Weyl Type Theorems for Restrictions of Bounded Linear Operators

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Abstract: In this paper we give sufficient conditions for which Weyl’s theorems for a bounded linear operator $T$, acting on a Banach space $X$, can be reduced to the study of Weyl’s theorems for some restriction of $T$.

Key words: Weyl’s theorem, a-Weyl’s theorem, semi-Fredholm operator, pole of the resolvent.

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1. Introduction

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space $X$. For $T \in L(X)$, we denote by $N(T)$ the null space of $T$ and by $R(T) = T(X)$ the range of $T$. We denote by $\alpha(T) := \dim N(T)$ the nullity of $T$ and by $\beta(T) := \text{codim } R(T) = \dim X/R(T)$ the defect of $T$. Other two classical quantities in operator theory are the ascent $p = p(T)$ of an operator $T$, defined as the smallest non-negative integer $p$ such that $N(T^p) = N(T^{p+1})$ (if such an integer does not exist, we put $p(T) = \infty$), and the descent $q = q(T)$, defined as the smallest non-negative integer $q$ such that $R(T^q) = R(T^{q+1})$ (if such an integer does not exist, we put $q(T) = \infty$). It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Furthermore, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if $\lambda$ is a pole of the resolvent, see [12, Proposition 50.2]. An operator $T \in L(X)$ is said to be Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm), if $\alpha(T), \beta(T)$ are both finite (respectively, $R(T)$ closed and $\alpha(T) < \infty$, $\beta(T) < \infty$). $T \in L(X)$ is said to be semi-Fredholm if $T$ is either an upper semi-Fredholm or a lower semi-Fredholm operator. If $T$ is semi-Fredholm the index of $T$ defined by $\text{ind } T := \alpha(T) - \beta(T)$. Other two
important classes of operators in Fredholm theory are the classes of semi-Browder operators. These classes are defined as follows, $T \in L(X)$ is said to be Browder (resp. upper semi-Browder, lower semi-Browder) if $T$ is a Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm) and both $p(T), q(T)$ are finite (respectively, $p(T) < \infty, q(T) < \infty$). A bounded operator $T \in L(X)$ is said to be upper semi-Weyl (respectively, lower semi-Weyl) if $T$ is an upper Fredholm operator (respectively, lower semi-Fredholm) and index $\text{ind} T \leq 0$ (respectively, $\text{ind} T \geq 0$). $T \in L(X)$ is said to be Weyl if $T$ is both upper and lower semi-Weyl, i.e. $T$ is a Fredholm operator having index 0. The Browder spectrum and the Weyl spectrum are defined, respectively, by

$$
\sigma_b(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder} \},
$$

and

$$
\sigma_w(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl} \}.
$$

Since every Browder operator is Weyl then $\sigma_w(T) \subseteq \sigma_b(T)$. Analogously, The upper semi-Browder spectrum and the upper semi-Weyl spectrum are defined by

$$
\sigma_{ub}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder} \},
$$

and

$$
\sigma_{uw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl} \}.
$$

Lemma 1.1. If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:

(i) There exists $n \geq p + 1$ such that $T^n(X)$ is closed;

(ii) $T^n(X)$ is closed for all $n \geq p$.

Proof. Define $c_i'(T) := \dim(N(T^i)/N(T^{i+1}))$. Clearly, $p = p(T) < \infty$ entails that $c_i'(T) = 0$ for all $i \geq p$, so $k_i(T) := c_i'(T) - c_{i+1}'(T) = 0$ for all $i \geq p$. The equivalence easily follows from [13, Lemma 12].

Now, we introduce an important property in local spectral theory. The localized version of this property has been introduced by Finch [11], and in the framework of Fredholm theory this property has been characterized in several ways, see [1, Chapter 3]. A bounded operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at
\( \lambda_0 \), if for every open disc \( \mathbb{D}_{\lambda_0} \subseteq \mathbb{C} \) centered at \( \lambda_0 \) the only analytic function \( f : \mathbb{D}_{\lambda_0} \to X \) which satisfies the equation

\[(\lambda I - T) f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0},\]

