Virtually \((r; r_1, \ldots, r_n; s)\)-nuclear multilinear operators

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**Abstract:** In this paper, the space of virtually \((r; r_1, \ldots, r_n; s)\)-nuclear multilinear operators between Banach spaces is introduced, some of its properties are described and its topological dual is characterized as a Banach space of multiple absolutely \((r'; r'_1, \ldots, r'_n; s')\)-summing multilinear operators.

**Key words:** Multilinear operators, nuclear operators, summing operators.

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**1. Introduction**

The nuclear operators between Banach spaces appeared in [5] when the author studied an infinite dimensional extension of the Malgrange theorem on existence and approximation of solutions for convolution equations (see also [7]). The concept of nuclear multilinear operators was extended and studied in [8]. For other related results we mention [9] and [10]. Matos [9] studied virtually \((r; r_1, \ldots, r_n)\)-nuclear \(n\)-linear operators from \(X_1 \times \cdots \times X_n\) into \(Y\), and proved that, if the spaces \(X_k^*\)'s \((k = 1, \ldots, n)\) have the \(\lambda_k\)-bounded approximation property; then for \(r, r_1, \ldots, r_n \in [1, +\infty]\) the topological dual of the space of these operators, endowed with a natural linear topology, is isomorphic isometrically to the space of all absolutely \((r', r'_1, \ldots, r'_n)\)-summing operators from \(X_1^* \times \cdots \times X_n^*\) into \(Y^*\) with \(1/r + 1/r = 1\) and \(1/r_k + 1/r'_k = 1\) for \(r, r_k\) and \(s \in [1, +\infty]\), \(k = 1, \ldots, n\).

In [3] Cerna established the definition of \((r; r_1, \ldots, r_n; s)\)-nuclear multilinear operators, which are the natural generalization of the concept of \((r, p, s)\)-nuclear linear operator introduced by Lapresté [6] (see also [11])

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Motivated by these ideas and developments, in this paper we introduce and study the virtually $(r; r_1, \ldots, r_n; s)$-nuclear $n$-linear operators and we will prove a relation between the topological dual of virtually $(r; r_1, \ldots, r_n; s)$-nuclear $n$-linear operators and the multiple $(r'; r'_1, \ldots, r'_n; s')$-summing operators [2]. As a consequence we get the same result between the topological dual of the space of $(r; r_1, \ldots, r_n; s)$-nuclear $n$-linear operators from $X_1 \times \cdots \times X_n$ into $Y$ [4] and to the space of all absolutely $(r', r'_1, \ldots, r'_n, s')$-summing operators from $X_1^* \times \cdots \times X_n^*$ into $Y^*$ [1], for $r, r_k$ and $s \in [1, +\infty]$, $k = 1, \ldots, n$.

The definitions and notations used in this paper are, in general, standard. Let $n \in \mathbb{N}$. As usual, an element $j$ from $\mathbb{N}^n$ will be represented by $(j_1, \ldots, j_n)$ with $j_k \in \mathbb{N}$ and $k = 1, \ldots, n$. We also consider the finite families $(y_j)_{j \in \mathbb{N}^m}$ of elements of a Banach space with $\mathbb{N}_m = \{1, \ldots, m\}$. If $n = 1$, we omit $\mathbb{N}^n$ in the preceding notations. Let $X_1, \ldots, X_n; Y$ be Banach spaces over $\mathbb{K}$ (either $\mathbb{C}$ or $\mathbb{R}$). The space of all continuous $n$-linear operators $T : X_1 \times \cdots \times X_n \to Y$ will be denoted by $L(X_1, \ldots, X_n; Y)$. It becomes a Banach space with the natural norm

$$\|T\| = \sup_{\|x^k\| \leq 1, \ k = 1, \ldots, n} \|T(x^1, \ldots, x^n)\|.$$

We recall that a $n$-linear mappings $T \in L(X_1, \ldots, X_n; Y)$ is said to be of finite type if it has a finite representation of the form

$$T = \sum_{i=1}^m \lambda_i \varphi_i^1 \times \cdots \times \varphi_i^n b_i,$$

where $\lambda_i \in \mathbb{K}$, $\varphi_i^k \in X_k^*$, $k = 1, \ldots, n$, $b_i \in Y$, $i = 1, \ldots, m$. We denote by $L_f(X_1, \ldots, X_n; Y)$ the vector subspace of $L(X_1, \ldots, X_n; Y)$ of all $n$-linear mappings of finite type.

If $r \in [0, +\infty[$, we denote by $l_r(Y; \mathbb{N}^n)$ or $(l_r(\mathbb{N}^n); \text{if } Y = \mathbb{K})$, the vector space of all families $(y_j)_{j \in \mathbb{N}^n}$ of elements of $Y$ such that

$$\left\| (y_j)_{j \in \mathbb{N}^n} \right\|_r = \left( \sum_{j \in \mathbb{N}^n} \|y_j\|^r_Y \right)^{1/r} < \infty.$$

We observe that $\|\cdot\|_r$ is a norm (r-norm, if $r < 1$) on $l_r(Y; \mathbb{N}^n)$ and defines a complete metrizable linear topology on it. We denote by $l_\infty(Y; \mathbb{N}^n)$ (or $l_\infty(\mathbb{N}^n)$, if $Y = \mathbb{K}$) the Banach space of all bounded families $(y_j)_{j \in \mathbb{N}^n}$ of
elements of $Y$, with the norm

$$\left\| (y_j)_{j \in \mathbb{N}^n} \right\|_\infty = \sup_{j \in \mathbb{N}^n} \|y_j\|.$$ 

The Banach subspace of all families $(y_j)_{j \in \mathbb{N}^n}$ such that

$$\lim_{j_k \to +\infty, k=1,\ldots,n} \|y_j\| = 0$$

is denoted by $c_0(Y;\mathbb{N}^n)$ (or $c_0(\mathbb{N}^n)$, if $Y = \mathbb{K}$).

