A Note on Rational Approximation with Respect to Metrizable Compactifications of the Plane

M. Fragoulopoulou, V. Nestoridis

Department of Mathematics, University of Athens
Panepistimiopolis, Athens 157 84, Greece
fragoulop@math.uoa.gr, vnestor@math.uoa.gr

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Abstract: In the present note we examine possible extensions of Runge, Mergelyan and Arakelian Theorems, when the uniform approximation is meant with respect to the metric \( \rho \) of a metrizable compactification \((S, \rho)\) of the complex plane \( \mathbb{C} \).

Key words: compactification, Arakelian’s theorem, Mergelyan’s theorem, Runge’s theorem, uniform approximation in the complex domain.

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1. Introduction

It is well known that the class of uniform limits of polynomials in \( \overline{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) coincides with the disc algebra \( A(D) \). A function \( f : \overline{D} \to \mathbb{C} \) belongs to \( A(D) \) if and only if it is continuous on \( \overline{D} \) and holomorphic in the open unit disc \( D \). It is less known (see [3, 7]) what is the corresponding class when the uniform convergence is not meant with respect to the usual Euclidean metric on \( \mathbb{C} \), but it is meant with respect to the chordal metric \( \chi \) on \( \mathbb{C} \cup \{\infty\} \). The class of \( \chi \)-uniform limits of polynomials on \( \overline{D} \) is denoted by \( \hat{A}(D) \) and contains \( A(D) \). A function \( f : \overline{D} \to \mathbb{C} \cup \{\infty\} \) belongs to \( \hat{A}(D) \) if and only if \( f \equiv \infty \), or it is continuous on \( \overline{D} \), \( f(D) \subset \mathbb{C} \) and \( f|_D \) is holomorphic. The function \( f(z) = \frac{1}{1-z} \), \( z \in D \), belongs to \( \hat{A}(D) \), but not to \( A(D) \); thus, it cannot be uniformly approximated on \( D \), by polynomials with respect to the usual Euclidean metric on \( \mathbb{C} \), but it can be uniformly approximated by polynomials with respect to the chordal metric \( \chi \).

More generally, if \( K \subset \mathbb{C} \) is a compact set with connected complement, then according to Mergelyan’s theorem [10] polynomials are dense in \( A(K) \) with respect to the usual Euclidean metric on \( \mathbb{C} \). We recall that a function \( f : K \to \mathbb{C} \) belongs to \( A(K) \) if and only if it is continuous on \( K \) and holomorphic in the interior \( K^\circ \) of \( K \).
An open problem is to characterize the class $\tilde{A}(K)$ of $\chi$–uniform limits of polynomials on $K$.

**Conjecture.** ([1, 6]) Let $K \subset \mathbb{C}$ be a compact set with connected complement $K^c$. A function $f : K \to \mathbb{C} \cup \{\infty\}$ belongs to $\tilde{A}(K)$ if and only if it is continuous on $K$ and for each component $V$ of $K^c$, either $f(V) \subset \mathbb{C}$ and $f|_V$ is holomorphic, or $f|_V \equiv \infty$.

Extensions of this result have been obtained in [5] when $K^c$ has a finite number of components and $K$ is bounded by a finite set of disjoint Jordan curves. In this case, the $\chi$–uniform approximation is achieved using rational functions with poles out of $K$ instead of polynomials. Furthermore, extensions of Runge’s theorem are also proved in [5]. Finally a first result has been obtained in [5] concerning an extension of the approximation theorem of Arakelian ([2]).

Instead of considering the one point compactification $\mathbb{C} \cup \{\infty\}$ of the complex plane $\mathbb{C}$, we can consider an arbitrary metrizable compactification $(S, \varrho)$ of $\mathbb{C}$ and investigate the analogues of all previous results. This is the content of the present paper.

2. Preliminaries

We say that $(S, \varrho)$ is a *metrizable compactification of the plane* $\mathbb{C}$, if $\varrho$ is a metric on $S$, $S$ is compact, $S \supset \mathbb{C}$ and $\mathbb{C}$ is an open dense subset of $S$. Obviously, $S \setminus \mathbb{C}$ is a closed subset of $S$. We say that the points in $S \setminus \mathbb{C}$ are the points at infinity.