is the function \( f \equiv 0 \) on \( \mathbb{D}_{\lambda_0} \). The operator \( T \) is said to have SVEP if \( T \) has the SVEP at every point \( \lambda \in \mathbb{C} \). Evidently, \( T \in \mathcal{L}(X) \) has SVEP at every point of the resolvent \( \rho(T) := \mathbb{C} \setminus \sigma(T) \). Moreover, from the identity theorem for analytic functions it is easily seen that \( T \) has SVEP at every point of the boundary \( \partial \sigma(T) \) of the spectrum. In particular, \( T \) has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

\[ p(\lambda I - T) < \infty \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \quad (1.1) \]

and dually

\[ q(\lambda I - T) < \infty \quad \Rightarrow \quad T^* \text{ has SVEP at } \lambda. \quad (1.2) \]

Recall that \( T \in \mathcal{L}(X) \) is said to be bounded below if \( T \) is injective and has closed range. Denote by \( \sigma_{ap}(T) \) the classical approximate point spectrum defined by

\[ \sigma_{ap}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below} \}. \]

Note that if \( \sigma_{s}(T) \) denotes the surjectivity spectrum

\[ \sigma_{s}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not onto} \}, \]

then \( \sigma_{ap}(T) = \sigma_{s}(T^*) \) and \( \sigma_{s}(T) = \sigma_{ap}(T^*). \)

It is easily seen from definition of localized SVEP that

\[ \lambda \notin \text{acc } \sigma_{ap}(T) \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \quad (1.3) \]

where \( \text{acc } K \) means the set of all accumulation points of \( K \subseteq \mathbb{C} \), and if \( T^* \) denotes the dual of \( T \), then

\[ \lambda \notin \text{acc } \sigma_{ap}(T) \quad \Rightarrow \quad T \text{ has SVEP at } \lambda. \quad (1.4) \]

Remark 1.2. The implications (1.1), (1.2), (1.3) and (1.4) are actually equivalences whenever \( T \in \mathcal{L}(X) \) is semi-Fredholm (see [1, Chapter 3]).
Denote by iso $K$ the set of all isolated points of $K \subseteq \mathbb{C}$. Let $T \in L(X)$, define

$$
\pi_{00}(T) = \{ \lambda \in \sigma(T) : 0 < \alpha(\lambda I - T) < \infty \}, \\
\pi_{00}^a(T) = \{ \lambda \in \sigma_{ap}(T) : 0 < \alpha(\lambda I - T) < \infty \}.
$$

Clearly, for every $T \in L(X)$ we have $\pi_{00}(T) \subseteq \pi_{00}^a(T)$.

Let $T \in L(X)$ be a bounded operator. Following Coburn [8], $T$ is said to satisfy Weyl’s theorem, in symbol (W), if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. According to Rakočević [15], $T$ is said to satisfy a-Weyl’s theorem, in symbol (aW), if $\sigma_{ap}(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T)$.

Note that

$$\text{a-Weyl’s theorem} \quad \Rightarrow \quad \text{Weyl’s theorem},$$

see for instance [1, Chapter 3]. The converse of these implication in general does not hold.

Weyl type theorems have been recently studied by several authors ([2], [3], [5], [6], [8], [9], [10], [15] and [16]). In these papers several results are obtained, by considering an operator $T \in L(X)$ in the whole space $X$. In this paper we give sufficient conditions for which Weyl type theorems holds for $T$, if and only if there exists $n \in \mathbb{N}$ such that the range $R(T^n)$ of $T^n$ is closed and Weyl type theorems holds for $T_n$, where $T_n$ denote the restriction of $T$ on the subspace $R(T^n) \subseteq X$.

2. Preliminaries

In this section we establish several lemmas that will be used throughout the paper. We begin examining some algebraic relations between $T$ and $T_n$, $T_n$ viewed as a operator from the space $R(T^n)$ in to itself.