If $0 < s \leq \infty$, we will write $l^w_s(Y;\mathbb{N}^n)$ (or $l^w_s(\mathbb{N}^n)$, if $Y = \mathbb{K}$) for the vector space of all families $(y_j)_{j \in \mathbb{N}^n}$ of elements of $Y$ such that

$$\left\| (y_j)_{j \in \mathbb{N}^n} \right\|_{w,s} := \sup_{\|\psi\|_{Y^*} \leq 1} \left( \sum_{j \in \mathbb{N}^n} |\psi(y_j)|^s \right)^{\frac{1}{s}} = \sup_{\|\psi\|_{Y^*} \leq 1} \left\| (\psi(y_j))_{j \in \mathbb{N}^n} \right\|_s < \infty,$$

where $Y^*$ denotes the topological dual of $Y$.

It is well-known that for $1 \leq s < \infty$ and $(\varphi_j)_{j \in \mathbb{N}^n} \in l^w_s(Y^*;\mathbb{N}^m)$, we have

$$\left\| (\varphi_j)_{j \in \mathbb{N}^n} \right\|_{w,s} = \sup_{\varphi \in B_{Y^*}} \left( \sum_{j \in \mathbb{N}^n} |\phi(\varphi_j)|^s \right)^{\frac{1}{s}} = \sup_{y \in B_Y} \left\| (\varphi_j(y))_{j \in \mathbb{N}^n} \right\|_s.$$

Let $0 < r, \ 1 < p, \ s \leq \infty$ such that

$$\frac{1}{t} = \frac{1}{r} + \frac{1}{p} + \frac{1}{s}, \text{ with } t \in ]0,1].$$

An operator $T \in \mathcal{L}(X;Y)$ is $(r;p;s)$-nuclear (see, e.g., [6, 11]) if it has a representation of the form

$$T = \sum_{i=1}^\infty \lambda_i x_i \otimes y_i,$$ \hspace{1cm} (1)

with $(\lambda_i)_i \in l_r$, if $r < \infty$ (or $(\lambda_i)_i \in c_0$, if $r = +\infty$), $(x_i)_i \in l^w_p(X^*)$ and $(y_i)_i \in l^w_s(Y)$. The vector space of all such operators is denoted by $\mathcal{N}_{(r;p;s)}(X;Y)$ and it is a complete metrizable topological vector space under the $t$-norm

$$\mu_{(r;p;s)}(T) = \inf \left\{ \| (\lambda_i)_i \|_r \| (x_i)_i \|_{w,p} \| (y_i)_i \|_{w,s} \right\}$$

where the infimum is taken over all representations of $T$ as in (1).
The definition of the virtually \((r; r_1, \ldots, r_n)\)-nuclear operators below was first given in [9].

We consider \(r \in [0, +\infty], r_k \in [1, +\infty]\), such that \(r \leq r_k, k = 1, \ldots, n\) and

\[
1 \leq \frac{1}{t_n} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n}.
\]

**Definition 1.1.** An operator \(T \in \mathcal{L}(X_1, \ldots, X_n; Y)\) is said to be virtually \((r; r_1, \ldots, r_n)\)-nuclear if there is a representation of the form

\[
T = \sum_{j \in \mathbb{N}^n} \lambda_j \phi_{j_1}^1 \times \cdots \times \phi_{j_n}^n b_j
\]

(2)

with \((\lambda_j)_{j \in \mathbb{N}^n} \in l_r(\mathbb{N}^n)\), if \(r < \infty\) (or \((\lambda_j)_{j \in \mathbb{N}^n} \in c_0(\mathbb{N}^n)\), if \(r = +\infty\),

\[
(\phi_{i_k}^k)_{i_k=1}^{\infty} \in \ell_{r_k}^0(X_k^s), \quad \text{for} \quad k = 1, \ldots, n
\]

and \((b_j)_{j \in \mathbb{N}^n} \in l_\infty(Y; \mathbb{N}^n)\).

The vector space of these operators is denoted by \(\mathcal{L}_{VN}(X_1, \ldots, X_n; Y)\) and we consider on it the \(t_n\)-norm

\[
\|T\|_{VN,(r;r_1,\ldots,r_n)} = \inf \left\{ \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_r \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{\infty} \prod_{k=1}^n \left\| (\phi_{i_k}^k)_{i_k=1}^{\infty} \right\|_{w,r_k}^{t_k} \right\},
\]

where the infimum is taken over all representations of \(T\) as in (2).

The notion of absolutely \((r; r_1, \ldots, r_n; s)\)-summing multilinear operators was introduced by the first author in [1].

**Definition 1.2.** For \(0 < r, r_1, \ldots, r_n < \infty\) and \(0 < s \leq \infty\) with \(\frac{1}{r} \leq \frac{1}{r_1} + \cdots + \frac{1}{r_n} + \frac{1}{s}\), an \(n\)-linear operator \(T \in \mathcal{L}(X_1, \ldots, X_n; Y)\) is \((r; r_1, \ldots, r_n; s)\)-summing if there is a constant \(C > 0\) such that for any \(x_1^k, \ldots, x_m^k \in X_k, \quad (1 \leq k \leq n)\), and any \(\varphi_1, \ldots, \varphi_m \in Y^*\), we have

\[
\left( \sum_{i=1}^m \left| \varphi_i \left( T \left( x_1^i, \ldots, x_n^i \right) \right) \right|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^n \left\| \left( x_i^k \right)_{i=1}^m \right\|_{w,r_k} \left\| \left( \varphi_i \right)_{i=1}^m \right\|_{w,s}.
\]

We denote the vector space of these operators by \(\mathcal{L}_{as,(r;r_1,\ldots,r_n;s)}(X_1, \ldots, X_n; Y)\) and the smallest \(C\) satisfying the above inequality by \(\pi_{rn,(r_1,\ldots,r_n,s)}^n(T)\) which defines a norm \((r\text{-}norm \text{ if } r < 1)\) on \(\mathcal{L}_{as,(r;r_1,\ldots,r_n;s)}(X_1, \ldots, X_n; Y)\).

