

## On the Moduli Space of Donaldson-Thomas Instantons

YUUI TANAKA

*Graduate School of Mathematics, Nagoya University  
Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan  
yu2tanaka@gmail.com*

Presented by Oscar García Prada

Received May 19, 2015

*Abstract:* In alignment with a programme by Donaldson and Thomas, Thomas [48] constructed a deformation invariant for smooth projective Calabi-Yau threefolds, which is now called the Donaldson-Thomas invariant, from the moduli space of (semi-)stable sheaves by using algebraic geometry techniques. In the same paper [48], Thomas noted that certain perturbed Hermitian-Einstein equations might possibly produce an analytic theory of the invariant. This article sets up the equations on symplectic 6-manifolds, and gives the local model and structures of the moduli space coming from the equations. We then describe a Hitchin-Kobayashi style correspondence for the equations on compact Kähler threefolds, which turns out to be a special case of results by Álvarez-Cónsul and García-Prada [1].

*Key words:* gauge theory; the Donaldson-Thomas theory.

*AMS Subject Class.* (2010): 53C07.

### 1. INTRODUCTION

In [21], Donaldson and Thomas suggested higher-dimensional analogues of gauge theories, and proposed the following two directions: gauge theories on  $Spin(7)$  and  $G_2$ -manifolds; and gauge theories in complex 3 and 4 dimensions. The first ones could be related to “Topological M-theory” proposed by Nekrasov and others [39], [15]. The second ones are a “complexification” of the lower-dimensional gauge theories. In this direction, Thomas [48] constructed a deformation invariant of smooth projective Calabi-Yau threefolds from the moduli space of (semi-)stable sheaves, which he called the *holomorphic Casson invariant* because it can be viewed as a complex analogue of the Taubes-Casson invariant [47]. It is now called the Donaldson-Thomas invariant (D-T invariant for short), and further developed by Joyce-Song [28] and Kontsevich-Soibelman [32], [33], [34]. Later, Donaldson and Segal [20] further promoted the programme, taking into account the progress made after the proposal. Recently, more breakthroughs concerning the “categorification” of the D-T invariant by using perverse sheaves were made by a group led by Joyce [7], [27], [8], [9], [4], also by Kiem-Li [29].

Let us mention here a conjecture (called the MNOP conjecture) posed by Maulik-Nekrasov-Okounkov-Pandharipande [37], [38], which insists that the rank one D-T invariants (“counting” of ideal sheaves on a Calabi-Yau threefold) can be determined by only the Betti numbers and the Gromov-Witten invariants. Assuming the conjecture is true, one can observe that the rank one D-T invariants are symplectic invariants, as the Gromov-Witten invariants are symplectic invariants. One might further speculate that the full D-T invariants defined by Joyce and Song could be also symplectic invariants. One of our goals is to work toward proving this by using a gauge-theoretic equation (we call it the *Donaldson-Thomas equation*) on a compact symplectic 6-manifold, which ought to be an analytic counterpart of the notion of stable holomorphic vector bundles, as the problem is analytic in nature.

Perhaps, one might think of that a gauge-theoretic equation which would describe the D-T invariant could be the Hermitian-Einstein equations, as the Hitchin-Kobayashi correspondence [17], [18], [50], [51] (see also [31], [36]) insists that there is a one-to-one correspondence between the existence of the Hermitian-Einstein connection and the Mumford-Takemoto stability of an irreducible vector bundle over a compact Kähler manifold. However, the Hermitian-Einstein equations do not form an elliptic system even with a gauge fixing equation in complex dimension three and more (see Section 2.1), so this might cause a little problem.

In order to work out this issue, Donaldson and Thomas [48] suggested a perturbation of the Hermitian-Einstein equations described below. This perturbation was also brought in by Baulieu-Kanno-Singer [3] and Iqbal-Nekrasov-Okounkov-Vafa [26] in String Theory context.

