Abstract: In this paper, we define the multivalued entire series in a Banach algebra $\mathcal{A}$ as well as the exponential, the spectrum and the numerical range of a compact set of $\mathcal{A}$. We provide properties for these two sets which are also verified in the univalued case.

Key words: Banach algebra, Hausdorff distance, spectrum and numerical range.

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1. Introduction

The concept of the exponential of a set has been useful in the study of differential inclusions and Lipschitz selections. Firstly, it was considered (independently) by A. L. Dontchev and E. M. Farkhi [9] in 1989 and P. R. Wolenski [19] in 1990. In 2003, E. O. Ayoola has developed this concept for the study of quantum stochastic differential inclusions [3]. In 2006 and in various ways the extension of multivalued case exponential function was developed in [1], [5] and [6].

At the beginning of this paper, we study the multivalued entire series $S(K) = \sum a_n K^n$ (where $K$ is in $\mathbb{K}(\mathcal{A})$, the set of all compact sets of a Banach algebra $\mathcal{A}$) which is used to define $e^K$.

Then, for $K \in \mathbb{K}(\mathcal{A})$, we define $\sigma(K)$, the spectrum of $K$, as the union of all spectrum $\sigma(a)$ when $a$ runs $K$. If $\mathcal{A} = \mathcal{B}(H)$, i.e., the set of all bounded linear operators on a complex Hilbert space $H$, and $K$ is in $\mathbb{K}(\mathcal{B}(H))$, we define $W(K)$, the numerical range of $K$, as the convex hull of the union of $W(A)$ when $A$ varies over $K$ and

$$W(A) = \{ \langle Ax, x \rangle : \|x\| = 1 \}.$$ 

The last set is called the numerical range of $A$ which is always a convex set of $\mathbb{C}$ whose closure contains the convex hull of $\sigma(A)$ or $\cos(A)$ [14]. In general, in the noncommutative case, the spectrum is not continuous with respect to the
Hausdorff metric [2]. (For more recent work on this topic, see, for example, [18]). We show a range of properties for $\sigma(K)$ and $W(K)$ which are verified in the single valued case, such as continuity of the numerical range in the sense of Hausdorff [8] and the continuity of the spectrum in the case where $\mathcal{A}$ is commutative. We also show for $K \in \mathbb{K}(\mathcal{B}(H))$ that:

$$|K| \leq 2\omega(K) - \frac{\omega^2(K)}{|K|},$$

(1)

where

$$\omega'(K) = \inf \{ \|z\| : z \in W(A), A \in K \},$$

and

$$\omega(K) = \sup \{ \|z\| : z \in W(A), A \in K \},$$

is the $K$ numerical radius. The last inequality is optimal and generalized in the single valued case the following classical inequality [13]:

$$\|A\| \leq 2\omega(A), \ A \in \mathcal{B}(H).$$

As an application of (1) we show that for $K$ and $K'$ in $\mathbb{K}(\mathcal{B}(H))$

$$|KK'| \leq \left( w(K) - \frac{w^2(K)}{2|K|} \right) |K'| + \left( w(K') - \frac{w^2(K')}{2|K'|} \right) |K|.$$

(2)

The previous inequality is an improvement in the single valued case of the following theorem from Dragomir [10]:

**Theorem 1.** ([10]) Let $A, B \in \mathcal{B}(H)$ and $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that for every $x \in H$,

$$\langle (A^* - \tilde{\alpha}I)(\beta I - A)x, x \rangle \geq 0 \text{ and } \langle (B^* - \tilde{\gamma}I)(\lambda I - B)x, x \rangle \geq 0.$$

Then,

$$\|AB\| \leq w(A) \|B\| + w(B) \|A\| + w(A) w(B) + \frac{1}{4} |\beta - \alpha| |\lambda - \gamma|. \quad (3)$$

In [11], and [12], Dragomir said that’s an open problem whether or not the constant $\frac{1}{4}$ is best possible in the inequality (3). The inequality (2) is the solution of this problem.

Dragomir in 2008 [11] showed that

$$\|A\|^2 \leq \omega^2(A) + d^2(A), \ A \in \mathcal{B}(H),$$
with
\[ d(A) = \sup \left\{ \| \langle Ax, y \rangle \| : \| x \| = \| y \| = 1, \langle x, y \rangle = 0 \right\}. \]
We also generalize this result in the set valued case by showing that for \( K, K' \in \mathcal{B}(H) \)
\[ \omega(KK') \leq \omega(K)\omega(K') + d(K)d(K'), \]
where
\[ d(K) = \sup \{ d(A) : A \in K \}. \]
Finally, when
\[ K_1(A) = \{ K \in \mathbb{K}(A) : \forall a, b \in K, \ ab = ba \}, \]
we show the following spectral theorem:

**Theorem 2.** For each \( K \in K_1(A) \), we have
\[ \sigma(S(K)) \subset S(\sigma(K)). \]

2. Definitions and preliminaries

In this paper \( A \) is a Banach algebra over \( \mathbb{C} \), with unit element \( I \). The following definitions are useful in the sequel.