**Lemma 2.1.** Let $T \in L(X)$ and $T_n$, $n \in \mathbb{N}$, be the restriction of the operator $T$ on the subspace $R(T^n) = T^n(X)$. Then, for all $\lambda \neq 0$, we have:

(i) $N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$, for any $m$;

(ii) $R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n)$, for any $m$;

(iii) $\alpha(\lambda I - T_n) = \alpha(\lambda I - T)$;

(iv) $p(\lambda I - T_n) = p(\lambda I - T)$;

(v) $\beta(\lambda I - T_n) = \beta(\lambda I - T)$.
Proof. (i) For \(m = 0\),
\[
N((\lambda I - T_n)^m) = N((\lambda I - T)^m)
\]
holds trivially. Let \(x \in N((\lambda I - T)^m), m \geq 1\), then
\[
0 = (\lambda I - T)^m x = \sum_{k=0}^{m} \frac{m!}{k!(m - k)!} (-1)^k \lambda^{m-k} T^k x
\]
\[
= \lambda^m x + \sum_{k=1}^{m} \frac{m!}{k!(m - k)!} (-1)^k \lambda^{m-k} T^k x.
\]
Thus \(0 = \lambda^m x + h(T)x\), where
\[
h(T) = \sum_{k=1}^{m} \frac{m!}{k!(m - k)!} (-1)^k \lambda^{m-k} T^k.
\]
Hence \(-\lambda^m x = h(T)x\), and since \(\lambda \neq 0\), then \(x = -\lambda^{-m} h(T)x\). From this equality, it follows that
\[
(-\lambda^{-m} h(T))^2 x = -\lambda^{-m} h(T)(-\lambda^{-m} h(T)x)
\]
\[
= -\lambda^{-m} h(T)x = x.
\]
Consequently \(x = (-\lambda^{-m} h(T))^2 x\). By repeating successively the same argument, we obtain that \(x = (-\lambda^{-m} h(T))^j x\), for all \(j \in \mathbb{N}\). But since \(-\lambda^{-m} h(T)x \in R(T)\), then \((-\lambda^{-m} h(T))^j x \in R(T^j)\), for all \(j \in \mathbb{N}\). Therefore \(x = (-\lambda^{-m} h(T))^n x \in R(T^n)\), and since \(R(T^n)\) is \(T\)-invariant subspace, we conclude that
\[
0 = (\lambda I - T)^m x = \sum_{k=0}^{m} \frac{m!}{k!(m - k)!} (-1)^k \lambda^{m-k} T^k x
\]
\[
= \sum_{k=0}^{m} \frac{m!}{k!(m - k)!} (-1)^k \lambda^{m-k} (T_n)^k x = (\lambda I - T_n)^m x.
\]
So \(x \in N((\lambda I - T_n)^m)\), and we get the inclusion
\[
N((\lambda I - T)^m) \subseteq N((\lambda I - T_n)^m).
\]
On the other hand, since \(T_n\) is the restriction of \(T\) on \(R(T^n)\), and \(R(T^n)\) is invariant under \(T\), it then follows the inclusion
\[
N((\lambda I - T_n)^m) \subseteq N((\lambda I - T)^m).
\]
From which, we obtain that $N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$.

(ii) Since $T_n$ is the restriction of $T$ on $R(T^n)$, and $R(T^n)$ is invariant under $T$, then

$$R((\lambda I - T_n)^m) \subseteq R((\lambda I - T)^m) \cap R(T^n).$$

Now, we show the inclusion $R((\lambda I - T)^m) \cap R(T^n) \subseteq R((\lambda I - T_n)^m)$. For this, it will suffice to show that for $m \in \mathbb{N}$, the implication

$$(\lambda I - T)^m x \in R(T^n) \quad \Rightarrow \quad x \in R(T^n),$$

holds. For $m = 1$. Let $y \in R(\lambda I - T) \cap R(T^n)$, then there exists $x \in X$ such that $\lambda x - Tx = (\lambda I - T)x = y \in R(T^n)$, so $\lambda^2 x - \lambda Tx = \lambda y \in R(T^n)$. But since $\lambda Tx - T^2 x = Ty \in R(T^n)$, because $\lambda x - T x = y$ and $R(T^n)$ is invariant under $T$, we have that $\lambda^2 x - \lambda Tx, \lambda Tx - T^2 x \in R(T^n)$. Then

$$\lambda^2 x - T^2 x = \lambda^2 x - \lambda Tx + \lambda Tx - T^2 x \in R(T^n).$$

