from Theorems 7 and 10 we obtain the following two results.

**THEOREM 14.** *Let  $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$  be a separable Hilbert space and let  $f : \Delta \cap (-1, 1) \rightarrow \mathcal{E}$  be a positive definite function such that  $f(0) = I_{\mathcal{E}}$ , then  $f$  can be extended to a positive definite function  $F : \Delta \rightarrow \mathcal{E}$ .*

**THEOREM 15.** *Suppose that  $a \in [0, +\infty)$  and that  $a \Delta \subset \Gamma$ . Let  $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$  be a separable Hilbert space and let  $f : \Gamma \cap (-a, a) \rightarrow \mathcal{E}$  be a strongly continuous positive definite function such that  $f(0) = I_{\mathcal{E}}$ , then  $f$  can be extended to a strongly continuous positive definite function  $F : \mathbb{R} \rightarrow \mathcal{E}$ .*

*Remark 16.* Note that it was not necessary to assume the strong continuity of the function in Theorem 14.

#### REFERENCES

- [1] R. BRUZUAL, Local semigroups of contractions and some applications to Fourier representation theorems, *Integr. Equ. Oper. Theory* **10** (1987), 780–801.
- [2] R. BRUZUAL, M. DOMÍNGUEZ, Extensions of operator valued positive definite functions and commutant lifting on ordered groups, *J. Funct. Anal.* **185** (2) (2001), 456–473.
- [3] R. BRUZUAL, M. DOMÍNGUEZ, M. MONTILLA, On unitary dilations of two-parameter semigroups of contractions and continuous commutant lifting, *Acta Sci. Math.* **77** (3-4) (2011), 607–620.

- [4] M.L. GORBACHUCK, Representation of positive definite operator functions, *Ukrainian Math. J.* **17** (1965), 29–46.
- [5] M. GROSSMAN, H. LANGER, Über indexerhaltende Erweiterungen eines hermiteschen operators in Pontrjaginraum, *Math. Nachr.* **64** (1974), 289–317.
- [6] M.G. KREĬN , Sur le problème du prolongement des fonctions hermitiennes positives et continues, *Dokl. Akad. Nauk. SSSR* **26** (1940), 17–22.
- [7] M.A. NEUMARK, Spectral functions of a symmetric operator, *Bull. Acad. Sci. URSS, Sér. Math.* **4** (1940), 277–318.
- [8] M. PTAK, Unitary dilations of multiparameter semigroups of operators, *Ann. Polon. Math.* **45** (1985), 237–243.
- [9] Z. SASVÁRI, “Positive Definite and Definitizable Functions”, Akademie Verlag, Berlin, 1994.
- [10] B. SZ.-NAGY, C. FOIAS, “Harmonic Analysis of Operators on Hilbert Space”, North Holland Publishing Co., Amsterdam-London, 1970.