**Definition 3.** Let \( K \) and \( K' \) be two elements of \( \mathbb{K}(A) \) and \( \alpha \) a complex number. We denote
\[
K \cdot K' = \{ x \cdot y : x \in K, y \in K' \}, \\
K + K' = \{ x + y : x \in K, y \in K' \}, \\
\alpha K = \{ \alpha I \} \cdot K = \{ \alpha \cdot x : x \in K \}, \\
\alpha + K = \{ \alpha I \} + K = \{ \alpha I + x : x \in K \}, \\
|K| = \sup_{X \in K} \| X \|, \\
K^0 = \{ I \}, \quad K^n = K \cdot K^{n-1}, \quad \forall n \in \mathbb{N}^*. 
\]
We note that in general \( K \cdot K' \) is not equal to \( K' \cdot K \) and \( K^n = K^p K^q \), with \( p + q = n \) and \( p, q, n \in \mathbb{N} \).

**Definition 4.** Let \( K, K' \in \mathbb{K}(A) \). The Hausdorff distance between \( K \) and \( K' \) denoted by \( h(K, K') \) is the maximum of the excess \( e(K, K') \) and \( e(K', K) \) where
\[ e(K, K') = \sup_{X \in K} \inf_{Y \in K'} \| X - Y \|. \]
**Definition 5.** Let $F$ be a multifunction from $\mathcal{A}$ into $\mathbb{K}(\mathcal{A})$ and let $X_0 \in \mathcal{A}$. $F$ is called Hausdorff upper semicontinuous at $X_0$ ("Hscs" at $X_0$) if for any sequence $(X_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{A}$, which converges to $X_0$, we have

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, F(X_n) \subset F(X_0) + B(0, \epsilon),$$

where $B(0, \epsilon)$ is the open ball in $\mathcal{A}$ with center 0 and radius $\epsilon$.

It follows immediately from (4) that

$$\forall \epsilon > 0, \exists \eta > 0 \text{ such that } \forall X \in B(X_0, \eta), e(F(X), F(X_0)) \leq \epsilon.$$  

(5)

3. Multivalued power series in $\mathcal{A}$

**Definition 6.** Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers, and let $K \in \mathbb{K}(\mathcal{A})$. We set

$$S_n(K) = \sum_{i=0}^{n} a_i K^i = a_0 + a_1 K + a_2 K^2 + \cdots + a_n K^n = \left\{ \sum_{i=0}^{n} a_i x_i : x_i \in K^i \right\}.$$

**Definition 7.** Let $K \in \mathbb{K}(\mathcal{A})$ be such that the sequences $\sum_{i=0}^{n} a_i x_i$ converges for all $x_i \in K^i$. We set

$$S(K) = \left\{ \sum_{n=0}^{+\infty} a_n x_n : x_n \in K^n \right\} = \sum_{i=0}^{\infty} a_n K^n.$$

In the remainder of this section, $K$ denotes an element of $\mathbb{K}(\mathcal{A})$ and $(a_n)_{n \in \mathbb{N}}$ a sequence of complex numbers such that

$$\sum_{n=0}^{+\infty} a_n x_n \text{ converges and } \forall n \in \mathbb{N}, x_n \in K^n.$$

**Theorem 8.** Let $r$ be the radius of convergence of the complex power series $\sum a_n z^n$. If $K \in \mathbb{K}(\mathcal{A})$, with $K \subset B(0, \delta)$ and $0 < \delta < r$, then $S(K)$ is a compact set of $\mathcal{A}$.

**Proof.** Let $(Y_p)_{p \in \mathbb{N}}$ be a sequence of elements of $S(K)$. We show that $(Y_p)_{p \in \mathbb{N}}$ admits a subsequence $(Y_{\varphi(p)})_{p \in \mathbb{N}}$ which converges in $S(K)$. For all $p \in \mathbb{N}$, we have

$$Y_p = \sum_{i=0}^{+\infty} a_i X_{i,p}.$$
with \(X_{i,p} \in K^i\) and \(X_{0,p} = I\). We set

\[Z_p = (a_0X_{0,p}, a_1X_{1,p}, \ldots, a_iX_{i,p}, \ldots) \in \prod_{i=0}^{\infty} a_iK^i.\]

This set is a compact set product. By Tychonov theorem [17], this is a compact set for the norm \(\|\cdot\|_\pi\), where for all \(p\) in \(\mathbb{N}\),

\[\|Z_p\|_\pi = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \min\{1, \|a_iX_{i,p}\|\}.\]

We extract a subsequence \((Z_{\varphi(p)})_{p \in \mathbb{N}}\) which converges to

\[Z = (a_0X_0, a_1X_1, \ldots, a_iX_i, \ldots) \in \prod_{i=0}^{\infty} a_iK^i.\]

Let us show that \((Y_{\varphi(p)})_{p \in \mathbb{N}}\) converges to \(Y = \sum_{i=0}^{\infty} \alpha_iX_i\). Let \(\varepsilon \in ]0, 1[\). The sequence \((Z_{\varphi(p)})_{p \in \mathbb{N}}\) converges to \(Z\), and then, for all \(\varepsilon_1 > 0\), there exists \(p_1 > 0\) such that for all \(p > p_1\),

\[\sum_{n=0}^{+\infty} \frac{1}{2^{n+1}} \min\{1, \|a_nX_{\varphi(p)}(p) - a_nX_n\|\} \leq \varepsilon_1,\]

and then,

\[\frac{1}{2^{n+1}} \min\{1, \|a_nX_{\varphi(p)}(p) - a_nX_n\|\} \leq \varepsilon_1\]

for any \(n \geq 0\). Since \(\delta < r\), \(\sum_{n=0}^{+\infty} |a_n\delta^n|\) is convergent. Thus, there exists \(n_2 > 0\) such that for all \(n \geq n_2\),

\[\sum_{i=n+1}^{+\infty} |a_i\delta^i| \leq \frac{\varepsilon}{3}.\]