The following multilinear generalization of \((r; r_1, \ldots, r_n; s)\)-summing operators was recently introduced by Bernardino et al. in [2].
DEFINITION 1.3. Let $n \in \mathbb{N}$, $r, s, r_1, \ldots, r_n \geq 1$ and $X_1, \ldots, X_n, Y$ be Banach spaces. A continuous multilinear operator $T : X_1 \times \cdots \times X_n \rightarrow Y$ is multiple $(r; r_1, \ldots, r_n; s)$-summing if there is a $C > 0$ such that

$$
\left( \sum_{j \in \mathbb{N}_m} \left| \varphi_j \left( T \left( x_{j,1}, \ldots, x_{j,n} \right) \right) \right|^r \right)^{\frac{1}{r}} \leq C \left\| \left( \varphi_j \right)_{j \in \mathbb{N}_m} \right\|_{w,s} \prod_{k=1}^{n} \left\| x_{k}^k \right\|_{w,r_k}^{m_k}
$$

where $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_n} + \frac{1}{s}$. $x_1^k, \ldots, x_m^k \in X_k$, $k = 1, \ldots, n$ and $(\varphi_j)_{j \in \mathbb{N}_m} \in l^w_s \left( Y^*; \mathbb{N}_m \right)$.

We denote by $L^{(r;r_1,\ldots,r_n;s)}_{\text{mas}} \left( X_1, \ldots, X_n; Y \right)$ the vector space of these operators. The smallest $C$ satisfying the above inequality defines a norm ($r$-norm if $r < 1$) on $L^{(r;r_1,\ldots,r_n;s)}_{\text{mas}} \left( X_1, \ldots, X_n; Y \right)$; it is denoted by $\| T \|_{\text{mas}(r;r_1,\ldots,r_n;s)}$.

Remark 1.4. By choosing $(s = \infty)$ in Definition 1.3, we obtain the definition of fully (or multiple) $(r; r_1, \ldots, r_n)$-summing $n$-linear operators presented in [9].

We also need the definition of the $(r; r_1, \ldots, r_n; s)$-nuclear $n$-linear operators. The ideal of $(r; r_1, \ldots, r_n; s)$-nuclear operators was introduced by Cerna [3] (see also [4]).

DEFINITION 1.5. For $0 < r \leq \infty$, $1 \leq s$, $r_1, \ldots, r_n \leq \infty$, such that $1 \leq \frac{1}{r} + \frac{1}{r_1} + \cdots + \frac{1}{r_n} + \frac{1}{s}$, $T \in L \left( X_1, \ldots, X_n; Y \right)$ is called $(r; r_1, \ldots, r_n; s)$-nuclear if it has the form

$$
T = \sum_{i=1}^{+\infty} \lambda_i \phi_i^1 \times \cdots \times \phi_i^n b_i,
$$

with $(\lambda_i)_{i \in \mathbb{N}} \in l_r \left( \mathbb{N} \right)$, if $r < \infty$ (or $(\lambda_i)_{i \in \mathbb{N}} \in c_0 \left( \mathbb{N} \right)$, if $r = +\infty$), $(\phi_i^k)_{i \in \mathbb{N}} \in l^w_{r_i} \left( X_k \right)$ for $k = 1, \ldots, n$ and $(b_i)_{i \in \mathbb{N}} \in l^w_{s} \left( Y \right)$. The set of $(r; r_1, \ldots, r_n; s)$-nuclear operators satisfying the definition is a vector space and is denoted by $N^{(r;r_1,\ldots,r_n;s)} \left( X_1, \ldots, X_n; Y \right)$. Considering that

$$
N^{(r;r_1,\ldots,r_n;s)} \left( T \right) = \inf \left\{ \left\| (\lambda_i)_{i \in \mathbb{N}} \right\|_r \left\| (b_i)_{i \in \mathbb{N}} \right\|_{w,s} \prod_{k=1}^{n} \left\| \phi_i^k \right\|_{w,r_k}^{m_k} \right\},
$$

where the infimum is taken over all possible representations of $T$ described in (3), we obtain a $t$-norm with

$$
\frac{1}{t} = \frac{1}{r} + \frac{1}{r_1} + \cdots + \frac{1}{r_n} + \frac{1}{s'}.
$$
2. Virtually \((r; r_1, \ldots, r_n; s)-nuclear\) \(n\)-linear operators

We consider \(r \in [0, +\infty], s, r_k \in [1, +\infty], k = 1, \ldots, n\), such that \(1 \leq \frac{1}{r_n} = \frac{1}{r} + \frac{1}{r_1} + \cdots + \frac{1}{r_n} + \frac{1}{s} = k \in \mathbb{N}\).

