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A Function on Exponential Convergence in a Fréchet Metric Space

D.K. Ganguly, Bablu Biswas

Department of Pure Mathematics, 35, *Ballygunge Circular road, Calcutta*-700019, *India*
gangulydk@yahoo.co.in, bablubiswas100@yahoo.com

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Abstract: This paper deals with some fundamental properties of a function defined on exponential convergence connecting with a monotonic increasing divergent sequence in a Fréchet metric space.

Key words: Borel classification of sets, first category, Baire class of sets, Lebesgue Measure and dense set.

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1. Introduction

In the paper [1] the author investigated some properties on the exponential convergence of all real non-decreasing sequences. Being inspired by this paper we consider a positive non-decreasing sequence $\{a_n\}_n$ with $\lim_{n\to\infty} a_n = +\infty$ and we denote $A = \{a_n\}_n$. Let $\mathbf{S}$ be the collection of all the infinite subsequences of A. We consider $\mathbf{S}$ as a metric space endowed with the Fréchet metric $d(x, y)$ given by,

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

where $x = \{x_k\}$, $y = \{y_k\} \in \mathbf{S}$. The convergence in this space is considered as point-wise convergence. It is well known from the monograph [2] that for any monotonic increasing divergent sequence $A = \{a_n\}_n$, $a_n > 0$ there exists a unique real number $\lambda \geq 0$ such that

$$\sum_{n=1}^{\infty} {a_n}^{-\sigma} = +\infty, \quad \text{for each } \sigma < \lambda$$
$$\sum_{n=1}^{\infty} {a_n}^{-\sigma} < +\infty, \quad \text{for each } \sigma > \lambda,$$

where $\sigma > 0$ is a real number.

Here the number $\lambda = \lambda(A)$ is called the exponent of convergence of the sequence A. It is formulated by

$$\inf\left\{\sigma > 0 : \sum_{n=1}^{\infty} {a_n}^{-\sigma} < +\infty\right\} = \lambda.$$

It is known [3] that

$$\lambda(A) = \lim_{n\to\infty}\sup \frac{\log n}{\log a_n}\cdot$$

We now consider a function $\lambda : \mathbf{S} \to [0, \lambda(A)]$ defined as

$$\lambda(x) = \lambda(A(x)) = \inf\left\{\sigma > 0 : \sum_{k=1}^{\infty} a_{n_k}^{-\sigma} < +\infty\right\},$$

where $x = \{a_{n_k}\}_{k=1}^{\infty} \in \mathbf{S}$. It is clear that $\lambda(x) = \lambda(A(x)) \le \lambda(A)$ for every $x \in \mathbf{S}$.

2. Some set theoretic properties of the function λ.

Theorem 2.1. *The function $\lambda : \boldsymbol{S} \to [0, \lambda(A)]$ is onto but not one to one.*

Proof. Case 1: Let $t = 0$. We can choose $x = \{a_{k_n}\}$ in $\mathbf{S}$ such that $a_{k_n} > n^n$ for each natural number n. Then $\lambda(x) = t$.

Case 2: For $t = \lambda(A)$ clearly we choose $x = \{a_n\}$ so that $\lambda(x) = t$.

Case 3: Let $t \in (0, \lambda(A))$. We have

$$\lim_{k\to\infty}\sup \frac{\log k}{\log\left(a_{l+k}\right)} = \lim_{k\to\infty}\sup \frac{\log(l+k)}{\log\left(a_{l+k}\right)} \cdot \frac{\log k}{\log(l+k)} = \lambda(A) > t, \qquad (1)$$

for any natural number l.

Now we can choose an integer $P_1 \ge 1$ such that $a_{P_1+2} > 1$ and $\frac{\log 2}{\log(a_{P_1+2})} < t$. By the result (1) we have

$$\lim_{k\to\infty}\sup \frac{\log k}{\log\left(a_{P_1+k}\right)} > t.$$

Then there exists a least positive integer $P_2 > 2$ such that $\frac{\log P_2}{\log\left(a_{P_1+P_2}\right)} \ge t$. So for each n with $2 \le n < P_2$ we have $\frac{\log n}{\log\left(a_{P_1+n}\right)} < t$. Again choose $P_3 >$