Let \(\varepsilon_1 = \frac{1}{3} \frac{1}{2^{n_2+1}} \frac{\varepsilon}{n_2+1}\). Then, there exists \(p_{n_2}\) such that \(\frac{1}{2^{n+1}} > \varepsilon_1\) and

\[\frac{1}{2^{n+1}} \min\{1, \|a_nX_{\varphi(p)}(p) - a_nX_n\|\} = \frac{\|a_nX_{\varphi(p)}(p) - a_nX_n\|}{2^{n+1}} \leq \varepsilon_1,\]

for all \(p > p_{n_2}\) and \(n \leq n_2\). Then, for all \(n \leq n_2\),

\[\|a_nX_{\varphi(p)}(p) - a_nX_n\| \leq \frac{\varepsilon}{3(n_2+1)}.\]
and thus, for all \( p > p_{n_2} \),

\[
\|Y_{\varphi(p)} - Y\| \leq \sum_{n=0}^{n_2} \|a_n X_{n,\varphi(p)} - a_n X_n\| + \sum_{n=n_2+1}^{+\infty} \|a_n X_{n,\varphi(p)}\| + \sum_{n=n_2+1}^{+\infty} \|a_n X_n\|
\]

\[
\leq \sum_{n=0}^{n_2} \|a_n X_{n,\varphi(p)} - a_n X_n\| + \frac{2}{3}\varepsilon
\]

\[
\leq \sum_{n=0}^{n_2} (n_2 + 1) \frac{\varepsilon}{3(n_2 + 1)} + \frac{2}{3}\varepsilon = \varepsilon.
\]

**Definition 9.** Let \( K \in \mathbb{K}(\mathcal{A}) \). We define the set valued exponential of \( K \), denoted \( e^K \), by

\[
e^K = \sum_{n=0}^{+\infty} \frac{1}{n!} K^n = \left\{ \sum_{n=0}^{+\infty} \frac{1}{n!} x_n : \forall n \in \mathbb{N}, \ x_n \in K^n \right\}.
\]

**Remark 10.** Since the radius of convergence of complex series \( \sum z^n \) is infinite, then for every \( K \in \mathbb{K}(\mathcal{A}) \), \( e^K \) is well defined. Using Theorem 8, \( e^K \) is in \( \mathbb{K}(\mathcal{A}) \).

**Theorem 11.** Let \( K \in \mathbb{K}(\mathcal{A}) \), with \( K \subset B(0, \delta) \), \( r \) the radius of convergence of the complex power series \( \sum a_n z^n \) and \( 0 < \delta < r \). Then, the sequence \( S_n(K) \) converges in the sense of Hausdorff to \( S(K) \).

**Proof.** Let \( Y_n \in S_n(K) \) and \( Y \in S(K) \), with \( Y_n = \sum_{i=0}^{n} a_i x_i \), \( Y = Y_n + \sum_{i=n+1}^{\infty} a_i x_i \), and \( x_i \in K^i \) for all \( i \in \mathbb{N} \). We have

\[
\|Y - Y_n\| \leq \sum_{i=n+1}^{\infty} |a_i| \delta^i,
\]

and then

\[
h(S(K), S_n(K)) \leq \sum_{i=n+1}^{\infty} |a_i| \delta^i.
\]

Hence the result.

The following lemma is useful in the proof of Theorem 13.
Lemma 12. Let $\sum_{n} a_{n}z^{n}$ be a complex entire series. Then for any $n \in \mathbb{N}$, the mapping $S_{n}$ from $\mathbb{K}(\mathcal{A})$ to $\mathbb{K}(\mathcal{A})$, which associates to each $K$ the set $S_{n}(K)$, is continuous in the sense of Hausdorff.

Proof. It is easy to see that the product and sum of two compact sets of $\mathcal{A}$ are compact sets. For the continuity of $S_{n}$, it suffices to show that if $(K_{p})_{p \in \mathbb{N}}$ and $(K'_{p})_{p \in \mathbb{N}}$ are two sequences of compact set of $\mathcal{A}$ which converge in the sense of Hausdorff respectively to two compact set $K$ and $K'$ then the sequences $(K_{p}K'_{p})_{p \in \mathbb{N}}$ and $(K_{p} + K'_{p})_{p \in \mathbb{N}}$ converge in the sense of Hausdorff respectively to $KK'$ et $K + K'$.

By the triangle inequality, we have

$$h(K_{p}K'_{p}, KK') \leq |K_{p}| h(K'_{p}, K') + |K'| h(K_{p}, K).$$

The sequence $(K_{p})_{p \in \mathbb{N}}$ is convergent, and therefore $(|K_{p}|)_{p \in \mathbb{N}}$ is bounded from above. As a result, $(K_{p}K'_{p})_{p \in \mathbb{N}}$ converges to $KK'$.