**Definition 2.1.** An operator \(T \in \mathcal{L}(X_1, \ldots, X_n; Y)\) is said to be virtually \((r; r_1, \ldots, r_n; s)-nuclear\) if there are \((\lambda_j)_{j \in \mathbb{N}} \in l_r(\mathbb{N}^n)\), if \(r < \infty\) (or \((\lambda_j)_{j \in \mathbb{N}} \in c_0(\mathbb{N}^n)\), if \(r = +\infty\), \((\phi^k_i)_{i=1}^\infty \in l_r^w(X^*_k)\), for \(k = 1, \ldots, n\) and \((b_j)_{j \in \mathbb{N}} \in l_r^w(Y; \mathbb{N}^m)\) such that

\[
T = \sum_{j \in \mathbb{N}^n} \lambda_j \phi^1_{j_1} \times \cdots \times \phi^n_{j_n} b_j. \tag{4}
\]

We denote the vector space of all such operators by \(\mathcal{L}^{(r; r_1, \ldots, r_n; s)}_{VN}(X_1, \ldots, X_n; Y)\), with the \(t_n\)-norm

\[
\|T\|_{VN,(r; r_1, \ldots, r_n; s)} = \inf \left\| (\lambda_j)_{j \in \mathbb{N}} \right\|_r \left\| (b_j)_{j \in \mathbb{N}} \right\|_{w,s'} \prod_{k=1}^n \left\| (\phi^k_i)_{i=1}^\infty \right\|_{w,s'_k},
\]

where the infimum is taken over all representations of \(T\) as in (4). This \(t_n\)-normed space is a complete metrizable topological vector space.

**Remarks 2.2.** (a) By choosing \(s' = \infty\) in Definition 2.1, we obtain virtually \((r; r_1, \ldots, r_n)-nuclear\) \(n\)-linear operators presented in Definition 1.1.

(b) We have \(N_{(r; r_1, \ldots, r_n; s)}(X_1, \ldots, X_n; Y) \subset \mathcal{L}^{(r; r_1, \ldots, r_n; s)}_{VN}(X_1, \ldots, X_n; Y)\) and

\[
\|T\| \leq \|T\|_{VN,(r; r_1, \ldots, r_n; s)} \leq N_{(r; r_1, \ldots, r_n; s)}(T),
\]

for every \(T\) in \(N_{(r; r_1, \ldots, r_n; s)}(X_1, \ldots, X_n; Y)\).

By definition every \(T\) in \(\mathcal{L}_f(X_1, \ldots, X_n; Y)\) has a finite representation

\[
T = \sum_{j \in \mathbb{N}^{n_m}} \lambda_j \phi^1_{j_1} \times \cdots \times \phi^n_{j_n} b_j. \tag{5}
\]

It is clear that we have a \(t_n\)-norm on \(\mathcal{L}_f(X_1, \ldots, X_n; Y)\) defined by

\[
\|T\|_{VN_f,(r; r_1, \ldots, r_n; s)} = \inf \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_r \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'} \prod_{k=1}^n \left\| (\phi^k_i)_{i=1}^m \right\|_{w,s'_k},
\]

where the infimum is taken over all finite representations of \(T\) as in (5).
The next result collects some elementary facts about virtually $(r; r_1, \ldots, r_n; s)$-nuclear $n$-linear operators.

**Proposition 2.3.** (i) The vector space $\mathcal{L}_f(X_1, \ldots, X_n; Y)$ of the continuous $n$-linear operators of finite type is dense in $\mathcal{L}^{(r;r_1,\ldots,r_n;s)}(X_1, \ldots, X_n; Y)$.

(ii) Ideal property: If $E_1, \ldots, E_n$, and $Y_0$ are Banach spaces and $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$, $S_k \in \mathcal{L}(E_k, X_k)$, $k = 1, \ldots, n$, and $R \in \mathcal{L}(Y, Y_0)$ with $T$ virtually $(r; r_1, \ldots, r_n; s)$-nuclear, then $R \circ T \circ (S_1, \ldots, S_n)$ is virtually $(r; r_1, \ldots, r_n; s)$-nuclear and

$$\|R \circ T \circ (S_1, \ldots, S_n)\|_{\mathcal{L}^{(r;r_1,\ldots,r_n,s)}(X_1, \ldots, X_n; Y)} \leq \|R\| \|T\|_{\mathcal{L}^{(r;r_1,\ldots,r_n,s)}(Y; Y_0)} \prod_{k=1}^{n} \|S_k\|.$$

(iii) $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is virtually $(r; r_1, \ldots, r_n; s)$-nuclear if and only there are bounded linear operators $A_k \in \mathcal{L}(X_k; l_{r_k}^1)$, $k = 1, \ldots, n$, $B \in \mathcal{L}(l_1^r(N^n); Y)$ and $(\lambda_j)_{j \in N^n} \in l_r^r(N^n)$, if $r < \infty$ (or $(\lambda_j)_{j \in N^n} \in c_0(N^n)$, if $r = +\infty$), such that

$$T = B \circ D(\lambda_j)_{j \in N^n} \circ (A_1, \ldots, A_n),$$

where $D(\lambda_j)_{j \in N^n} : l_{r_1}^1 \times \cdots \times l_{r_n}^1 \rightarrow l_1^r(N^n)$ defined by $D(\lambda_j)_{j \in N^n}(\xi_1^1, \xi_2^1, \ldots) = (\lambda_j^1 \xi_{j_1}^1 \cdots \xi_{j_n}^n)_{j \in N^n}$ for $(\xi_j^1)_{j_1=1}^\infty \in l_{r_1}^1$, is a virtually $(r; r_1, \ldots, r_n; s)$-nuclear with

$$\|D(\lambda_j)_{j \in N^n}\|_{\mathcal{L}^{(r;r_1,\ldots,r_n,s)}(l_1^r(N^n); Y)} = \|\lambda_j\|_{l_r^r(N^n)}.$$ 

In this case

$$\|T\|_{\mathcal{L}^{(r;r_1,\ldots,r_n,s)}(X_1, \ldots, X_n; Y)} = \inf \|B\| \|\lambda_j\|_{l_r^r(N^n)} \prod_{k=1}^{n} \|A_k\|,$$

where the infimum is taken over all such factorizations.