Thus $\lambda^2 x - T^2 x \in R(T^n)$. Hence $\lambda^3 x - \lambda T^2 x = \lambda(\lambda^2 x - T^2 x) \in R(T^n)$, and since $\lambda T^2 x - T^3 x = T^2 y \in R(T^n)$, we have that $\lambda^3 x - \lambda T^2 x, \lambda T^2 x - T^3 x \in R(T^n)$. From which,

$$\lambda^3 x - T^3 x = \lambda^3 x - \lambda T^2 x + \lambda T^2 x - T^3 x \in R(T^n).$$

That is, $\lambda^3 x - T^3 x \in R(T^n)$. Now, suppose that $\lambda^j x - T^j x \in R(T^n)$, for some $j \in \mathbb{N}$. From this, $\lambda^{j+1} x - \lambda T^j x = \lambda(\lambda^j x - T^j x) \in R(T^n)$, and $\lambda T^j x - T^{j+1} x = T^j y \in R(T^n)$, thus $\lambda^{j+1} x - \lambda T^j x, \lambda T^j x - T^{j+1} x \in R(T^n)$. From which,

$$\lambda^{j+1} x - T^{j+1} x = \lambda^{j+1} x - \lambda T^j x + \lambda T^j x - T^{j+1} x \in R(T^n).$$

Consequently, by mathematical induction, we obtain that $\lambda^j x - T^j x \in R(T^n)$ for all $j \in \mathbb{N}$. In particular, $\lambda^n x - T^n x \in R(T^n)$, and since $\lambda \neq 0$, then

$$x = \lambda^{-n}((\lambda^n x - T^n x) + T^n x) \in R(T^n).$$

By the above reasoning, we conclude that, for $m = 1$, the implication

$$(\lambda I - T)x \in R(T^n) \quad \Rightarrow \quad x \in R(T^n)$$

holds.

Now, suppose that for $m \geq 1$,

$$(\lambda I - T)^m x \in R(T^n) \quad \Rightarrow \quad x \in R(T^n).$$
If \((\lambda I - T)^{m+1}x \in R(T^n)\), then \((\lambda I - T)((\lambda I - T)^m x) \in R(T^n)\). From the proof of case \(m = 1\), we conclude that \((\lambda I - T)^m x \in R(T^n)\). Therefore by inductive hypothesis, \(x \in R(T^n)\). Then, by mathematical induction, we conclude that for all \(m \in \mathbb{N}\)
\[(\lambda I - T)^m x \in R(T^n) \implies x \in R(T^n)\]
holds.

Finally, if \(y \in R((\lambda I - T)^m) \cap R(T^n)\) there exists \(x \in X\) such that \((\lambda I - T)^m x = y \in R(T^n)\), then \((\lambda I - T)^m x \in R(T^n)\). As the above proof, we conclude that \(x \in R(T^n)\). Thus
\[
y = (\lambda I - T)^m x = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \lambda^{m-k}T^k x
\]
then \(y \in R((\lambda I - T_n)^m)\). This shows that,
\[R((\lambda I - T)^m) \cap R(T^n) \subseteq R((\lambda I - T_n)^m)\].
Consequently, \(R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n)\).

(iii) and (iv), it follows immediately from the equality
\[N((\lambda I - T_n)^m) = N((\lambda I - T)^m)\] for all \(m \in \mathbb{N}\).

(v) Observe that \(R(\lambda I - T_n)\) is a subspace of \(R(T^n)\). Let \(M\) be a subspace of \(R(T^n)\) such that \(R(T^n) = R(\lambda I - T_n) \oplus M\). Since \(R(\lambda I - T_n) = R(\lambda I - T) \cap R(T^n)\), we have
\[R(\lambda I - T) \cap M = R(\lambda I - T) \cap R(T^n) \cap M = R(\lambda I - T_n) \cap M = 0\].
Thus \(R(\lambda I - T) \cap M = \{0\}\). Now, we show that \(X = R(\lambda I - T) + M\).