Let  $Z$  be a compact symplectic 6-manifold with symplectic form  $\omega$ ,  $P$  a principal  $U(r)$ -bundle on  $Z$ , and  $E$  the associated unitary vector bundle on  $Z$ . The equations we consider are ones for a connection  $A$  of  $P$  and an  $\text{Ad}(P)$ -valued  $(0,3)$ -form  $u$  on  $Z$  of the following form:

$$F_A^{0,2} + \bar{\partial}_A^* u = 0, \quad F_A^{1,1} \wedge \omega^2 + [u, \bar{u}] + 2\pi i \mu(E) \text{Id}_E \omega^3 = 0,$$

where  $F_A^{0,2}$  and  $F_A^{1,1}$  are the  $(0,2)$  and  $(1,1)$  components of the curvature  $F_A$  of  $A$ , and

$$\mu(E) := \frac{1}{r} \int_Z c_1(E) \wedge \omega^2.$$

Here we picked up an almost complex structure compatible with  $\omega$  to get the splitting of the space of the complexified two forms. We call the equations the *Donaldson-Thomas equations* (*D-T equations* for short) and a solution

to the equations a *Donaldson-Thomas instanton* (*D-T instanton* for short). These equations with a gauge fixing equation form an elliptic system. We aim at developing an analytic theory concerning the D-T invariant by using the moduli space coming from these equations.

In [44], [45], we studied some analytic properties of solutions to the equations on compact Kähler threefolds. In [44], we proved that a sequence of solutions to the D-T equation has a subsequence which smoothly converges to a solution to the D-T equation outside a closed subset of the Hausdorff dimension two. In [45], we proved some of singularities which appeared in the above weak limit can be removed.

In this article, we describe the infinitesimal deformation and the Kuranishi model of the moduli space of D-T instantons by using familiar techniques in gauge theory, for example, the corresponding results for the anti-self-dual instantons in real four dimensions were studied by Atiyah-Hitchin-Singer [2] (see also [22], [19]), and for the Hermitian-Einstein connections by Kim [30] (see also [31], [36]). We then describe a Hitchin-Kobayashi style correspondence for the D-T instanton on compact Kähler threefolds, which turns out to be a special case of results by Álvarez-Cónsul and García-Prada [1].

The organisation of this article is as follows. In Section 2, we briefly recall the Hermitian-Einstein connections, subsequently, we introduce the D-T equations on symplectic 6-manifolds. We also mention a relation between the D-T equations and the complex anti-self-dual equations by dimensional reduction argument. In Section 3, we give the Kuranishi model of the space of the D-T instantons. In Section 4, we describe a Hitchin-Kobayashi style correspondence for the D-T instanton on compact Kähler threefolds.

## 2. THE DONALDSON-THOMAS INSTANTONS

2.1. THE HERMITIAN-EINSTEIN CONNECTIONS ON COMPACT KÄHLER MANIFOLDS. We first recall the notion of the Hermitian-Einstein connections on compact Kähler manifolds. General references for the Hermitian-Einstein connections are [31] and [36].

Let  $X$  be a compact Kähler manifold of complex dimension  $n$  with Kähler form  $\omega$ ,  $E$  a hermitian vector bundle over  $X$  with hermitian metric  $h$ . A metric preserving connection  $A$  of  $E$  is said to be a *Hermitian-Einstein connection* if  $A$  satisfies the following equations:

$$F_A^{0,2} = 0, \quad i\Lambda F_A^{1,1} = 2n\pi\mu(E)Id_E, \quad (2.1)$$

where  $F_A^{0,2}$  and  $F_A^{1,1}$  are the  $(0, 2)$  and  $(1, 1)$  components of the curvature  $F_A$

of  $A$ ,  $\Lambda := (\omega)^*$ , and

$$\mu(E) := \frac{1}{r} \int_X c_1(E) \wedge \omega^{n-1}.$$

The existence of a solution to the equations (2.1) is related to the notion of stability for holomorphic vector bundles. In fact, Donaldson [17], [18] and Uhlenbeck-Yau [50], [51] proved that there is a one-to-one correspondence between the existence of the Hermitian-Einstein connection and the Mumford-Takemoto stability of an irreducible vector bundle over a compact Kähler manifold (see also [31], [36]).