3. **Duality**

The natural question is to find out when we have

$$\|T\|_{\mathcal{L}^{(r;r_1,\ldots,r_n,s)}(X_1, \ldots, X_n; Y)} = \|T\|_{\mathcal{L}^{(r;r_1,\ldots,r_n,s)}(Y; Y_0)},$$

for each $T \in \mathcal{L}_f(X_1, \ldots, X_n; Y)$.
Of course we have

\[ \|T\|_{VN,(r;r_1,\ldots,r_n:s)} \leq \|T\|_{VN_f,(r;r_1,\ldots,r_n:s)} . \]

Below we will see that the reverse implication holds to be true for some certain Banach spaces \(X_k\)'s \((k = 1, \ldots, n)\). We start with finite dimensional spaces \(X_k\)'s. The following theorem can be proved as in [9, Proposition 4.6].

**Theorem 3.1.** If the spaces \(X_k\) (\(k = 1, \ldots, n\)) are finite dimensional vector spaces, then

\[ \|T\|_{VN_f,(r;r_1,\ldots,r_n:s)} \leq \|T\|_{VN,(r;r_1,\ldots,r_n:s)} , \]

for every \(T \in L_f(X_1, \ldots, X_n; Y)\).

As in [9, Proposition 4.8], we get the following, which extends Theorem 3.1 to infinite dimensional Banach spaces with the \(\lambda\)-bounded approximation property (\(\lambda\)-BAP, for short).

**Proposition 3.2.** If the spaces \(X_k^*\)'s \((k = 1, \ldots, n)\) have the \(\lambda_k\)-BAP, then

\[ \|T\|_{VN,(r;r_1,\ldots,r_n:s)} \geq \|T\|_{VN_f,(r;r_1,\ldots,r_n:s)} , \]

for all \(T \in L_f(X_1, \ldots, X_n; Y)\).

**Proof.** We consider \(T_k \in L(X_k; L(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n; Y))\), defined by

\[ T_k(x^k)(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n) = T(x^1, \ldots, x^{k-1}, x^k, x^{k+1}, \ldots, x^n) , \]

for \(x^k \in X_k\), \(k = 1, \ldots, n\).

Since \(X_k^*\) has the \(\lambda_k\)-bounded approximation property for some \(\lambda_k > 0\), given \(\epsilon > 0\), we can find \(S_k \in L_f(D_k, X_k)\), such that \(T_k = T_k \circ S_k\) and \(\|S_k\| \leq (1 + \epsilon) \lambda_k\). Hence, for all \(x^k \in X_k\), for \(k = 1, \ldots, n\), we have

\[ T(x^1, \ldots, x^{k-1}, S_k(x^k), x^{k+1}, \ldots, x^n) = T(x^1, \ldots, x^{k-1}, x^k, x^{k+1}, \ldots, x^n) . \]

Now, we can write

\[ T(x^1, \ldots, x^n) = T \circ (S_1, \ldots, S_n)(x^1, \ldots, x^n) , \quad \forall x^k \in X_k\) \(, k = 1, \ldots, n\).

If \(J_k\) denotes the natural injection from \(S_k(D_k)\) into \(X_k\), we can write \(S_k = J_k \circ \hat{S}_k\) \((\hat{S}_k \in L_f(D_k, S_k(D_k)))\), with \(\|\hat{S}_k\| = \|S_k\|.\) Therefore we can
say that $T \circ (J_1, \ldots, J_n) \in \mathcal{L}_f ((S_1 (D_1), \ldots, S_n (D_n)); Y)$. By Theorem 3.1 and Proposition 2.3 (ii) we have

$$
\|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} = \|T \circ (S_1, \ldots, S_n)\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} \\
\leq \|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} \prod_{k=1}^{n} \|S_k\| \\
\leq \|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} (1 + \epsilon)^{n} \prod_{k=1}^{n} \lambda_k.
$$

This implies that

$$
\|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} \leq \left( \prod_{k=1}^{n} \lambda_k \right) \|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)}.
$$

For each $\epsilon > 0$, we choose a representation

$$
T = \sum_{j \in \mathbb{N}^n} \sigma_j \phi_{j_1}^{1} \times \cdots \times \phi_{j_n}^{n} y_j
$$

such that

$$
\left\| (\sigma_j)_{j \in \mathbb{N}^n} \left\| ((y_j)_{j \in \mathbb{N}^n} \left\| \prod_{k=1}^{n} \left( \phi_k^{i} \right)_{i=1}^{\infty} \right\|_{w, r'_k} \right\|_{w, r_k} \right\|_{w, r'_k} \leq (1 + \epsilon) \|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)}.
$$

We can find $m \in \mathbb{N}$ such that

$$
\left( \prod_{k=1}^{n} \lambda_k \right) \left\| \sum_{j \in \mathbb{N}^n / \mathbb{N}^n_m} \sigma_j \phi_{j_1}^{1} \times \cdots \times \phi_{j_n}^{n} y_j \right\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} \leq \epsilon \|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)}.
$$

We use the triangular inequality for $t_n$-norms in order to write

$$
\left( \|T\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} \right)^{t_n} \leq \left( \left\| \sum_{j \in \mathbb{N}^n_m} \sigma_j \phi_{j_1}^{1} \times \cdots \times \phi_{j_n}^{n} y_j \right\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} \right)^{t_n} \\
+ \left( \left\| \sum_{j \in \mathbb{N}^n / \mathbb{N}^n_m} \sigma_j \phi_{j_1}^{1} \times \cdots \times \phi_{j_n}^{n} y_j \right\|_{\mathcal{V}N_f, (r; r_1, \ldots, r_n; s)} \right)^{t_n}.
$$
\[
\leq (1 + \epsilon)^{t_n} \left( \| T \|_{V_{N_1}(r_1, \ldots, r_n; s)} \right)^{t_n} \\
+ \left( \prod_{k=1}^{n} \lambda_k \right)^{t_n} \left( \left\| \sum_{j \in N_m} \sigma_j \phi_{j_1}^1 \times \cdots \times \phi_{j_n}^n (y_j) \right\|_{V_{N_1}(r_1, \ldots, r_n; s)} \right)^{t_n} \\
\leq \left[ (1 + \epsilon)^{t_n} + \epsilon^{t_n} \right] \left( \| T \|_{V_{N_1}(r_1, \ldots, r_n; s)} \right)^{t_n} .
\]