Let \(\mu \in \mathbb{C}\) such that \(\mu I - T\) is invertible in \(L(X)\), then \((\mu I - T)^j\) is invertible in \(L(X)\), for all \(j \in \mathbb{N}\). In particular \((\mu I - T)^m\) is invertible in \(L(X)\), for all \(m \geq n\). Thus, if \(y \in X\) there exists \(x \in X\) such that \(y = (\mu I - T)^m x\). Thus,
\[
y = (\mu I - T)^m x = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!}(-1)^j \mu^{m-j}T^j x
\]
\[
= \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!}(-1)^j \mu^{m-j}T^j x + \sum_{j=n}^{m} \frac{m!}{j!(m-j)!}(-1)^j \mu^{m-j}T^j x.
\]
Since $R(T^j) \subseteq R(T^n)$, for $n \leq j \leq m$, then we can write $y = u + v$, where:

$$
u = \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!}(-1)^j \mu^{m-j}T^jx \in X,$$

$$v = \sum_{j=n}^{m} \frac{m!}{j!(m-j)!}(-1)^j \mu^{m-j}T^jx \in R(T^n).$$

Now, from the above decomposition and for any $\lambda \neq 0$, we obtain a sequence $(y_k)_{k=0}^{\infty}$, where $y_k = \lambda^{-k-1}(\lambda I - T)T^k u$, for $k = 0, 1, \ldots$, such that

$$u = y_0 + y_1 + \cdots + y_{n-1} + \lambda^{-n}T^n u \in R(\lambda I - T) + R(T^n),$$

because $y_k = \lambda^{-k-1}(\lambda I - T)T^k u \in R(\lambda I - T)$ and $\lambda^{-n}T^n u \in R(T^n)$.

On the other hand,

$$v + \lambda^{-n}T^n u \in R(T^n) + R(T^n) = R(T^n) = R(\lambda I - T_n) + M.$$ 

Thus $v + \lambda^{-n}T^n u = z + m$, where $z \in R(\lambda I - T_n)$ and $m \in M$. From this, and since $R(\lambda I - T_n) \subseteq R(\lambda I - T)$, we obtain that

$$y = u + v = y_0 + y_1 + \cdots + y_{n-1} + \lambda^{-n}T^n u + v$$

$$= y_0 + y_1 + \cdots + y_{n-1} + z + m$$

$$= (y_0 + y_1 + \cdots + y_{n-1} + z) + m \in R(\lambda I - T) + M.$$ 

Therefore, we have that $X \subseteq R(\lambda I - T) + M$, consequently $X = R(\lambda I - T) + M$. But since $R(\lambda I - T) \cap M = \{0\}$, and hence it follows that $X = R(\lambda I - T) \oplus M$, which implies that

$$\beta(\lambda I - T) = \dim M = \beta(\lambda I - T_n).$$

This shows that $\beta(\lambda I - T) = \beta(\lambda I - T_n)$. □

The following result concerning the ranges of the powers of $\lambda I - T$, where $\lambda \in \mathbb{C}$ and $T \in L(X)$, plays an important role in this paper. In the proof of this corollary we use the notion of paraclosed (or paracomplete) subspace and the Neubauer Lemma (see [14]).

**Lemma 2.2.** If $R(T^n)$ is closed in $X$ and $R((\lambda I - T_n)^m)$ is closed in $R(T^n)$, then there exists $k \in \mathbb{N}$ such that $R((\lambda I - T)^k)$ is closed in $X$. 
Proof. Observe that for $\lambda = 0$,
\[
R((0I - T_n)^m) = R((T_n)^m) = R(T^{m+n}) .
\]
Then $R(T^{m+n})$ is a closed subspace of $R(T^n)$. Since $R(T^n)$ is closed, we have that $R((0I - T)^{m+n}) = R(T^{m+n})$ is closed. On the other hand, if $\lambda \neq 0$ and $R((\lambda I - T)^{m})$ is a closed subspace of $R(T^n)$, since $R(T^n)$ is closed in $X$, we have that $R((\lambda I - T_n)^m)$ is closed in $X$. But, from the incise (ii) in Lemma 2.1,
\[
R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n) .
\]
Thus $R((\lambda I - T)^m) \cap R(T^n)$ is closed in $X$. Also, if $\lambda \neq 0$ the polynomials $(\lambda - z)^m$ and $z^n$ have no common divisors, so there exist two polynomials $u$ and $v$ such that $1 = (\lambda - z)^m u(z) + z^n v(z)$, for all $z \in \mathbb{C}$. Hence $I = (\lambda I - T)^m u(T) + T^n v(T)$ and so $R((\lambda I - T)^m) + R(T^n) = X$. Since both $R((\lambda I - T)^m)$ and $R(T^n)$ are paraclosed subspaces, and $R((\lambda I - T)^m) \cap R(T^n)$ and $R((\lambda I - T)^m) + R(T^n)$ are closed, using Neubauer Lemma [14, Proposition 2.1.2], we have that $R((\lambda I - T)^m)$ is closed. 