For the other convergence, by triangle inequality, we have

$$h(K_{p} + K'_{p}, K + K') \leq h(K'_{p}, K') + h(K_{p}, K).$$

Theorem 13. Let $r$ be the radius of convergence of the complex entire series $\sum_{n} a_{n}z^{n}$ and $\delta < r$. Then the mapping $S : \mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{A})$, which to $K \subset B(0, \delta)$ associates $S(K)$, is continuous in the sense of Hausdorff.

Proof. Let us consider a sequence $(K_{p})_{p \in \mathbb{N}}$ of compact sets of $\mathcal{A}$ included in $B(0, \delta)$, which converges in the sense of Hausdorff to a compact set $K$. Let us show that $h(S(K_{p}), S(K))$ tends to 0.

The series $\sum_{p} |a_{p}| \delta^{p}$ is convergent, and so the sequence $R_{n} = \sum_{p=n}^{\infty} |a_{p}| \delta^{p}$ tends to 0. Thus, for all $\varepsilon > 0$, there exists $n_{0} \in \mathbb{N}$ such that for all $n \geq n_{0}$,

$$\sum_{p=n}^{\infty} |a_{p}| \delta^{p} \leq \frac{\varepsilon}{3}.$$

Hence

$$h(S(K_{p}), S(K)) \leq h(S(K_{p}), S_{n_{0}}(K_{p})) + h(S_{n_{0}}(K_{p}), S_{n_{0}}(K))$$

$$+ h(S_{n_{0}}(K), S(K)).$$
By Lemma 12, the mapping $S_{n_0}$ is continuous, and so for all $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$,

$$h(S_{n_0}(K_p), S_{n_0}(K)) \leq \frac{\varepsilon}{3}.$$ 

We have

$$h(S(K_p), S_{n_0}(K)) \leq \sum_{p=n}^{\infty} |a_p| \delta^p \leq \frac{\varepsilon}{3}.$$ 

And, similarly, for $h(S_{n_0}(K), S(K))$. Thus, for every $p \geq p_0$, $h(S(K_p), S(K)) \leq \varepsilon$. 

4. Spectrum and numerical range of a compact set

**Definition 14.** Let $K$ be an element of $\mathbb{K}(\mathcal{A})$. We define the spectrum of $K$, denoted $\sigma(K)$, and the algebraic numerical range of $K$, denoted $V(K)$, by:

$$\sigma(K) = \{ \lambda \in \mathbb{C} : \exists X \in K, \lambda \in \sigma(X) \} = \bigcup_{X \in K} \sigma(X)$$

and

$$V(K) = \text{co}\{ \emptyset(t) : \emptyset \in S(\mathcal{A}), t \in K \},$$

respectively, with

$$S(\mathcal{A}) = \{ \emptyset \in \mathcal{A}^* : \emptyset(I) = \|\emptyset\| = 1 \},$$

and $\sigma(X)$ the spectrum of $X$. Therefore, we have

$$V(K) = \text{co} \bigcup_{t \in K} V(t),$$

where

$$V(t) = \{ \emptyset(t) : \emptyset \in S(\mathcal{A}) \}.$$ 

The last set is called the algebraic numerical range of $t$ in the single-valued case, which is always a closed and convex set in $\mathbb{C}$ [16]. It is also located in the disk with center 0 and radius $\|t\|$, and satisfies $V(\mathcal{A}) = W(\mathcal{A})$ for all $\mathcal{A} \in \mathcal{B}(H)$ [4].

**Definition 15.** If $\mathcal{A} = \mathcal{B}(H)$, we define the numerical domain of $K$ by:

$$W(K) = \text{co}\{ \langle Ax, x \rangle : \|x\| = 1, A \in \mathcal{K} \} = \text{co} \bigcup_{A \in \mathcal{K}} W(A).$$
For $K \in \mathbb{K}(\mathcal{A})$, we define the numerical radius of $K$, denoted $\omega(K)$, and the spectral radius of $K$, denoted $\rho(K)$, by:

$$\omega(K) = |V(K)| \quad \text{and} \quad \rho(K) = |\sigma(K)|.$$ 

Similarly, if $\mathcal{A} = \mathcal{B}(H)$, the numerical radius of $K$ is

$$\omega(K) = |W(K)|.$$ 

**Theorem 16.** If $K \in \mathbb{K}(\mathcal{A})$, then $\sigma(K)$ is a compact set in $\mathbb{C}$.

The proof of this theorem is a consequence of Lemma 17 since in the single valued case, the spectrum mapping from $\mathcal{A}$ to $\mathbb{K}(\mathcal{C})$ is Husc [2].

**Lemma 17.** Let $(E, \|\cdot\|)$ be a normed space, $F$ a Husc multifunction from $\mathcal{A}$ into $\mathbb{K}(E)$ and $K$ a compact set of $\mathcal{A}$. Assume that there exists $\alpha > 0$ such that for all $x \in K$, $|F(x)| \leq \alpha \|x\|$. Then, $D = \cup F(x)$ is a closed bounded subset of $E$.