Since \( \epsilon > 0 \) is arbitrary we have

\[
\| T \|_{V_{N_1}(r_1, \ldots, r_n; s)} \leq \| T \|_{V_{N_1}(r_1, \ldots, r_n; s)} ,
\]

and this proves the theorem. \( \square \)

For Banach spaces with \( \lambda \)-bounded approximation property, Proposition 3.2 can be seen as a generalization of a result obtained by B. Cerna [4, Lemma 2.1].

Now, we also give another generalization of [4, Lemma 2.1].

**Proposition 3.3.** Let \( T : X_1 \times \cdots \times X_n \to L_s(\Omega, \mu) \) be defined by

\[
T (x^1, \ldots, x^n) = \sum_{j \in N_m} \lambda_j \phi_{j_1}^1 (x^1) \cdots \phi_{j_n}^n (x^n) b_j ,
\]

where \( \frac{1}{s} = \frac{1}{r_1} + \cdots + \frac{1}{r_n} \). Then, \( \| T \|_{V_{N_1}(\infty; r_1, \ldots, r_n; s)} = \| T \|_{V_{N_1}(\infty; r_1, \ldots, r_n; s)} = \| T \| . \)

**Proof.** It is clear that for \( \frac{1}{s} = \frac{1}{r_1} + \cdots + \frac{1}{r_n} \), we have

\[
\| T \| \leq \| T \|_{V_{N_1}(\infty; r_1, \ldots, r_n; s)} \leq \| T \|_{V_{N_1}(\infty; r_1, \ldots, r_n; s)} .
\]

Moreover,

\[
\| T \| \| x^1 \| \cdots \| x^n \| \geq \left( \int_{\Omega} \left| \sum_{j \in N_m} \lambda_j \phi_{j_1}^1 (x^1) \cdots \phi_{j_n}^n (x^n) b_j (t) \right|^s d\mu(t) \right)^{1/s} .
\]

Since \( \phi_{j_i}^i \) is surjective there exists \( \xi_i \in X_i \) such that \( \phi_{j_i}^i (\xi_i) = M_i / 2^{j_i / r_i} \), where

\[
M_i = \sup_{\| x^i \|_{X_i} \leq 1} \left( \sum_{j_i=1}^{m} |\langle \phi_{j_i}^i, x^i \rangle|^r_i \right)^{1/r_i} ,
\]
We will show that $\|\xi_i\| \leq 1$ and $M_i < +\infty$ for $i = 1, \ldots, n$. From the definition of $M_i$ for a fixed $i$ and for $\epsilon > 0$ we have

$$M_i \|\xi_i\| < (1 + \epsilon) \left( \sum_{j_i=1}^{m_i} M_i^{r_i} / 2^{j_i} \right)^{1/r_i'},$$

which implies that $\|\xi_i\| < (1 + \epsilon)$, for all $\epsilon > 0$.

So, considering $\|\xi_i\| < 1$ in equation (6) we have

$$\|T\| \geq \left( \int_{\Omega} \left| \sum_{j \in \mathbb{N}_m^n} \frac{\lambda_j M_j / 2^{j/r_j} \cdots M_n / 2^{j_n/r_n} b_j(t)}{\prod_{i=1}^n M_i} \right|^s d\mu(t) \right)^{1/s},$$

if $k = \max \{j_1, \ldots, j_n\}$ we get

$$\|T\| \geq \left( \int_{\Omega} \left| \sum_{j \in \mathbb{N}_m^n} \frac{\lambda_j b_j(t)}{2^{j/k}} \right|^s d\mu(t) \right)^{1/s} \prod_{i=1}^n M_i. \quad (7)$$

Let $z(t) = \sum_{j \in \mathbb{N}_m^n} \frac{\lambda_j b_j(t)}{2^{j/k}}$, then for all $s \geq 1$ we have

$$|\langle \varphi, z \rangle| = \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j \left\langle \varphi, \frac{b_j}{2^{j/k}} \right\rangle \right| \leq \|\varphi\| \|z\|. \quad (8)$$

By renumbering multi-finite indices $j \in \mathbb{N}_m^n$, we can rewrite this finite sum as

$$z(t) = \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/k}}.$$

In addition, let $M = \text{span}_{k \in \{1, \ldots, f(m,n)\} - k_0} \left\{ \frac{b_k}{2^{k/k}} \right\}$ where $k_0$ is a fixed number belongs to $\{1, \ldots, f(m,n)\}$, and $f(m,n) \in \mathbb{N}$. Moreover, as a consequence of the Hahn-Banach theorem there exists $\varphi$ such that $\|\varphi\| = \frac{1}{d}$, $\langle \varphi, x \rangle = 0$ for all $x \in M$ and $\left\langle \varphi, \frac{b_{k_0}}{2^{k_0/k}} \right\rangle = 1$, where $d = \inf_{x \in M} \|x - \frac{b_{k_0}}{2^{k_0/k}}\|$ and further one can choose $\lambda_{k_0}$ such that
\[ |\lambda_{k_0}| = \max_{k=1,\ldots,f(m,n)} |\lambda_k| = \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty \text{; where } j = (j_1, \ldots, j_n). \]

Taking into account these last relations in equation (8) we can get,

\[ \|z\| \geq |\lambda_{k_0}| d. \tag{9} \]