Let $(S, \varrho)$ be a metrizable compactification of $\mathbb{C}$ with metric $\varrho$. Many such compactifications can be found in [1]. The one point compactification $\mathbb{C} \cup \{\infty\}$ with the chordal metric $\chi$ is a distinct one of them. We note that in this case, the continuous function $\pi : S \to \mathbb{C} \cup \{\infty\}$, such that $\pi(c) = c$, for every $c \in \mathbb{C}$ and $\pi(x) = \infty$, for every $x \in S \setminus \mathbb{C}$, is useful.

Another metrizable compactification is the one defined in [8] and constructed as follows: consider the map

$$\phi : \mathbb{C} \quad \longrightarrow \quad D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

$$z \quad \longmapsto \quad \frac{z}{1 + |z|},$$

which is a homeomorphism. A compactification of the image $D$ of $\phi$ is $\overline{D}$, the closure of $D$, with the usual metric. This leads to the following compactifica-
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tion of \( \mathbb{C} \)

\[
S_1 := \mathbb{C} \cup \{ \infty e^{i\vartheta} : 0 \leq \vartheta \leq 2\pi \},
\]

with metric \( d \) given by

\[
d(z, w) = \left| \frac{z}{1 + |z|} - \frac{w}{1 + |w|} \right| \quad \text{if } z, w \in \mathbb{C},
\]

\[
d(z, \infty e^{i\vartheta}) = \left| \frac{z}{1 + |z|} - e^{i\vartheta} \right| \quad \text{if } z \in \mathbb{C}, \vartheta \in \mathbb{R},
\]

\[
d(\infty e^{i\vartheta}, \infty e^{i\varphi}) = |e^{i\vartheta} - e^{i\varphi}| \quad \text{if } \vartheta, \varphi \in \mathbb{R}.
\]

In what follows, with a compactification \((S, \varrho)\) of \( \mathbb{C} \), we shall always mean a metrizable compactification.

An important question for a given compactification of \( \mathbb{C} \) is, whether for \( c \in \mathbb{C} \) and \( x \in S \setminus \mathbb{C} \), the addition \( c + x \) is well defined. In other words, having two convergent sequences \( \{z_n\}, \{w_n\} \) in \( \mathbb{C} \), such that \( z_n \to c \) and \( w_n \to x \) does the sequence \( \{z_n + w_n\} \) have a limit in \( S \)?

If the answer is positive for any such sequences \( \{z_n\}, \{w_n\} \) in \( \mathbb{C} \), then the limit \( y \in S \) of the sequence \( \{z_n + w_n\} \) is uniquely determined and we write \( c + x \). We are interested in compactifications \((S, \varrho)\), where \( c + x \) is well defined for any \( c \in \mathbb{C} \) and \( x \in S \setminus \mathbb{C} \). In this case, the map \( \mathbb{C} \times S \to S, (c, x) \mapsto c + x \), is automatically continuous.

Indeed, let \( x \in S \setminus \mathbb{C}, y \in \mathbb{C} \) and \( w = x + y \in S \setminus \mathbb{C} \). Let \( \{z_n\} \) in \( S \) and \( \{y_n\} \) in \( \mathbb{C} \), such that \( z_n \to x \) and \( y_n \to y \). If all but finitely many \( z_n \) belong to \( \mathbb{C} \), then by our assumption \( z_n + y_n \to x + y \). Suppose that infinitely many \( z_n \) belong to \( S \setminus \mathbb{C} \) and by compactness we can assume that \( z_n + y_n \to \ell \neq w = x + y \).

Let \( d = \varrho(\ell, w) > 0 \). Then there exists \( n_0 \in \mathbb{N} \), such that

\[
\varrho(z_n + y_n, \ell) < \frac{d}{2} \quad \text{for all } n \geq n_0.
\]

Fix \( n \geq n_0 \). Since, \( z_n + y_n \) is well defined, there exists \( z'_n \in \mathbb{C} \), such that

\[
\varrho(z_n, z'_n) < \frac{1}{n} \quad \text{and} \quad \varrho(z_n + y_n, z'_n + y_n) < \frac{1}{n}.
\]