$\max\{P_1, P_2\}$ so that $\frac{\log(P_2+1)}{\log(a_{P_2+P_3+1})} < t$. Result (1) implies that

$$\limsup_{k\to\infty} \frac{\log k}{\log(a_{P_3+k})} > t.$$

Then there exists a least positive integer $P_4 > P_3$ such that $\frac{\log P_4}{\log(a_{P_3+P_4})} \geq t$. Then for $P_2 < n < P_4$ we have $\frac{\log n}{\log(a_{P_3+n})} < t$. Proceeding this way we construct a sequence $\{P_n\}$ of natural numbers such that

$$\frac{\log P_{2i}}{\log(a_{P_{2i-1}+P_{2i}})} \geq t, \quad i = 1, 2, 3, \dots$$

and

$$\frac{\log n}{\log(a_{P_{2i-1}+n})} < t,$$

for $P_{2i-2} < n < P_{2i}$, $i = 2, 3, 4, \dots$ and $P_{2i+1} > \max\{P_1, P_2, \dots, P_{2i}\}$. Now for $i \geq 2$,

$$\begin{aligned} 0 &\leq \frac{\log P_{2i}}{\log(a_{P_{2i-1}+P_{2i}})} - \frac{\log(P_{2i}-1)}{\log(a_{P_{2i-1}+P_{2i}-1})} \\ &\leq \frac{\log P_{2i}}{\log(a_{P_{2i-1}+P_{2i}})} - \frac{\log(P_{2i}-1)}{\log(a_{P_{2i-1}+P_{2i}})} \\ &= \frac{\log \frac{P_{2i}}{P_{2i}-1}}{\log(a_{P_{2i-1}+P_{2i}})} \to 0 \text{ as } i \to \infty. \end{aligned}$$

Then clearly,

$$\limsup_{i\to\infty} \frac{\log P_{2i}}{\log(a_{P_{2i-1}+P_{2i}})} = \limsup_{i\to\infty} \frac{\log(P_{2i}-1)}{\log(a_{P_{2i-1}+P_{2i}-1})} = t.$$

So if we choose $x = \{a_{k_n}\}$ where,

$$\begin{aligned} &k_1 = P_1,\ k_2 = P_1 + 2,\ k_3 = P_1 + 3,\ \dots,\ k_{P_2-1} = P_1 + P_2 - 1, \\ &k_{P_2} = P_1 + P_2,\ k_{P_2+1} = P_2 + P_3 + 1,\ \dots,\ k_{P_4-1} = P_3 + P_4 - 1, \\ &\dots \\ &k_{P_{2i}} = P_{2i-1} + P_{2i},\ k_{P_{2i}+1} = P_{2i} + P_{2i+1} + 1,\ \dots \end{aligned}$$

then we have

$$\limsup_{n\to\infty} \frac{\log n}{\log(a_{k_n})} = t, \text{ i.e., } \lambda(x) = t.$$

We now show that λ is not one to one.

Let $a \in [0, \lambda(A)]$. Then there exists $x = \{a_{n_k}\}_{k=1}^{\infty} \in \mathbf{S}$ such that $\lambda(x) = a$, i.e.,

$$a = \inf\{\sigma > 0 : \sum_{k=1}^{\infty} a_{n_k}^{-\sigma} < +\infty\}.$$

Let $y_k = a_{n_{k+1}}$, for $k = 1, 2, 3, \dots$, then $y = \{y_k\}_{k=1}^{\infty} \in \mathbf{S}$. Clearly

$$\inf\Big\{\sigma > 0 : \sum_{k=1}^{\infty} {y_k}^{-\sigma} < +\infty\Big\} = a, \text{ i.e., } \lambda(y) = a.$$

So $\lambda(x) = \lambda(y)$ when $x \neq y$. Therefore λ is not one to one. ∎

We are interested about the measurability of the function λ. For this purpose here we shall study some properties in terms of Borel classification and Baire category of the level sets of λ defined as follows:

$$K_t = \{x \in \mathbf{S}\colon \lambda(x) \leq t\}; \quad K^t = \{x \in \mathbf{S}\colon \lambda(x) > t\},$$

for $t \in \mathbf{R} = (-\infty, \infty)$.

Theorem 2.2. *The set* $K_t = \{x \in \mathbf{S}\colon \lambda(x) \leq t\}$

(i) *belongs to the second multiplicative Borel class for each* $t \in (-\infty, \infty)$.

(ii) *is dense in* $\mathbf{S}$ *for* $0 \leq t \leq \lambda(A)$.

(iii) *is of first category for* $t < \lambda(A)$.