The infinitesimal deformation of a Hermitian-Einstein connection  $A$  was studied by Kim [30] (see also [31], [41]), and it is described by the following:

$$\begin{aligned} 0 \longrightarrow \Omega^0(X, \mathbf{u}(E)) \xrightarrow{d_A} \Omega^1(X, \mathbf{u}(E)) \xrightarrow{d_A^+} \Omega^+(X, \mathbf{u}(E)) \\ \xrightarrow{\bar{D}'_A} A^{0,3}(X, \mathbf{u}(E)) \xrightarrow{\bar{D}_A} A^{0,4}(X, \mathbf{u}(E)) \\ \xrightarrow{\bar{D}_A} \dots \xrightarrow{\bar{D}_A} A^{0,n}(X, \mathbf{u}(E)) \longrightarrow 0, \end{aligned} \quad (2.2)$$

where  $A^{0,q}(X, \mathbf{u}(E)) := C^\infty(\mathbf{u}(E) \otimes A^{0,q})$ ,  $\mathbf{u}(E) = \text{End}(E, h)$  is the bundle of skew-Hermitian endomorphisms of  $E$ ,  $A^{0,p}$  is the space of real  $(0, p)$ -forms (see [42, pp. 32-33]) over  $X$ , defined by  $A^{0,p} \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{0,p} \oplus \Lambda^{p,0}$ ,

$$\begin{aligned} \Omega^+(X, \mathbf{u}(E)) &:= A^{0,2}(X, \mathbf{u}(E)) \oplus \Omega^0(X, \mathbf{u}(E))\omega \\ &= \{ \phi + \bar{\phi} + f\omega : \phi \in \Omega^{0,2}(X, \mathbf{u}(E)), f \in \Omega^0(X, \mathbf{u}(E)) \}, \end{aligned}$$

$\bar{D}_A : A^{0,p}(X, \mathbf{u}(E)) \rightarrow A^{0,p+1}(X, \mathbf{u}(E))$  is defined by

$$\bar{D}_A \alpha = \bar{\partial}_A \alpha^{0,p} + \partial_A \overline{\alpha^{0,p}}$$

for  $\alpha = \alpha^{0,p} + \overline{\alpha^{0,p}}$ , where  $\alpha^{0,p} \in \Omega^{0,p}(X, \mathbf{u}(E))$ , and

$$d_A^+ := \pi^+ \circ d_A, \quad \bar{D}'_A := \bar{D}_A \circ \pi^{0,2},$$

where  $\pi^+$  and  $\pi^{0,2}$  are respectively the orthogonal projections from  $\Omega^2$  to  $\Omega^+$  and  $A^{0,2}$ .

Kim proved that (2.2) is an elliptic complex if  $A$  is a Hermitian-Einstein connection. However, it is obviously not the Atiyah-Hitchin-Singer type complex [2] if  $n \geq 3$ , since there are additional terms such as  $A^{0,3}(X, \mathbf{u}(E))$  and so on. Hence, the Hermitian-Einstein connections would not work for an analytic construction of the Donaldson-Thomas invariant just as it is. But, in

[48], Thomas noted a perturbed Hermitian-Einstein equation, which basically corresponds to a “holding” of the extra term  $A^{0,3}(X, u(E))$  in (2.2) (we shall see it in Section 3.1), could possibly work for an analytic definition of the Donaldson-Thomas invariant. We introduce that perturbed equation in the next subsection.