It follows that

\[
\varrho(z'_n, x) \leq \varrho(z'_n, z_n) + \varrho(z_n, x) < \frac{1}{n} + \varrho(z_n, x) \to 0.
\]
Hence, \( z'_n \to x \), \( y_n \to y \) and \( z'_n, y_n \in \mathbb{C} \). By our assumption, it follows that \( z'_n + y_n \to x + y = w \). But
\[
\rho(z'_n + y_n, l) \leq \rho(z'_n + y_n, z_n + y_n) + \rho(z_n + y_n, l) \\
\leq \frac{1}{n} + \rho(z_n + y_n, l) < \frac{1}{n} + \frac{d}{2} \to \frac{d}{2}.
\]
Thus, for all \( n \) large enough we have
\[
\rho(z'_n + y_n, l) \leq \frac{3d}{4} < d = \rho(l, w).
\]
It follows that \( \rho(z'_n + y_n, w) \geq \frac{d}{4} \), for all \( n \) large enough. Therefore, we cannot have \( z'_n + y_n \to w \).

Consequently, one concludes that the addition map is continuous at every \((x, y)\) with \( x \in S \setminus C \) and \( y \in \mathbb{C} \). Obviously, it is also continuous at every \((x, y)\) with \( x \) and \( y \) in \( \mathbb{C} \). Thus, addition is continuous on \( S \times \mathbb{C} \). Furthermore, the following holds:

Let \( K \subset \mathbb{C} \) be compact. Obviously, the map \( K \times S \to S, (c, x) \mapsto c + x \), is uniformly continuous.

Remark 1. The preceding certainly holds for the compactification \((S_1, d)\) (see (2.1)), since
\[
c + \infty e^{i\vartheta} = \infty e^{i\vartheta} \quad \text{for all } c \in \mathbb{C} \text{ and } \vartheta \in \mathbb{R},
\]
and we have continuity.

Remark 2. If we identify \( \mathbb{R} \) with the interval \((-1, 1)\), up to a homeomorphism, then \( \mathbb{C} \cong \mathbb{R}^2 \) is identified with the square \((-1, 1) \times (-1, 1)\). An obvious compactification of \( \mathbb{C} \) is then the closed square with the usual metric. The points at infinity are those on the boundary of the square, for instance, those points on the side \( \{1\} \times [-1, 1] \). If \( x \in \{1\} \times (-1, 1) \) and \( c \in \mathbb{C}, \) then \( c + x \) is a point in the same side; if \( \text{Im } c \neq 0 \), then \( c + x \neq x \). If \( x = (1, 1) \) and \( c \in \mathbb{C}, \) then \( x + c = x \). If \( \text{Im } c > 0 \), then \( c + x \) lies higher than \( x \) in the side \( \{1\} \times (-1, 1) \).

In this example, the addition is well defined and continuous, but the points at infinity are not stabilized as in Remark 1.

Question. Is there a metrizable compactification of \( \mathbb{C} \) such that the addition \( c + x \) is not well defined for some \( c \in \mathbb{C} \) and \( x \in S \setminus \mathbb{C} \)?
The answer is “yes”. An example comes from the previous square in Remark 2, if we identify all the points of \( \{1\} \times [-\frac{1}{2}, \frac{1}{2}] \) and make them just one point.

3. Runge and Mergelyan type theorems

In this section using a compactification of \( \mathbb{C} \) satisfying all properties discussed in the Preliminaries, we obtain the following theorem, that extends [5, Theorem 3.3].

**Theorem 3.1.** Let \( \Omega \subset \mathbb{C} \) be a bounded domain, whose boundary consists of a finite set of pairwise disjoint Jordan curves. Let \( K = \overline{\Omega} \) and \( A \) a set containing one point from each component of \((\mathbb{C} \cup \{\infty\}) \setminus K\). Let \((S, \varrho)\) be a compactification of \( \mathbb{C} \), such that the addition \(+ : \mathbb{C} \times S \to S\) is well defined. Let \( f : K \to S\) be a continuous function, such that \( f(\Omega) \subset \mathbb{C} \) and \( f |_{\Omega} \) is holomorphic. Let \( \varepsilon > 0 \). Then, there exists a rational function \( R \) with poles only in \( A \) and such that \( \varrho(f(z), R(z)) < \varepsilon \), for all \( z \in K \).