Recall that for an operator $T \in L(X)$, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when $\lambda$ is a pole of the resolvent of $T$ (see [12, Proposition 50.2]).

Lemma 2.3. If $0$ is not a pole of the resolvent of $T \in L(X)$ and $R(T^n)$ is closed, then $\pi_00(T) \subseteq \pi_00(T_n)$.

Proof. By Lemma 2.1, $\sigma(T_n) \setminus \{0\} = \sigma(T) \setminus \{0\}$. Also, $0 \notin \sigma(T)$ implies $T$ bijective, thus $T = T_n$. Hence $\sigma(T_n) \subseteq \sigma(T)$. Moreover, $\sigma(T) \subseteq \sigma(T_n)$.

Since, if $\lambda \in \sigma(T)$, then $\sigma(T) \cap \mathbb{D}_\lambda = \{\lambda\}$ for some open disc $\mathbb{D}_\lambda \subseteq \mathbb{C}$ centered at $\lambda$. Thus,
\[
\sigma(T_n) \cap \mathbb{D}_\lambda \subseteq \sigma(T) \cap \mathbb{D}_\lambda = \{\lambda\} .
\]
Consequently $\sigma(T_n) \cap \mathbb{D}_\lambda = \{\lambda\}$ or $\sigma(T_n) \cap \mathbb{D}_\lambda = \emptyset$. If $\sigma(T_n) \cap \mathbb{D}_\lambda = \emptyset$, then $\lambda \notin \sigma(T_n)$, so that $p(\lambda I - T_n) = q(\lambda I - T_n) = 0$. For the case $\lambda \neq 0$, from Lemma 2.1, $p(\lambda I - T) = 0$ and $\beta(\lambda I - T) = 0$, then $\lambda \notin \sigma(T)$ a contradiction. In the case where $\lambda = 0$, $p(T_n) = q(T_n) = 0$ implies, by [7, Lemma 2 and Lemma 3] and [12, Proposition 38.6], that $0 < p(T) = q(T) < \infty$, which is impossible, because 0 is not a pole of the resolvent of $T$. Consequently, $\sigma(T_n) \cap \mathbb{D}_\lambda = \{\lambda\}$, so we have that $\lambda \in \sigma(T_n)$.

Now, the following argument shows that $\pi_00(T) \subseteq \pi_00(T_n)$. If $\lambda \in \pi_00(T)$, we have that $\lambda \in \sigma(T_n)$, because $\lambda \in \sigma(T)$. On the other hand, for
$\lambda \neq 0$, Lemma 2.1 implies that $\alpha(\lambda I - T) = \alpha(\lambda I - T_n)$, so $0 < \alpha(\lambda I - T_n) < \infty$.

For $\lambda = 0$, we claim that $\alpha(T_n) > 0$. If $\alpha(T_n) = 0$, we have that $p(T_n) = 0$.

By [7, Lemma 2], $p(T) < \infty$. Moreover [7, Remark 1],

$$p(T) = \inf\{k \in \mathbb{N} : T_k \text{ is injective}\} \leq n.$$  

Thus, by Lemma 1.1, $T_n$ is bounded below, because $T_n$ is injective and $R(T_n) = R(T^{n+1})$ is closed, so $T_n$ is semi-Fredholm. Also $(T_n^*)^\dagger$ has SVEP at 0, because 0 $\in$ iso $\sigma(T_n)$, then $q(T_n) < \infty$ ([1, Chapter 3]), which implies that $q(T) < \infty$ ([7, Lemma 3]). Hence $0 < p(T) = q(T) < \infty$, a contradiction, since 0 is not a pole of the resolvent of $T$. Thus $0 < \alpha(T_n) = \alpha(0I - T_n)$. Finally, since $N(T_n) \subseteq N(T)$ and $\alpha(T) < \infty$ it then follows the equality $\alpha(T_n) = \alpha(0I - T_n) < \infty$. Thus, $0 \in$ iso $\sigma(T_n)$ and $0 < \alpha(0I - T_n) < \infty$.