**2.2. THE DONALDSON-THOMAS INSTANTONS ON COMPACT SYMPLECTIC 6-MANIFOLDS.** Let  $Z$  be a compact symplectic 6-manifold with symplectic form  $\omega$ , and  $E$  a unitary vector bundle of rank  $r$  over  $Z$ . We take an almost complex structure on  $Z$  compatible with the symplectic form  $\omega$ . Then the almost complex structure induces the splitting of the complexified two forms as  $\Lambda^2 \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}$ . We consider the following equations for a connection  $A$  of  $E$ , which preserves the hermitian structure of  $E$ , and a  $u(E)$ -valued  $(0,3)$ -form  $u$  on  $Z$ .

$$F_A^{0,2} + \bar{\partial}_A^* u = 0, \tag{2.3}$$

$$F_A^{1,1} \wedge \omega^2 + [u, \bar{u}] + 2\pi i \mu(E) Id_E \omega^3 = 0, \tag{2.4}$$

where  $F_A^{0,2}$  and  $F_A^{1,1}$  are the  $(0, 2)$  and  $(1, 1)$  components of the curvature  $F_A$  of  $A$ , and

$$\mu(E) := \frac{1}{r} \int_Z c_1(E) \wedge \omega^2.$$

We call these equations (2.3), (2.4) the *Donaldson-Thomas equations*, and a solution  $(A, u)$  to these equations a *Donaldson-Thomas instanton* (*D-T instanton* for short).

One may think of these equations as the Hermitian-Einstein equations with a perturbation  $u$ . However, we think of  $u$  as a Higgs field, namely, a new variable. One of advantages of bringing in the new field  $u$  is that the Donaldson-Thomas equations form an elliptic system after fixing a gauge transformation, despite the fact that the Hermitian-Einstein equations on compact Kähler threefolds do not form it in the same way.

These equations (2.3), (2.4) were also studied in physics such as in [3]. In that context, these equations are interpreted as a bosonic part of dimensional reduction equations of the  $N = 1$  super Yang-Mills equation in 10 dimensions to 6 dimensions (see also [26], [40]).

**THE EQUATIONS IN THE KÄHLER CASE.** If the almost complex structure is integrable, then we have  $\bar{\partial}_A F_A^{0,2} = 0$  by the Bianchi identity. Hence

$\bar{\partial}_A \bar{\partial}_A^* u = 0$  by (2.3), thus we have  $\bar{\partial}_A^* u = 0$  on compact Kähler threefolds. Therefore, the Donaldson-Thomas equations (2.3), (2.4) becomes

$$\begin{aligned} \bar{\partial}_A^* u &= 0, & F_A^{0,2} &= 0, \\ F_A^{1,1} \wedge \omega^2 + [u, \bar{u}] + 2\pi i \mu(E) Id_E \omega^3 &= 0. \end{aligned}$$

The above equations could be thought of as a generalization of the Hitchin equation on Riemann surfaces [24] to Kähler threefolds in the same way as the Vafa-Witten equations on Kähler surfaces as mentioned in [46]. In Section 4 to this article, we describe the corresponding Hitchin-Kobayashi correspondence in this setting, which turns out to be a special case of results by Álvarez-Cónsul and García-Prada [1].

### 2.3. THE COMPLEX ASD AND THE DONALDSON-THOMAS INSTANTONS.

In this section, we see that the Donaldson-Thomas equations on Calabi-Yau threefolds can be thought of as the dimensional reduction of the complex ASD equations on Calabi-Yau fourfolds, this was pointed out by Tian [49], and it is analogous to the Hitchin pair [24].

COMPLEX ASD EQUATIONS ON CALABI-YAU FOURFOLDS. Let  $X$  be a compact Calabi-Yau fourfold with Kähler form  $\omega$  and holomorphic  $(4, 0)$ -form  $\theta$ . We assume the normalization condition  $\theta \wedge \bar{\theta} = \frac{16}{4!} \omega^4$  on  $\omega$  and  $\theta$ . Let  $E$  be a hermitian vector bundle over  $X$ . By using the holomorphic  $(4, 0)$ -form  $\theta$ , we define the complex Hodge operator  $*_\theta : \Lambda^{0,2} \rightarrow \Lambda^{0,2}$  by

$$\text{tr}(\phi \wedge *_\theta \psi) = \langle \phi, \psi \rangle \bar{\theta}, \quad \phi, \psi \in \Lambda^{0,2}.$$

Then  $*_\theta^2 = 1$ , and the space of  $(0, 2)$ -forms further decomposes into  $\Lambda^{0,2} = \Lambda_+^{0,2} \oplus \Lambda_-^{0,2}$ , where

$$\Lambda_+^{0,2} = \{\phi \in \Lambda^{0,2} : *_\theta \phi = \phi\}, \quad \Lambda_-^{0,2} = \{\phi \in \Lambda^{0,2} : *_\theta \phi = -\phi\}.$$

Note that the operator  $*_\theta$  is an anti-holomorphic map, hence  $\Lambda_+^{0,2}$  and  $\Lambda_-^{0,2}$  are real subspaces of  $\Lambda^{0,2}$ .