**Proof.** If \( \Omega \) is a disk, the proof has been given in [1]. If \( \Omega \) is the interior of a Jordan curve, the proof is given again in [1], but also in [6]. In the general case, we imitate the proof of [5, Theorem 3.3]. Namely, we consider the Laurent decomposition of \( f \), given by \( f = f_0 + f_1 + \cdots + f_N \) (see [4]). The function \( f_0 \) is defined on a simply connected domain, bounded by a Jordan curve, and it can be uniformly approximated by a polynomial or a rational function \( R_0 \) with pole in the unbounded component. Similarly, \( f_1 \) is approximated by a rational function \( R_1 \) with pole in \( A \) and so on. Thus, the function \( R_0 + R_1 + \cdots + R_N \) approximates, with respect to \( \varrho \), the function \( f = f_0 + f_1 + \cdots + f_N \). This is due to the fact that at every point \( z \) all the \( f_i \)’s, \( i = 1, 2, \ldots, N \), except maybe one, take values in \( \mathbb{C} \) and the one, maybe has as a value, an infinity point in \( S \setminus \mathbb{C} \). In this way, the addition map \( \mathbb{C} \times S \to S \), \((c, x) \mapsto c + x\), is well defined and uniformly continuous on compact sets and so we are done. 

Another Runge-type theorem is the following, where we do not need any assumption for the compactification \( S \), or the addition map \(+ : \mathbb{C} \times S \to S\).

**Theorem 3.2.** Let \( \Omega \subset \mathbb{C} \) be open, \( f : \Omega \to \mathbb{C} \) be holomorphic and \((S, \varrho)\) a compactification of \( \mathbb{C} \). Let \( A \) be a set containing one point from each component of \((\mathbb{C} \cup \infty) \setminus \Omega\). Let \( \varepsilon > 0 \) and \( L \subset \Omega \) compact. Then, there
exists a rational function $R$ with poles in $A$, such that $g(f(z), R(z)) < \varepsilon$ for all $z \in L$.

Proof. Clearly the subset $f(L)$ of $\mathbb{C}$ is compact. Then, from the classical theorem of Runge, there exist rational functions $\{R_n\}$, with poles only in $A$, converging uniformly to $f$ on $L$, with respect to the Euclidean metric $|\cdot|$. Hence, there is a positive integer $n_0$ and a compact $K$, such that

$$f(L) \subset K \subset \mathbb{C} \quad \text{and} \quad R_n(L) \subset K \quad \text{for all} \quad n \geq n_0.$$ 

But on $K$ the metrics $|\cdot|$ and $\varrho$ are uniformly equivalent. Therefore, $R_n \to f$ uniformly on $L$, with respect to $\varrho$. To conclude the proof, it suffices to put $R = R_n$, for $n$ large enough. \qed

Theorem 3.2 easily yields the following

**Corollary 3.3.** Under the assumptions of Theorem 3.2 there exists a sequence $\{R_n\}$ of rational functions with poles in $A$, such that $R_n \to f$, $\varrho$–uniformly, on each compact subset of $\Omega$.

**Remark.** According to Corollary 3.3, some of the $\varrho$–uniform limits, on compacta, of rational functions with poles in $A$, are the holomorphic functions $f: \Omega \to \mathbb{C}$. Those are limits of the finite type. The other limits of sequences $\{R_n\}$ as above may be functions $f: \Omega \to S\setminus\mathbb{C}$ of infinite type, continuous (but maybe not all of them, as the Example $(S_1, d)$ shows; cf. [8]).

**Question.** Is a characterization possible for such limits $f: \Omega \to S_1\setminus\mathbb{C}$?

An imitation of the arguments in [8, p. 1007] gives that $f$ must be of the form $f(z) = \infty e^{\vartheta(z)}$, $z \in \Omega$, where $\vartheta$ is a multivalued harmonic function.

The following extends [5, Section 5].

**Theorem 3.4.** Let $\Omega \subset \mathbb{C}$ be open and $f$ a meromorphic function on $\Omega$. Let $B$ denote the set of poles of $f$. Let $(S, \varrho)$ be a compactification of $\mathbb{C}$, such that the addition $+: \mathbb{C} \times S \to S$ is well defined. Let $\varepsilon > 0$ and $K \subset \Omega$ be a compact set. Then, there is a rational function $g$, such that $\varrho(f(z), g(z)) < \varepsilon$, for every $z \in K \setminus B$.