Consequently $\lambda \in \pi_{00}(T_n)$, for each $\lambda \in \pi_{00}(T)$, so we have the inclusion $\pi_{00}(T) \subseteq \pi_{00}(T_n)$. 

The result of Lemma 2.3 may be extended as follows.

**Lemma 2.4.** If 0 is not a pole of the resolvent of $T \in L(X)$ and $R(T^n)$ is closed, then $\pi_{00}^0(T) \subseteq \pi_{00}^0(T_n)$.

**Proof.** If $\lambda \notin \sigma_{ap}(T)$, then $\lambda I - T$ is injective and $R(\lambda I - T)$ is closed. Now, here we consider the two different cases $\lambda \neq 0$ and $\lambda = 0$. If $\lambda \neq 0$, by Lemma 2.1, $N(\lambda I - T_n) = N(\lambda I - T)$ and $R(\lambda I - T_n) = R(\lambda I - T) \cap R(T^n)$ is closed. Hence $\lambda I - T_n$ is bounded below, and so $\lambda \notin \sigma_{ap}(T_n)$. In the other case, $-T$ bounded below implies that $0 = p(T) = p(T_n)$ and $R(T)$ is closed. Thus $T_n$ is injective and, by Lemma 1.1, $R(T_n) = R(T^{n+1})$ is closed. From this we obtain that $T_n$ is bounded below. Consequently, $\sigma_{ap}(T_n) \subseteq \sigma_{ap}(T)$. Similarly, as in the proof of Lemma 2.3 and taking into account Lemma 2.2, we can prove that $\text{iso } \sigma_{ap}(T) \subseteq \text{iso } \sigma_{ap}(T_n)$.

Finally, to show $\pi_{00}^0(T) \subseteq \pi_{00}^0(T_n)$. Observe that, if $\lambda \in \pi_{00}^0(T)$ then $\lambda \in \text{iso } \sigma_{ap}(T)$ and $0 < \alpha(\lambda I - T) < \infty$. Thus $\lambda \in \text{iso } \sigma(T_n)$. For $\lambda \neq 0$, by Lemma 2.1, $\alpha(\lambda I - T_n) = \alpha(\lambda I - T_n)$, and so $0 < \alpha(\lambda I - T_n) < \infty$. In the case $\lambda = 0$, $p(T_n) = 0$, and $R(T^n)$ is closed. Similarly to the case $p(T_n) = 0$ and $R(T^n)$ closed in the proof of Lemma 2.3, one shows that $0 < \alpha(0I - T_n) < \infty$. Consequently $\pi_{00}^0(T) \subseteq \pi_{00}^0(T_n)$. 

3. **Weyl’s theorems and restrictions**

In this section we give conditions for which Weyl’s theorem (resp. a-Weyl’s theorem) for an operator $T \in L(X)$ is equivalent to Weyl’s theorem (resp. a-
Weyl’s theorem) for certain restriction $T_n$ of $T$.

It is well known that if $\lambda$ is a pole of the resolvent of $T$, then $\lambda$ is an isolated point of the spectrum $\sigma(T)$. Thus, the following result is an immediate consequence of Lemma 2.1 and Lemma 2.3.

**Theorem 3.1.** Suppose that $0$ is not an isolated point of $\sigma(T)$. Then $T$ satisfies (W) if and only if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies (W).

**Proof.** (Necessity) Assume that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies (W). Let $\lambda \in \pi_{00}(T)$, i.e. $\lambda \in \text{iso}(T)$ and $0 < \alpha(\lambda I - T) < \infty$. By hypothesis and Lemma 2.3, $0 \neq \lambda \in \pi_{00}(T_n) = \sigma(T_n) \setminus \sigma_w(T_n)$. Then $\alpha(\lambda I - T_n) = \beta(\lambda I - T_n) < \infty$ since $\lambda I - T_n$ is a Weyl operator, and so by Lemma 2.1

$$\alpha(\lambda I - T) = \alpha(\lambda I - T_n) = \beta(\lambda I - T_n) = \beta(\lambda I - T) < \infty.$$  

Furthermore, $\lambda \in \sigma(T)$ because $\lambda \in \sigma(T_n) \subseteq \sigma(T)$. Thus $\lambda I - T$ is Weyl, and hence $\lambda \in \sigma(T) \setminus \sigma_w(T)$. But since $\sigma(T) \setminus \sigma_w(T) \subseteq \pi_{00}(T)$, it then follows that $\pi_{00}(T) = \sigma(T) \setminus \sigma_w(T)$, which implies that $T$ satisfies (W).