We consider the following equations for connections of  $E$ :

$$(1 + *_\theta)F_A^{0,2} = 0, \quad i\Lambda F_A^{1,1} = 8\pi\mu(E)Id_E, \quad (2.5)$$

where

$$\mu(E) := \frac{1}{r} \int_X c_1(E) \wedge \omega^3.$$

We call these equations *complex ASD equations*, and a solution to these equations a *complex ASD instanton*. These were brought in by Donaldson and Thomas in [21]. These equations with a gauge fixing equation form an elliptic system. Analytic properties of the complex ASD instantons were studied by Tian [49].

Note that the complex ASD instantons are special cases of  $Spin(7)$ -instantons on  $Spin(7)$ -manifolds (see [43, §3.1]).

More recently, Donaldson-Thomas style invariants for Calabi-Yau fourfolds, which concerns the moduli space of the solutions to the above complex ASD equations, were defined by Borisov-Joyce [5], Cao [10] and Cao-Leung [11] (see also [12], [13], [14]).

**DIMENSIONAL REDUCTION.** We describe a relation between the Donaldson-Thomas equations (2.3), (2.4) and the complex ASD equations (2.5) by dimensional reduction argument. This was pointed out by Tian [49].

Let  $Z$  be a compact Calabi-Yau threefold with Kähler form  $\omega_0$  and holomorphic  $(3, 0)$ -form  $\theta_0$ , and  $T^2$  a torus of complex dimension one. We consider the direct product of  $Z$  and  $T^2$ , and denote it by  $X$ , namely,  $X := Z \times T^2$ . We define a Kähler form  $\omega$  and a holomorphic  $(4, 0)$ -form on  $X$  by

$$\omega := \omega_0 + dz \wedge d\bar{z}, \quad \theta := \theta_0 \wedge dz,$$

where  $dz$  is the standard flat  $(1, 0)$  form on  $T^2$ .

Let  $E$  be a hermitian vector bundle with structure group  $SU(r)$  over  $Z$ , and  $p : X = Z \times T^2 \rightarrow Z$ . We then consider  $T^2$ -invariant solutions to the complex ASD equations (2.5) on  $p^*E \rightarrow X$ . Then these solutions satisfy the Donaldson-Thomas equations on  $Z$ . In fact, if we write a connection  $A$  on  $X = Z \times T^2$  as  $A_X = A + \phi dz + \bar{\phi} d\bar{z}$ , where  $A$  is the  $Z$ -component of the connection  $A_X$  and  $\phi \in \Gamma(Z, \mathfrak{su}(E))$ , then the curvature becomes

$$F_{A_X} = F_A + d_A \phi \wedge dz + d_A \bar{\phi} \wedge d\bar{z} + [\phi, \bar{\phi}] dz \wedge d\bar{z}.$$

Hence, if we put  $u := \phi \bar{\theta}_0 \in \Omega^{0,3}(Z, \mathfrak{su}(E))$ , then  $A$  and  $u$  satisfy the Donaldson-Thomas equations, provided that this  $A_X$  is a  $T^2$ -invariant solution to the complex ASD equations.

### 3. LOCAL MODEL FOR THE MODULI SPACE OF DONALDSON-THOMAS INSTANTONS

Let  $Z$  be a compact symplectic 6-manifold with symplectic form  $\omega$ ,  $(E, h)$  a hermitian vector bundle over  $Z$  with hermitian metric  $h$ .