Proof. (i) If $t < 0$, then $K_t = \phi$ and K_t belongs to the second multiplicative Borel class. Let $t \geq 0$. Then

$$\begin{aligned} K_t &= \{x = \{x_k\} = \{a_{n_k}\} \in \mathbf{S}\colon \lambda(x) \leq t\} \\ &= \bigcap_{m=1}^{\infty} \Big\{x \in \mathbf{S}\colon \sum_{k=1}^{\infty} a_{n_k}^{-(t+\frac{1}{m})} < +\infty\Big\} \\ &= \bigcap_{m=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcap_{r=1}^{\infty} S(m, i, p, r), \end{aligned}$$

where

$$S(m, i, p, r) = \Big\{x \in \mathbf{S}\colon \sum_{k=i}^{i+p} a_{n_k}^{-(t+\frac{1}{m})} \leq \frac{1}{r}\Big\}.$$

Let $x^{(r)} = \{x_k{}^{(r)}\}_{k=1}^{\infty} \in S(m,i,p,r)$ and $\lim_{r\to\infty} x^{(r)} = x$. It is clear that

$$\lim_{r\to\infty} \{x_k{}^{(r)}\}^{-(t+\frac{1}{m})} = x_k^{-(t+\frac{1}{m})}$$

for each $k = i, i+1, i+2, \dots, i+p$ whence $x \in S(m,i,p,r)$. Consequently each set $S(m,i,p,r)$ is closed. This proves that K_t is an $F_{\sigma\delta}$ set. Hence the set K_t belongs to the second multiplicative Borel class.

(ii) We show that K_t is dense in $\mathbf{S}$ for $0 \le t \le \lambda(A)$. We have

$$\begin{aligned} K_t &= \bigcap_{m=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcap_{r=1}^{\infty} S(m,i,p,r) \\ &= \bigcap_{m=1}^{\infty} F(m), \end{aligned}$$

where

$$F(m) = \Big\{ x \in \mathbf{S} : \exists_{i=1}^{\infty} \forall_{p=1}^{\infty} \forall_{r=1}^{\infty} \sum_{k=i}^{i+p} a_{n_k}^{-(t+\frac{1}{m})} \le \frac{1}{r} \Big\}.$$

Let $y = \{a_{p_k}\} \in \mathbf{S}$ and let $\varepsilon > 0$. Consider the open ball $S(y,\varepsilon)$ with centre at y and ε as the radius. Let l be the smallest positive integer such that $\sum_{i=l+1}^{\infty} 1/2^i < \varepsilon$. For $t \ge 0$ let s_1 be the least positive integer such that $2^{\frac{s_1}{t+1}} > a_{p_l}$. Now choose the least positive integer q_1 such that $a_{q_1} > 2^{\frac{s_1}{t+1}}$, $a_{q_1} \in A$. Again let s_2 be the least positive integer such that $2^{\frac{s_2}{t+1}} > a_{q_1}$ and further we can choose the least positive integer q_2 with $a_{q_2} > 2^{\frac{s_2}{t+1}}$, $a_{q_2} \in A$ and proceeding this way we have two subsequences $\{s_k\}$ and $\{q_k\}$ of natural numbers such that

$$a_{q_k} > 2^{\frac{s_k}{t+1}}, \quad k = 1,2,3,\dots$$

Consider the sequence $z = \{z_k\}_{k=1}^{\infty}$ as follows:

$$z_i = a_{p_i},\ i = 1,2,\dots,l\,; \quad z_{l+k} = a_{q_k},\ k = 1,2,3,\dots$$

Then it can be verified that $d(y,z) < \varepsilon$ and hence $z \in S(y,\varepsilon)$. Again

$$\begin{aligned}\sum_{k=1}^{\infty} z_k{}^{-(t+\frac{1}{m})} &\le \sum_{k=1}^{l} a_{p_k}{}^{-(t+\frac{1}{m})} + \sum_{k=l+1}^{\infty} 2^{\frac{-s_k(t+\frac{1}{m})}{t+1}} \\ &= \sum_{k=1}^{l} a_{p_k}{}^{-(t+\frac{1}{m})} + \sum_{k=l+1}^{\infty} (\frac{1}{2^{\alpha}})^{s_k}, \ \alpha = \frac{t+\frac{1}{m}}{t+1} \\ &\le \sum_{k=1}^{l} a_{p_k}{}^{-(t+\frac{1}{m})} + \sum_{k=l+1}^{\infty} (\frac{1}{2^{\alpha}})^{k} \\ &< \infty, \text{ since } 0 < \alpha \le 1.\end{aligned}$$

Clearly $z \in F(m)$ and then $z \in F(m) \cap S(y,\varepsilon)$. Therefore, $F(m)$ is dense in $\mathbf{S}$ and consequently K_t is dense in $\mathbf{S}$.

(iii) If $t < \lambda(A)$ we show that the set K_t is of first category.

Case 1: If $t < 0$, then $K_t = \phi$ and so K_t is of first category.