Since \( x = \sum_{k=1, k \neq k_0}^{f(m,n)} \frac{b_k}{2^{k/s}} \in M \), then for a given \( \epsilon > 0 \), we have

\[ (1 + \epsilon) d > \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\|. \]

Therefore, from (9) we get

\[ (1 + \epsilon) \|z\| > \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\|. \tag{10} \]

We know that

\[ \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'} = \sup_{\|\psi\|_s \leq 1} \left( \sum_{j \in \mathbb{N}_m^n} |\psi(b_j)|^{s'} \right)^{1/s'} = \sup_{a \in B_{l^f(m,n)}} \left\| \sum_{k=1}^{f(m,n)} a_k b_k \right\|, \]

and since \( a_k = \frac{1}{2^{k/s}} \) for \( k = 1, \ldots, f(m,n) \), given \( \epsilon' > 0 \) we have

\[ (1 + \epsilon') \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\| \geq \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'}. \]

From the last relation and the equation (10) we obtain

\[ (1 + \epsilon) (1 + \epsilon') \|z\| > \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'} \text{ for all } \epsilon \text{ and } \epsilon' > 0. \tag{11} \]

Therefore, from the relations (7) and (11) we get

\[ \|T\| \geq \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'} \prod_{i=1}^{n} M_i \]

\[ \geq \|T\|_{VN_{f(\infty; r_1, \ldots, r_n; s)}}. \]
We will prove a new link between the topological dual of virtually $(r; r_1, \ldots, r_n; s)$-nuclear $n$-linear operators and multiple $(r'; r'_1, \ldots, r'_n; s')$-summing operators. The proof of the next theorem is similar to the proof of Theorem 7.3.1 in [10]. We included the detailed proof here for completeness.

**Theorem 3.4.** If the spaces $X_k^*$'s ($k = 1, \ldots, n$) have the $\lambda_k$- BAP, then the topological dual of $\mathcal{L}_{VN}^{(r; r_1, \ldots, r_n; s)} (X_1, \ldots, X_n; Y)$ is isomorphic isometrically to $\mathcal{L}_{\text{mas}}^{(r'; r'_1, \ldots, r'_n; s')} (X_1^*, \ldots, X_n^*, Y^*)$, for $r, r_k \in [1, +\infty]$, $k = 1, \ldots, n$ through the mapping $\mathcal{B}$ defined by

$$\mathcal{B} (\Psi) (\phi^1, \ldots, \phi^n) (b) = \Psi (\phi^1 \times \cdots \times \phi^n b),$$

for all $b \in Y$, $\phi_k \in X_k^*$, $k = 1, \ldots, n$ and $\Psi \in \mathcal{L}_{VN}^{(r; r_1, \ldots, r_n; s)} (X_1, \ldots, X_n; Y)^*$. 

**Proof.** It is easy to see that the correspondence

$$\Psi \in \mathcal{L}_{VN}^{(r; r_1, \ldots, r_n; s)} (X_1, \ldots, X_n; Y)^* \rightarrow \mathcal{B} (\Psi) \in \mathcal{L}_{\text{mas}}^{(r'; r'_1, \ldots, r'_n; s')} (X_1^*, \ldots, X_n^*, Y^*)$$

defined by

$$\mathcal{B} (\Psi) (\phi^1, \ldots, \phi^n) (b) = \Psi (\phi^1 \times \cdots \times \phi^n b), \ \phi_k \in X_k^*, \ k = 1, \ldots, n \text{ and } b \in Y,$

is linear and injective. To show the surjectivity let $T \in \mathcal{L}_{\text{mas}}^{(r'; r'_1, \ldots, r'_n; s')} (X_1^*, \ldots, X_n^*, Y^*)$ and consider the linear functional $\Psi_T$ on the space $(\mathcal{L}_f (X_1, \ldots, X_n; Y) ; \|\cdot\|_{\text{VN}, (r; r_1, \ldots, r_n; s)})$ given by

$$\Psi_T (S) = \sum_{j \in \mathbb{N}_n} \lambda_j T (\varphi^1_j, \ldots, \varphi^n_j) (b_j)$$

for every $S \in \mathcal{L}_f (X_1, \ldots, X_n; Y)$ with a finite representation of the form

$$S = \sum_{j \in \mathbb{N}_n} \lambda_j \varphi^1_j \times \cdots \times \varphi^n_j b_j.$$

Hence, by Hölder’s inequality and Definition 1.3 it follows that

$$|\Psi_T (S)| \leq \left\| (\lambda_j)_{j \in \mathbb{N}_n} \right\|_r \left\| (T (\varphi^1_j, \ldots, \varphi^n_j) (b_j))_{j \in \mathbb{N}_n} \right\|_{r'},$$

$$\leq \|T\|_{\text{mas}}^{(r'; r'_1, \ldots, r'_n; s')} \left\| (\lambda_j)_{j \in \mathbb{N}_n} \right\|_r \prod_{k=1}^n \left\| (\varphi^k_j)_{i=1}^m \right\|_{w, r'_k} \| b_j \|_{\mathbb{N}_n} \| w, s',$$
This shows that
\[ |\Psi_T(S)| \leq \|T\|_{\text{max}(r',r_1',\ldots,r_n';s')} \|S\|_{\mathcal{L}(r;r_1,\ldots,r_n;s)} \]
for all \( S \in \mathcal{L}_f(X_1,\ldots,X_n;Y) \).