We denote by  $\mathcal{A}(E) = \mathcal{A}(E, h)$  the set of all connections of  $E$  which preserve the hermitian structure of  $E$ , and put  $\mathcal{C}(E) := \mathcal{A}(E) \times \Omega^{0,3}(Z, \mathbf{u}(E))$ . We denote by  $\mathcal{G}(E) = \mathcal{G}(E, h)$  the gauge group, the group of unitary automorphism of  $(E, h)$ , where the action of the gauge group on  $\mathcal{C}(E)$  is defined by  $g(A, u) = (A - (d_A g)g^{-1}, g^{-1}ug)$ . These spaces  $\mathcal{C}(E)$ ,  $\mathcal{G}(E)$  can be seen as Fréchet spaces with  $C^\infty$ -norms, but we shall use Sobolev completions of them in Section 3.2.

We denote by  $\Gamma_{(A,u)}$  the stabilizer at  $(A, u) \in \mathcal{C}(E)$  of the gauge group  $\mathcal{G}(E)$ , namely,

$$\Gamma_{(A,u)} := \{g \in \mathcal{G}(E) : g(A, u) = (A, u)\}.$$

We call  $(A, u) \in \mathcal{C}(E)$  *irreducible* if  $\Gamma_{(A,u)}$  coincides with the centre of the structure group of  $E$ , and *reducible* otherwise. We denote by  $\mathcal{C}^*(E)$  the set of all irreducible pair  $(A, u) \in \mathcal{C}(E)$ . Note that the action of  $\mathcal{G}(E)$  is not free on  $\mathcal{C}^*(E)$ , but the action of  $\hat{\mathcal{G}}(E) = \mathcal{G}(E)/U(1)$  is free on  $\mathcal{C}^*(E)$ .

We denote by  $\mathcal{D}(E)$  the set of all D-T instantons of  $E$ , and by  $\mathcal{D}^*(E)$  the set of all irreducible D-T instantons of  $E$ . We call  $\mathcal{M}(E) = \mathcal{D}(E)/\mathcal{G}(E)$  the *moduli space of the Donaldson-Thomas instantons*.

**3.1. LINEARIZATION.** The infinitesimal deformation of a D-T instanton  $(A, u)$  is described by the following sequence:

$$\begin{aligned} 0 \longrightarrow \Omega^0(Z, \mathbf{u}(E)) \xrightarrow{D_{(A,u)}} \Omega^1(Z, \mathbf{u}(E)) \oplus A^{0,3}(Z, \mathbf{u}(E)) \\ \xrightarrow{D_{(A,u)}^+} \Omega^+(Z, \mathbf{u}(E)) \longrightarrow 0, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} D_{(A,u)}(s) &= (d_A s, [\tilde{u}, s]), \quad \tilde{u} = u + \bar{u}, \\ D_{(A,u)}^+(\alpha, v) &= d_A^+ \alpha + \Lambda^2([u, \bar{v}] + [v, \bar{u}]) + \bar{D}_A^* v \end{aligned}$$

for  $s \in \Omega^0(Z, \mathbf{u}(E))$  and  $(\alpha, v) \in \Omega^1(Z, \mathbf{u}(E)) \oplus A^{0,3}(Z, \mathbf{u}(E))$ . If  $(A, u)$  is a D-T instanton, then (3.1) is a complex. In fact,  $D_{(A,u)}^+ D_{(A,u)} = 0$  follows directly from the equations (2.3), (2.4). The complex (3.1) can be seen as “holding” of the  $A^{0,3}(Z, \mathbf{u}(E))$ -term in (2.2), namely, it is equivalent to consider the following complex instead of (3.1):

$$\begin{aligned} 0 \longrightarrow \Omega^0(X, \mathbf{u}(E)) \xrightarrow{d_A} \Omega^1(X, \mathbf{u}(E)) \\ \xrightarrow{d_A^+} \Omega^+(X, \mathbf{u}(E)) \xrightarrow{\bar{D}'_A} A^{0,3}(X, \mathbf{u}(E)) \longrightarrow 0. \end{aligned} \quad (3.2)$$

This is the same as that of the Hermitian-Einstein connections in Section 2.2, but it still makes sense in the almost complex setting. Hence the following just reduces to the case in (3.2), and it was proved by Reyes Carrión [41].