Proof. Since $B \cap K$ is a finite set, the function $f$ decomposes to $f = h + w$, where $h$ is a rational function with poles in $B \cap K$ and $w$ is holomorphic on an open set containing $K$. By Runge’s theorem there exists a rational function $R$
with poles off $K$, such that $|w(z) - R(z)| < \varepsilon'$ on $K$. Since $w(K)$ is a compact subset of $\mathbb{C}$ and the addition $+ : \mathbb{C} \times S \to S$ is well defined, a suitable choice of $\varepsilon'$ gives

$$
\varrho\left( [h(z) + w(z)], [h(z) + R(z)] \right) < \varepsilon \quad \text{on } K \setminus B.
$$

We set $g = h + R$ and the result follows. \hfill \blacksquare

### 4. Arakelian sets

A closed set $F \subset \mathbb{C}$ is said a set of approximation if every function $f : F \to \mathbb{C}$ continuous on $F$ and holomorphic in $F^\circ$ can be approximated by entire functions, uniformly on the whole $F$. This is equivalent to the fact that $F$ is an Arakelian set (see [2]), that is $(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected and locally connected (at $\infty$).

We can now ask about an extension of the Arakelian theorem in the context of metrizable compactifications. A result in this direction is the following

**Proposition 4.1.** Let $F \subset \mathbb{C}$ be a closed Arakelian set with empty interior, i.e., $F^\circ = \emptyset$. We consider the compactification $(S_1, d)$ of $\mathbb{C}$ (see (2.1) and (2.2)) and let $f : F \to S_1$ be a continuous function. Let $\varepsilon > 0$. Then, there exists an entire function $g$ such that $d(f(z), g(z)) < \varepsilon$, for every $z \in F$.

**Proof.** According to (1.1), the compactification $S_1$ is homeomorphic to $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. For each $0 < R < 1$ let us define

$$
\phi_R : \overline{D} \longrightarrow \{z \in \mathbb{C} : |z| \leq R\} \subset \overline{D}
$$

$$
z \longrightarrow \begin{cases} 
z, & \text{if } |z| \leq R, \\
Rz/|z|, & \text{if } R \leq |z| \leq 1.
\end{cases}
$$

In other words, the whole line segment $[Re^{i\theta}, e^{i\theta}]$ is mapped at the end point $Re^{i\theta}$. The function $\phi_R$ is continuous and induces a continuous function $\tilde{\phi}_R : S_1 \to S_1$. It suffices to take $\tilde{\phi}_R := T^{-1} \circ \phi_R \circ T$, where $T : S_1 \to \{w \in \mathbb{C} : |w| \leq 1\}$ is defined as follows

$$
T(z) := \frac{z}{1 + |z|} \quad \text{for } z \in \mathbb{C} \subset S_1,
$$

$$
T(\infty e^{i\theta}) := e^{i\theta} \quad \text{for } \theta \in \mathbb{R}.
$$
If $\varepsilon > 0$ is given, then there exists $R_\varepsilon < 1$, such that for $R_\varepsilon \leq R < 1$ and $z \in S_1$, we have $d(z, \tilde{\phi}_R(z)) < \frac{\varepsilon}{2}$.

Let now $f$ be as in the statement of the Proposition 4.1. Then,

$$d\left(f(z), (\tilde{\phi}_R \circ f)(z)\right) < \frac{\varepsilon}{2} \quad \text{for all } z \in F.$$ 

Moreover, the function $\tilde{\phi}_R \circ f : F \to \mathbb{C}$ is continuous. Since $F$ is a closed Arakelian set, with empty interior, and $(\tilde{\phi}_R \circ f)(F) \subset K$, is included in a compact subset $K$ of $\mathbb{C}$, there exists $g$ entire, such that

$$\left| (\tilde{\phi}_R \circ f)(z) - g(z) \right| < \varepsilon' \quad \text{for all } z \in F.$$ 

Since $(\tilde{\phi}_R \circ f)(F)$ is contained in a compact subset $K$ of $\mathbb{C}$, for a suitable choice of $\varepsilon'$, it follows that

$$d\left((\tilde{\phi}_R \circ f)(z), g(z)\right) < \frac{\varepsilon}{2} \quad \text{for all } z \in F.$$ 

The triangle inequality completes the proof. 

An analogue of Proposition 4.1 for the one point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ has been established in [5].

References