**Proof.** $D$ is bounded since for all $\lambda \in D$ there exists $x \in K$ such that $\lambda \in F(x)$. Thus $\|\lambda\| \leq |F(x)| \leq \alpha |K|$. $D$ is closed since if $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence of elements of $D$ which converges to $\lambda \in E$, then for all $n \in \mathbb{N}$, there exists $x_n \in K$ such that $\lambda_n \in F(x_n)$. Let $(x_{n_k})$ be a subsequence of $(x_n)$ which converges to $\bar{x}$ in $K$. Let us show that $\lambda \in F(\bar{x})$. For this, it suffices to prove that $e(\{\lambda\}, F(\bar{x})) = 0$ since $F(\bar{x})$ is a compact set. Fix $\varepsilon > 0$.

1) Since $\lambda_{n_k} \to \lambda$, then there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$, $\|\lambda - \lambda_{n_k}\| \leq \frac{\varepsilon}{2}$.

2) By the inequality (5) and since $F$ is Hscs at $\bar{x}$, then there exists $\eta > 0$ such that for all $x \in B(\bar{x}, \eta)$, $e(F(x), F(\bar{x})) \leq \frac{\varepsilon}{2}$.

3) Also $x_{n_k} \to \bar{x}$ ensures that there exists $N_1 \in \mathbb{N}$ such that for all $k \geq N_1$, $x_{n_k} \in B(\bar{x}, \eta)$.

Take $k \geq \max(N_0, N_1) = N_2$, and use 1) and 2). We deduce that for all $k \geq N_2$, $e(F(x_{n_k}), F(\bar{x})) \leq \frac{\varepsilon}{2}$, and, consequently, for all $\varepsilon > 0$ and all $k \geq N_2$,

$$e(\{\lambda\}, F(\bar{x})) \leq \|\lambda - \lambda_{n_k}\| + e(\{\lambda_{n_k}\}, F(\bar{x})) \leq \|\lambda - \lambda_{n_k}\| + e(F(x_{n_k}), F(\bar{x})) \leq \varepsilon.$$ 

Thus, $\lambda \in F(\bar{x})$.  \[\blacksquare\]
Definition 18. Let $K \in \mathbb{K}(\mathcal{B}(H))$. We say that $K$ is positive (resp. self adjoint, normal) if each element of $K$ is positive (resp. self adjoint, normal).

In the following Propositions 19 and 20 we show some properties for the spectral mapping and the numerical range of a compact set in $\mathcal{A}$ which are also verified in the case of single valued mappings.

Proposition 19. Consider $K, K' \in \mathbb{K}(\mathcal{A})$ and $\alpha, \beta \in \mathbb{C}$. Then

1) $\sigma(\alpha K + \beta K') \subset \alpha \sigma(K) + \beta \sigma(K')$, if $ab = ba$ for all $(a, b) \in K \times K'$.

2) $V(\alpha K + \beta K') \subset \alpha V(K) + \beta V(K')$.

If $\mathcal{A} = \mathcal{B}(H)$, we further have

3) $W(\alpha K + \beta K') \subset \alpha W(K) + \beta W(K')$.

4) $w(K) = 0 \iff K = \{0\}$.

5) $\text{co} \sigma(K) \subset W(K)$.

6) If $K$ is positive (resp. self adjoint), then $W(K) \subset \mathbb{R}^+$ (resp. $W(K) \subset \mathbb{R}$).

Proof. Since $\sigma(\alpha a + \beta b) \subset \alpha \sigma(a) + \beta \sigma(b)$, $V(\alpha a + \beta b) \subset \alpha V(a) + \beta V(b)$ for $a, b \in \mathcal{A}$, and $W(\alpha A + \beta B) \subset \alpha W(A) + \beta W(B)$ for $A, B \in \mathcal{B}(H)$, then 1), 2) and 3) are fulfilled. Property 4) can be obtained from the fact that if $A \in \mathcal{B}(H)$, then $w(A) \leq \|A\| \leq 2w(A)$ [13]. Thus

$$w(A) = 0 \iff A = 0.$$  

Property 5) is deduced from $\text{co} \sigma(A) \subset W(A)$ if $A \in \mathcal{B}(H)$ [14]. Finally, the last property is trivial. 

Proposition 20. Let $K, K' \in \mathbb{K}(\mathcal{A})$ be such that $ab = ba$ for all $(a, b) \in K \times K'$. Then

1) $\sigma(KK') \subset \sigma(K)\sigma(K')$.

2) If further $\mathcal{A} = \mathcal{B}(H)$ and $K$ or $K'$ is normal, then we have $\overline{W(KK')} \subset \text{co} \overline{W(K)\overline{W(K')}}$.

Proof. 1) is deduced from $\sigma(ab) \subset \sigma(a)\sigma(b)$ if $(a, b) \in K \times K'$ and $ab = ba$. If $A, B \in \mathcal{B}(H)$, $AB = BA$ and $A$ or $B$ is normal, then $\overline{W(AB)} \subset \text{co} \overline{W(A)\overline{W(B)}}$ [7]. Thus, 2).
Example 21. In this example, we have $K = K'$, $KK' = K'K$, but the elements of $K$ do not commute with each other. As a consequence, Proposition 20 is not verified. Indeed, if $K = \{A, B\}$, with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we have $\sigma(KK') = \{0, 1, 3, 4\}$, $\sigma(K)\sigma(K') = \{0, 1, 2, 4\}$. If $x = \frac{1}{\sqrt{2}}$ and $y = \frac{i}{\sqrt{2}}$, then $\langle AB(x), (y) \rangle = \frac{3 + i}{2} \in W(AB) \subset W(KK')$ and $coW(K)W(K') = [0, 4]$.