**PROPOSITION 3.1.** *If  $(A, u) \in \mathcal{D}(E)$ , then the complex (3.1) is elliptic.*

We denote by  $H_{(A,u)}^i = H_{(A,u)}^i(Z, \mathbf{u}(E))$  the  $i$ -th cohomology of the complex (3.1) for  $i = 0, 1, 2$ .

The complex (3.2) has the associated Dolbeault complex as Kim [30] described it in the Kähler case (see also [31, Chapter VII, §2]):

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & \Omega^0 & \xrightarrow{d_A} & \Omega^1 & \xrightarrow{d_A^+} & \Omega^+ & \xrightarrow{\bar{D}'_A} & A^{0,3} & \xrightarrow{\bar{D}_A} & 0 \\
 & & \downarrow j_0 & & \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 & & \\
 0 & \longrightarrow & \Omega^{0,0} & \xrightarrow{\bar{\partial}_A} & \Omega^{0,1} & \xrightarrow{\bar{\partial}_A} & \Omega^{0,2} & \xrightarrow{\bar{\partial}_A} & \Omega^{0,3} & \xrightarrow{\bar{\partial}_A} & 0,
 \end{array} \tag{3.3}$$

where  $j_0$  is injective,  $j_1$  is bijective,  $j_2$  is surjective with the kernel  $\{\beta\omega : \beta \in \Omega^0\}$ , and  $j_3$  is bijective. Hence the index of the complex (3.2), thus that of the complex (3.1), can be expressed by that of the Dolbeault complex above, which is given by  $\int_Z \hat{A}(Z) \wedge ch(K_Z^{1/2}) \wedge ch(\mathbf{u}(E))$  (see [23, §3.5]). In the Kähler case, the index can be computed as

$$\int_Z c_1(Z) \wedge \left( \frac{r-1}{2} c_1(E)^2 - r c_2(E) \right) + r^2 \sum_{i=0}^3 (-1)^i \dim H^{0,i}(Z).$$

Note that the index is zero if  $Z$  is a Calabi-Yau threefold.

**3.2. KURANISHI MODEL AND THE LOCAL DESCRIPTION OF THE MODULI SPACE.** We denote by  $\mathcal{C}_k(E)$ ,  $\mathcal{C}_k^*(E)$ ,  $\mathcal{D}_k(E)$ ,  $\mathcal{D}_k^*(E)$  the  $L_k^2$ -completions of  $\mathcal{C}(E)$ ,  $\mathcal{C}^*(E)$ ,  $\mathcal{D}(E)$ ,  $\mathcal{D}^*(E)$  respectively, and by  $\mathcal{G}_{k+1}(E)$  the  $L_{k+1}^2$ -completion of  $\mathcal{G}(E)$ . We take  $k$  sufficiently large so that  $\mathcal{G}_{k+1}$  becomes a Hilbert Lie group acting smoothly on  $\mathcal{C}_k(E)$ , the quotient topology  $\mathcal{C}_k(E)/\mathcal{G}_{k+1}(E)$  becomes Hausdorff (see e.g. [22, §3]), and to use implicit function theorems for the Sobolev spaces. A general reference for the Sobolev spaces and the implicit function theorems on them for our purpose is, for example, [52].

SLICE. We define *slice*  $S_{(A,u),\varepsilon}$  at  $(A, u)$  in  $\mathcal{C}_k(E)$  by

$$S_{(A,u),\varepsilon} := \left\{ (\alpha, v) \in L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3})) \right. \\ \left. : D_{(A,u)}^*(\alpha, v) = 0, \|(\alpha, v)\|_{L_k^2} \leq \varepsilon \right\}.$$

This set  $S_{(A,u),\varepsilon}$  is transverse to the  $\mathcal{G}_{k+1}$ -orbit through  $(A, u)$  as  $\ker D_{(A,u)}^*$  is orthogonal, with respect to the  $L^2$ -norm in  $L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3}))$ , to  $\text{Im } D_{(A,u)}$ . There is a natural map

$$P_{(A,u),\varepsilon} : S_{(A,u),\varepsilon} \longrightarrow \mathcal{C}_k(E)/\mathcal{G}_{k+1}(E), \quad (\alpha, v) \longmapsto [(A + \alpha, u + v')],$$

where  $v' = j_3(v)$ , and  $j_3 : A^{0,3} \rightarrow \Omega^{0,3}$  is the map in (3.3).