(Sufficiency) Suppose that $T$ satisfies (W). Then for $n = 0$, $R(T^0) = X$ is closed and $T_0 = T$ satisfies (W). $\blacksquare$

In the same way as in Theorem 3.1, we have the following characterization of $a$-Weyl theorem for an operator throughout $a$-Weyl theorem for some restriction of the operator.

**Theorem 3.2.** Suppose that $0$ is not an isolated point of $\sigma(T)$. Then $T$ satisfies $(aW)$ if and only if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies $(aW)$.

**Proof.** (Necessity) Suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n$ satisfies $(aW)$. Let $\lambda \in \pi_{00}^a(T)$, by hypothesis and Lemma 2.4, $\lambda \in \pi_{00}^a(T_n) = \sigma_{ap}(T_n) \setminus \sigma_{uw}(T_n)$. Thus $\lambda I - T_n$ is a upper semi-Fredholm operator, because $\lambda I - T_n$ is a upper semi-Weyl operator. Since $\lambda I - T_n$ is upper semi-Fredholm, it follows that $R((\lambda I - T_n)^m)$ is closed in $R(T^n)$ for all $m \in \mathbb{N}$, and so by Lemma 2.2, there exists $k \in \mathbb{N}$ such that $R((\lambda I - T)^k)$ is closed. But since $\alpha((\lambda I - T)^k) < \infty$, then $\alpha((\lambda I - T)^k) < \infty$. That is, $(\lambda I - T)^k$ is a upper semi-Fredholm operator, which implies that $\lambda I - T$ is upper semi-Fredholm. Furthermore, $T$ has SVEP at $\lambda$ because $\lambda \in \text{iso}(\sigma_{ap}(T)$). Consequently, if
\[ \lambda \in \pi^a_{00}(T) \text{ then } \lambda I - T \text{ is upper semi-Fredholm and } p(\lambda I - T) < \infty. \] Hence \( \lambda I - T \) is upper semi-Weyl and \( \lambda \in \sigma_{ap}(T) \), thus \( \lambda \in \sigma_{ap}(T) \setminus \sigma_{uw}(T) \), and we obtain the inclusion \( \pi^a_{00}(T) \subseteq \sigma_{ap}(T) \setminus \sigma_{uw}(T) \). But since \( \sigma_{ap}(T) \setminus \sigma_{uw}(T) \subseteq \pi^a_{00}(T) \), it then follows that \( \pi^a_{00}(T) = \sigma_{ap}(T) \setminus \sigma_{uw}(T) \), which implies that \( T \) satisfies (aW).

(Sufficiency) If \( T \) satisfies (aW). Then for \( n = 0 \), trivially \( R(T^0) = X \) is closed and \( T_0 = T \) satisfies (aW).

Clearly, \( T \) has SVEP at every isolated point of \( \sigma(T) \). Thus, by Theorem 3.1 and Theorem 3.2, we have the following corollary.

**Corollary 3.3.** If \( T \) does not have SVEP at 0, then:

(i) there exists \( n \in \mathbb{N} \) such that \( R(T^n) \) is closed and \( T_n \) satisfies (W) if and only if \( T \) satisfies (W).

(ii) there exists \( n \in \mathbb{N} \) such that \( R(T^n) \) is closed and \( T_n \) satisfies (aW) if and only if \( T \) satisfies (aW).

**Remark 3.4.** There are more alternative ways to express Corollary 3.3. We may replace the assumption \( T \) does not have SVEP at 0 by: \( 0 \notin \partial \sigma(T), p(T) = \infty \) or \( q(T) = \infty \).

**References**