Definition 22. An operator $A$ in $B(H)$ is said to be convexoid (resp. normaloid, spectraloid) if $W(A) = co\sigma(A)$ (resp. $w(A) = \|A\|$, $|\sigma(A)| = w(A)$).

Definition 23. Let $K \in \mathbb{K}(B(H))$, we say that $K$ is a convexoid (resp. normaloid, spectraloid) if each element of $K$ is a convexoid (resp. normaloid, spectraloid).

The following lemma, whose proof is obvious, is useful to demonstrate Proposition 25.

Lemma 24. Let $(\Gamma_i)_{i \in J}$ be a family of subsets of $\mathbb{C}$ which indexed by a set $J$. We have:

$$co\bigcup_{i \in J} \Gamma_i = \bigcap_{i \in J} co\Gamma_i,$$

and

$$co \bigcup_{i \in J} co\Gamma_i = \bigcup_{i \in J} co\Gamma_i.$$

Proposition 25. Let $K \in \mathbb{K}(B(H))$ be a convexoid (resp. normaloid, spectraloid), then $W(K) = co\sigma(K)$ (resp. $w(K) = \|K\|$, $|\sigma(K)| = w(K)$).

Proof. In this proof we use the three equalities in the previous lemma. We consider only the case where $K$ is a convexoid. The other two cases are obvious. For every $A \in K$, we have $W(A) = co\sigma(A)$. So

$$\bigcup_{A \in K} co\sigma(A) = \bigcup_{A \in K} W(A),$$

and

$$\bigcup_{A \in K} co\sigma(A) = \bigcup_{A \in K} W(A).$$
As a result, we have
\[
\operatorname{co}\bigcup_{A \in K} \operatorname{co}\sigma(A) = \operatorname{co}\bigcup_{A \in K} \operatorname{co}\sigma(A).
\]
This means
\[
\operatorname{co}\bigcup_{A \in K} \operatorname{co}\sigma(A) = \operatorname{co}\bigcup_{A \in K} W(A),
\]
and thus
\[
\operatorname{co}\bigcup_{A \in K} \sigma(A) = \operatorname{co}\bigcup_{A \in K} W(A).
\]
This implies that
\[
\operatorname{co}\sigma(K) = W(K).
\]
By Theorem 16, \(\sigma(K)\) is closed, so it is the same for \(\operatorname{co}\sigma(K)\), and hence the desired equality.

The following theorem shows the continuity of the multifunction \(\overline{W(K)}\) and generalizes the univocal case [8].

**Theorem 26.** Let \(K_n\) be a sequence in \(\mathcal{K}(\mathcal{B}(H))\) which converges in the Hausdorff sense to an element \(K\) of \(\mathcal{K}(\mathcal{B}(H))\), then \(\overline{W(K_n)}\) converges to \(\overline{W(K)}\) in the sense of Hausdorff.

**Proof.** We have
\[
e(K_n, K) = \sup_{x \in K_n} d(x, K) \to 0, \quad \text{with} \quad d(x, K) = e(\{x\}, K).
\]
The continuity of the mapping \(x \mapsto d(x, K)\) and the fact that \(K_n\) and \(K\) are compact set imply the existence of \(x_n \in K_n\) and \(z_n \in K\) such that:
\[
e(K_n, K) = \|x_n - z_n\| \to 0.
\]
We also have
\[
e(\overline{W(K_n)}, \overline{W(K)}) \leq e(\overline{W(K_n)}, \overline{W(z_n)})
\]
\[
= \sup \{d(\alpha_n, \overline{W(z_n)}), \alpha_n \in \overline{W(K_n)}\}
\]
\[
= d(t_n, \overline{W(z_n)}),
\]
with
\[ t_n \in \overline{W(K_n)} = \bigcup_{A \in K_n} \overline{W\{A\}}. \]

Then
\[ e(\overline{W(K_n)}, \overline{W(K)}) \leq e(\overline{W(A)}, \overline{W\{z_n\}}), \]
where
\[ A \in K_n \quad \text{and} \quad t_n \in \overline{W(A)}. \]

And thus
\[ e(\overline{W(K_n)}, \overline{W(K)}) \leq \|A - z_n\| \leq \|y_n - z_n\| \to 0. \]

**Proposition 27.** Let \( K, K' \in \mathcal{K}(\mathcal{A}) \). Suppose that for all \( A \in K \) and \( B \in K' \), \( AB = BA \). Then
\[ h(\sigma(K), \sigma(K')) \leq h(K, K'). \]

**Proof.** The continuity of the norm in \( \mathcal{A} \) and the compactness of \( K \) and \( K' \) provide
\[ e(K, K') = \|y - z\|, \quad y \in K \text{ and } z \in K'. \]
We have
\[ e(\sigma(K), \sigma(K')) \leq e(\sigma(K), \sigma(z)) = e(\sigma(\{t_n\}), \sigma(z)), \]
where \( t_n \in \sigma(K) \). Then, there exists \( A \in K \) such that \( t_n \in \sigma(A) \), and
\[ e(\sigma(K), \sigma(K')) \leq e(\sigma(A), \sigma(z)), \]
\[ \leq \|A - z\| \quad ([2]) \]
\[ \leq \|y - z\| = e(K, K') \]
\[ \leq h(K, K'). \]

The following corollary is satisfied in the univocal case [2, page 49].