Since on \( \mathcal{L}_f(X_1,\ldots,X_n;Y) \), under our hypothesis for \( X_1,\ldots,X_n \), we have
\[ \|\|\|_{\mathcal{L}_f(r;r_1,\ldots,r_n;s)} = \|\|\|_{\mathcal{L}(r;r_1,\ldots,r_n;s)} \]
we conclude that \( \Psi_T \) is continuous on \( \mathcal{L}_f(X_1,\ldots,X_n;Y) \) for \( \|\|\|_{\mathcal{L}(r;r_1,\ldots,r_n;s)} \)
and
\[ \|\Psi_T\| \leq \|T\|_{\text{max}(r';r_1',\ldots,r_n';s')} \].

By Proposition 2.3 (i), \( \mathcal{L}_f(X_1,\ldots,X_n;Y) \) is dense in \( \mathcal{L}^{(r,r_1,\ldots,r_n;s)}_{VN} \) \( (X_1,\ldots,X_n;Y) \). Hence we can extend \( \Psi_T \) to a continuous functional \( \tilde{\Psi}_T \) on \( \mathcal{L}^{(r,r_1,\ldots,r_n;s)}_{VN} (X_1,\ldots,X_n;Y) \) in a unique way, with
\[ \|\tilde{\Psi}_T\| \leq \|T\|_{\text{max}(r';r_1',\ldots,r_n';s')} \).

Finally we note that \( \mathcal{B}(\tilde{\Psi}_T) = T \).

To show the reverse inequality let \( \Psi \in \left( \mathcal{L}^{(r,r_1,\ldots,r_n;s)}_{VN} (X_1,\ldots,X_n;Y) \right)^* \)
and consider the corresponding \( n \)-linear mapping \( \mathcal{B}(\Psi) \in \mathcal{L}(X_1^*,\ldots,X_n^*;Y^*) \),
defined by \( \mathcal{B}(\Psi)(\phi^1,\ldots,\phi^n)(b) = \Psi(\phi^1 \times \cdots \times \phi^n b) \), for \( \phi^k \in X_n^*, \ k = 1,\ldots,n \) and \( b \in Y \). Let us consider \( n \in \mathbb{N} \) and \( \varphi^k_j \in X_n^* \), for \( k = 1,\ldots,n \), and \( (b_j)_{j \in \mathbb{N}^n_m} \in I^n_r(Y;\mathbb{N}^n_m) \). There is \( (\lambda_j)_{j \in \mathbb{N}^n_m} \in I^r(\mathbb{N}^n_m) \) such that \( \|\lambda_j\|_{r} = 1 \)
and
\[ \left\| \mathcal{B}(\Psi)(\varphi^1_j,\ldots,\varphi^n_j)(b_j)_{j \in \mathbb{N}^n_m} \right\|_r = \sum_{j \in \mathbb{N}^n_m} \lambda_j \left| \mathcal{B}(\Psi)(\varphi^1_j,\ldots,\varphi^n_j)(b_j) \right| \]
Now we can choose \( \alpha_j, \ |\alpha_j| = 1, \ j \in \mathbb{N}^n_m \) such that
\[ \sum_{j \in \mathbb{N}^n_m} \lambda_j \left| \mathcal{B}(\Psi)(\varphi^1_j,\ldots,\varphi^n_j)(b_j) \right| = \sum_{j \in \mathbb{N}^n_m} \lambda_j \alpha_j \mathcal{B}(\Psi)(\varphi^1_j,\ldots,\varphi^n_j)(b_j) \]

\[ = \Psi\left( \sum_{j \in \mathbb{N}^n_m} \lambda_j \alpha_j \varphi^1_j \times \cdots \times \varphi^n_j b_j \right) = (*) \].
By the continuity of $\Psi$ and the Hölder’s inequality we have

\[
(*) \leq \|\Psi\| \left(\prod_{k=1}^{n} \left\| (\varphi_{j_k}^{k})_{j_k \in \mathbb{N}_m} \right\|_{w,r'_k} \left\| (b_j)_{j \in \mathbb{N}_m} \right\|_{w,s'} \right).
\]

This shows that $\mathcal{B}(\Psi) \in \mathcal{L}_{mas}(\mathbb{N}_n^r; \mathbb{N}_n^r; \mathbb{N}_s)$ and

\[
\|\mathcal{B}(\Psi)\|_{mas(r'_1, \ldots, r'_n; s')} \leq \|\Psi\|.
\]

If we replace $\mathbb{N}^n$ by $\mathbb{N}$ and $s'$ by $\infty$ in Theorem 3.4, we obtain the following known cases.

**Corollary 3.5.** If the spaces $X^*_k$’s ($k = 1, \ldots, n$) have the $\lambda_k$-bounded approximation property, then

(i) The topological dual of $\mathcal{N}_{(r; r_1, \ldots, r_n; s)}(X_1, \ldots, X_n; Y)$ is isometrically isomorphic to $\mathcal{L}_{as}(\mathbb{N}_n^r; \mathbb{N}_n^r; \mathbb{N}_s)$ ($X_1^*, \ldots, X_n^*; Y^*$), for $r$, $r_k$ and $s \in [1, +\infty]$, $k = 1, \ldots, n$ through the mapping $\mathcal{B}(\Psi)$ given as follows:

\[
\mathcal{B}(\Psi) (\phi^1, \ldots, \phi^n) (b) := \Psi (\phi^1 \times \cdots \times \phi^n b),
\]

where $\Psi$ is in the topological dual of $\mathcal{N}_{(r; r_1, \ldots, r_n; s)}(X_1, \ldots, X_n; Y)$, $\phi^k \in X^*_k$, $k = 1, \ldots, n$ and $b \in Y$.

(ii) The topological dual of $\mathcal{L}_{VN_{\lambda^n}}(X_1, \ldots, X_n; Y)$ is isometrically isomorphic to $\mathcal{L}_{mas}(\mathbb{N}_n^{r_1}, \ldots, \mathbb{N}_n^{r_n}; \mathbb{N}_s)$ ($X_1^*, \ldots, X_n^*; Y^*$).

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