Case 2: Let $0 \le t < \lambda(A)$. Since $t < \lambda(A)$, then there exists a natural number m_0 such that $t + \frac{1}{m} < \lambda(A)$ for all $m \ge m_0$. So we have

$$K_t = \bigcup_{m=m_0}^{\infty} F(m,r),$$

where

$$F(m,r) = \Big\{ x \in \mathbf{S} : \exists_{i=1}^{\infty} \forall_{p=1}^{\infty} \sum_{k=i}^{i+p} a_{n_k}^{-(t+\frac{1}{m})} \le \frac{1}{r} \Big\}.$$

In order to show that $F(m,r)$ is of first category in $\mathbf{S}$, it is sufficient to show that $F(m,r)$ is an F_σ set and its complement is dense in $\mathbf{S}$. Let $y = \{a_{p_k}\} \in \mathbf{S}$ and let $\varepsilon > 0$. Consider the open ball $S(y,\varepsilon)$ with centre at y and ε as the the radius. Let l be the smallest positive integer such that $\sum_{i=l+1}^{\infty} 1/2^i < \varepsilon$. Now we choose the smallest positive integer s so that $a_s > a_{p_l}$. Define a sequence $u = \{u_k\}$ in $\mathbf{S}$ as follows:

$$u_i = a_{p_i}, \ i = 1,2,\ldots,l\,; \ u_{l+k} = a_{s+k}, \ k = 1,2,3,\ldots$$

It is clear that $u \in S(y,\varepsilon)$ and for every positive integer i there exist integer p such that

$$\sum_{k=i+1}^{i+p} u_k{}^{-(t+\frac{1}{m})} > \frac{1}{r},$$

since the series

$$\sum_{k=1}^{\infty} {a_k}^{-(t+\frac{1}{m})}$$

is divergent for $(t+\frac{1}{m}) < \lambda(A)$. Thus the complement of $F(m,r)$ is dense in $\mathbf{S}$. Also each set $S(m,i,p,r)$ is closed and hence

$$F(m,r) = \bigcup_{i=1}^{\infty} \bigcap_{p=1}^{\infty} S(m,i,p,r)$$

is an F_σ set. Since every set $F(m,r)$ is of first category hence

$$K_t = \bigcup_{m=m_0}^{\infty} \bigcap_{r=1}^{\infty} F(m,r)$$

is of first category in $\mathbf{S}$. ∎

Corollary 2.3. *For each* $t \in \mathbf{R}$ *the set* $K^t = \{x \in \mathbf{S} \colon \lambda(x) > t\}$ *belongs to the second additive Borel class.*

Proof. It follows from the fact that $K^t = \mathbf{S} - K_t$ is a $G_{\delta\sigma}$ set for each $t \in \mathbf{R}$. ∎

Corollary 2.4. *The function* λ *is Lebesgue measurable on* $\boldsymbol{S}$

Proof. Here $\mathbf{S}$ is a subset of $[a_1, \infty)^{\mathbf{N}}$. Using Fubini's theorem we have from theorem 2.2 that λ is Lebesgue measurable. ∎

Theorem 2.5. *The function* λ *is discontinuous everywhere in* $\boldsymbol{S}$.

Proof. Let $b \in \mathbf{S}$ and $b = \{a_{p_1}, a_{p_2}, a_{p_3}, \ldots\}$. We can choose a sequence $c = \{a_{q_k}\}_{k=1}^{\infty} \in \mathbf{S}$ such that $\lambda(b) \neq \lambda(c)$. Let $\delta > 0$. It is sufficient to show that there exists a point z in the open ball $S(b,\delta)$ such that $\lambda(z) = \lambda(c)$. For $\delta > 0$ let l be the smallest positive integer such that $\sum_{i=l+1}^{\infty} 1/2^i < \delta$. Now we consider the sequence $z = \{z_k\}_{k=1}^{\infty} \in \mathbf{S}$ as follows:

$$z_k = \begin{cases} a_{p_k} & \text{for } k = 1,2,3,\ldots,l \\ a_{q_k} & \text{for } k > l. \end{cases}$$

Then clearly $z \in S(b, \delta)$ and

$$\begin{aligned}\lambda(z) &= \inf\Big\{\sigma > 0 : (\sum_{i=1}^{l} a_{p_i}^{-\sigma} + \sum_{i=l+1}^{\infty} a_{q_i}^{-\sigma}) < \infty\Big\}\\ &= \inf\Big\{\sigma > 0 : \sum_{i=1}^{\infty} a_{q_i}^{-\sigma} + (\sum_{i=1}^{l} a_{p_i}^{-\sigma} - \sum_{i=1}^{l} a_{q_i}^{-\sigma}) < \infty\Big\}\\ &= \inf\Big\{\sigma > 0 : \sum_{i=1}^{\infty} a_{q_i}^{-\sigma} < \infty\Big\}; \text{(since the sum in the first bracket is finite)}\\ &= \lambda(c).\end{aligned}$$

Hence λ is discontinuous everywhere in $\mathbf{S}$. ∎

COROLLARY 2.6. *The function λ is not a Darboux function.*

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