**Corollary 28.** Let \( K_n, K \in \mathcal{K}(\mathcal{A}) \) be such that for all \( a_n \in K_n \) and all \( b \in K \), \( a_n b = ba_n \). If the sequence \((K_n)\) converges in the sense of Hausdorff to \( K \), then \( \sigma((K_n)) \) converges in the sense of Hausdorff to \( \sigma(K) \).
Definition 29. For $K \in \mathbb{K}(\mathcal{B}(H))$ we set
\[ O(K) = \{ (Ax, y) : A \in K, \|x\| = \|y\| = 1, \langle x, y \rangle = 0 \} \]
and
\[ d(K) = \sup_{z \in O(K)} |z| = |O(K)|. \]

Proposition 30. $O(K)$ is a disk centered at the origin and with radius $d(K)$.

Proof. For all $A \in K$, $O(\{A\})$ is a disk centered at the origin and with radius $d(\{A\}) = \sup_{z \in O(\{A\})} |z|$, [8]. We have
\[ O(K) = \bigcup_{A \in K} O(\{A\}) \text{ and } d(K) \leq |K|. \]
Then $O(K)$ is a disk centered at the origin and with radius $d(K)$. \[\square\]

Proposition 31. For $K \in \mathbb{K}(\mathcal{B}(H))$, we have
\[ d(K) = \inf_{\lambda \in \mathbb{C}} |K - \lambda I|. \]

Proof. Since
\[ d(\{A\}) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\| \leq \inf_{\lambda \in \mathbb{C}} |K - \lambda I|, \]
then
\[ d(K) = \sup_{A \in K} d(\{A\}) \leq \inf_{\lambda \in \mathbb{C}} |K - \lambda I|. \]
For the reverse, we have that for all $\lambda \in \mathbb{C}$ and all $A \in K$,
\[ |K - \lambda I| \geq \|A - \lambda I\|, \]
and then, for all $A \in K$,
\[ \inf_{\lambda \in \mathbb{C}} |K - \lambda I| \geq d(A). \]
Thus
\[ d(K) \leq \inf_{\lambda \in \mathbb{C}} |K - \lambda I|. \] \[\square\]
Proposition 32. For $K \in \mathcal{K}(\mathcal{B}(H))$ we have

$$|K| \leq 2w(K) - \frac{w'(K)}{|K|},$$

where

$$w'(K) = \inf \{|z| \in W(A) : A \in K\}.$$

Proof. Remark that

$$Ax = \langle Ax, x \rangle x + \langle Ax, y \rangle y,$$

with $\langle x, y \rangle = 0$,

then

$$\langle Ax, Ax \rangle = \langle Ax, x \rangle \langle x, Ax \rangle + \langle Ax, y \rangle \langle y, Ax \rangle = |\langle Ax, x \rangle|^2 + |\langle Ax, y \rangle|^2.$$

The product operator $M_{2,A,B}$ defined on the Hilbert-Schmidt space $C_2(H)$, fitted with the scalar product

$$\langle X, Y \rangle = \text{tr}XY,$$

is given by

$M_{2,A,B}(X) = AXB, \ A, B \in \mathcal{B}(H)$,

and satisfies [15]

$$w(M_{2,A,B}) \leq w(A) \|B\|.$$

Set

$$X = \frac{\sqrt{2}}{2} x \otimes x + \frac{\sqrt{2}}{2} y \otimes y.$$

Then the norm of $X$ in $C_2(H)$ is equal to 1. Then we have

$$\langle M_{2,A,A}(X), X \rangle = \frac{1}{2} |\langle Ax, x \rangle|^2 + \frac{1}{2} |\langle Ax, y \rangle|^2 + \frac{1}{2} |\langle Ay, x \rangle|^2 + \frac{1}{2} |\langle Ay, y \rangle|^2$$

$$= \frac{1}{2} \|Ax\|^2 + \frac{1}{2} |\langle Ay, y \rangle|^2 + \frac{1}{2} |\langle Ax, x \rangle|^2$$

$$\leq w(A) \|A\|.$$

Thus

$$\|Ax\|^2 \leq 2w(A) \|A\| - |\langle Ay, y \rangle|^2,$$
and
\[ \|A\|^2 \leq 2w(A)\|A\| - w^2(A). \]

We conclude
\[ \|A\| \leq 2w(A) - \frac{w^2(A)}{\|A\|}, \]
and
\[ \sup_{A \in K} \|A\| \leq 2 \sup_{A \in K} w(A) - \frac{\inf_{A \in K} w^2(A)}{\sup_{A \in K} \|A\|}, \]
that is to say
\[ |K| \leq 2w(K) - \frac{w^2(K)}{|K|}. \] (6)

In the single valued case the inequality (6) generalizes the following inequality [13]:
\[ \|A\| \leq 2w(A). \] (7)

**Corollary 33.** If \( w'(K) \neq 0 \), then
\[ |K| < 2w(K). \]

In the following example we have equality in (6) but not in (7): let \( r > 0 \), then for \( K = \{re^{i\theta} I : \theta \in [0, 2\pi]\} \) we have \(|K| = r = w(A) = w'(A)\).

**Proposition 34.** For \( K, K' \in K(\mathcal{B}(H)) \) we have
\[ |KK'| \leq \left( w(K) - \frac{w^2(K)}{2|K|} \right) |K'| + \left( w(K') - \frac{w^2(K')}{2|K'|} \right) |K|. \]

**Proof.** By (6) we have \( \frac{1}{2}|K| \leq w(K) - \frac{w^2(K)}{2|K|} \) and \( \frac{1}{2}|K'| \leq w(K') - \frac{w^2(K')}{2|K'|} \). On the other hand, we have \( |KK'| \leq |K||K'| \), hence the desired inequality. \( \blacksquare \)

**Proposition 35.** Let \( K, K' \in K(\mathcal{B}(H)) \). Then
\[ W(KK') \subset I_{K,K'} + O(K)O(K'), \]
and
\[ w(KK') \leq w(K)w(K') + d(K)d(K'), \] (8)
where
\[ I_{K,K'} = \{ \langle Ax, x \rangle \langle Bx, x \rangle : \|x\| = 1, A \in K, B \in K' \}. \]
Proof. Let $x \in H$ be such that $\|x\| = 1$. Then, $Bx = \langle Bx, x \rangle x + \langle Bx, y \rangle y$, with $\|y\| = 1$ and $\langle x, y \rangle = 0$, and thus,

$$\langle ABx, x \rangle = \langle Bx, x \rangle \langle Ax, x \rangle + \langle Bx, y \rangle \langle Ay, x \rangle,$$

and the result follows.

Remark 36. If in the inequality (8) $K$ and $K'$ are, respectively, replaced by $A^*$ and $A$ we obtain the following inequality due to Dragomir [11]:

$$\|A\|^2 \leq w^2(A) + d^2(A).$$

Proposition 37. Let $K$ be an element of $\mathbb{K}_1(A)$, and let $P$ be the polynomial with complex coefficients defined by

$$P(X) = \sum_{i=0}^{n} a_i X^i = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n.$$

Then

$$\sigma(P(K)) \subset P(\sigma(K)).$$

If further $A = B(H)$ and $K$ is normal, then

$$W(P(K)) \subset \text{co } P(W(K)).$$

Proof. It suffices to use (1) and (3) of Propositions 19 and 20, respectively.

Finally we end with the following spectral theorem:

Theorem 38. Let $K$ be an element of $\mathbb{K}_1(A)$, then

$$\sigma(S(K)) \subset S(\sigma(K)).$$

If further, $A = B(H)$ and $K$ is normal, then

$$W(S(K)) \subset \text{co } S(W(K)).$$

Proof. Firstly, we prove (9). For this, let $\lambda \in \sigma(S(K))$ and verify $\lambda \in S(\sigma(K))$. There exists $A \in S(K)$ such that $A - \lambda I$ is not invertible. That is to say, $A = \sum_{i=0}^{\infty} a_i x_i$, $x_i \in K$ and $\lambda \in \sigma(A)$. However, $A = \lim A_n$ with $A_n = \sum_{i=0}^{n} a_i x_i$, $x_i \in K$ and $A_n \in S_n(K)$. Then $A_n A_p = A_p A_n$, for all $n, p \in \mathbb{N}$, $h(\sigma(A), \sigma(A_n)) \rightarrow 0$ [2]. We have

$$e(\{\lambda\}, \sigma(A_n)) \leq h(\sigma(A), \sigma(A_n)) \rightarrow 0.$$
Therefore \( e(\{\lambda\}, \sigma(A_n)) = \|\lambda - \lambda_n\| \), where \( \lambda_n \in \sigma(A_n) \) and \( \lambda = \lim \lambda_n \). Thus, 
\[
\lambda_n \in \sigma(A_n) \subset \sigma(S_n(K)) \subset S_n(\sigma(K)).
\]
The last inclusion is due to Proposition 37. Therefore, 
\[
e(\{\lambda\}, S(\sigma(K))) \leq e(\{\lambda\}, \{\lambda_n\}) + e(\{\lambda_n\}, S_n(\sigma(K))) + e(S_n(\sigma(K)), S(\sigma(K))).
\]
By Theorem 11, we have 
\[
e(S_n(\sigma(K)), S(\sigma(K))) \to 0.
\]
In addition, 
\[
e(\{\lambda\}, \{\lambda_n\}) = \|\lambda - \lambda_n\| \to 0,
\]
and 
\[
e(\{\lambda_n\}, S_n(\sigma(K))) = 0, \text{ since } \lambda_n \in S_n(\sigma(K)).
\]
So \( \lambda \in S(\sigma(K)) = S(\sigma(K)) \). The last equality follows from Theorem 8. Inclusion (10) is the same as (9) by replacing the multifunction \( \sigma(K) \) by the multifunction \( W(K) \), with values in \( \mathbb{K}(\mathbb{C}) \).

References

[8] M.K. CHRAÏBI, Domaine numérique de l’opérateur produit \( M_{2,A,B} \) et de la dérivation généralisée \( \delta_{2,A,B} \), *Extracta Math.* 17 (1) (2002), 59–68.