In the following, we take  $(A, u) \in \mathcal{C}_k^*(E)$  for simplicity.

PROPOSITION 3.2. *Let  $(A, u) \in \mathcal{C}_k^*(E)$ . Then there exists  $\varepsilon > 0$  such that  $S_{(A,u),\varepsilon}$  is diffeomorphic to  $P_{(A,u),\varepsilon}(S_{(A,u),\varepsilon})$  in  $\mathcal{C}_k^*(E)/\hat{\mathcal{G}}_{k+1}(E)$ .*

*Proof.* This is a familiar claim in gauge theory, the proof is a modification of known results for the ASD and the Hermitian-Einstein connections (cf. [16, Theorem 6], [22, Theorem 3.2 and Theorem 4.4], [31, Chapter VII, §4, Theorem 4.16], and [36, Proposition 4.2.1]). We divide the proof into two steps:

Step 1. We consider a map  $f_{(A,u)} : S_{(A,u),\varepsilon} \times \hat{\mathcal{G}}_{k+1}(E) \rightarrow \mathcal{C}_k^*(E)$  defined by  $f_{(A,u)}((\alpha, v), g) = g(A + \alpha, u + v')$ . Then the differential of  $f_{(A,u)}$  at  $((0, 0), id)$  is given by

$$Df_{(A,u)}|_{((0,0),id)}((\beta, \varphi), s) = (\beta, \varphi) + D_{(A,u)}(s).$$

As  $\text{Im } D_{(A,u)}$  and  $\ker D_{(A,u)}^*$  are  $L^2$ -orthogonal in  $L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3}))$ ,  $Df_{(A,u)}|_{((0,0),id)}$  is injective if  $(A, u)$  is irreducible.

On the other hand, associated to the operator

$$D_{(A,u)}^* D_{(A,u)} : L_{k+1}^2(\mathfrak{u}(E) \otimes \Lambda^0)/\mathfrak{u}(1) \longrightarrow L_{k-1}^2(\mathfrak{u}(E) \otimes \Lambda^0)/\mathfrak{u}(1),$$

where

$$L_{k+1}^2(\mathfrak{u}(E) \otimes \Lambda^0)/\mathfrak{u}(1) = \{s \in L_{k+1}^2(\mathfrak{u}(E) \otimes \Lambda^0) : \int_Z \text{tr}(s) \text{vol}_g = 0\},$$

there exist the Green operator  $G^0 : L_k^2(\mathfrak{u}(E) \otimes \Lambda^0)/\mathfrak{u}(1) \rightarrow L_k^2(\mathfrak{u}(E) \otimes \Lambda^0)/\mathfrak{u}(1)$  and the harmonic projection  $H^0 : L_k^2(\mathfrak{u}(E) \otimes \Lambda^0)/\mathfrak{u}(1) \rightarrow L_k^2(\mathfrak{u}(E) \otimes \Lambda^0)/\mathfrak{u}(1)$  with the identity:

$$Id = H^0 + D_{(A,u)}^* D_{(A,u)} \circ G^0$$

(see e.g. [52, Chapter IV, §5]). From the identity, we obtain

$$D_{(A,u)}^*((\gamma, \chi) - D_{(A,u)}G^0D_{(A,u)}^*(\gamma, \chi)) = 0$$

for any  $(\gamma, \chi) \in L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3}))$ . Thus, for a given  $(\gamma, \chi) \in L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3}))$ , we take  $(\beta, \varphi) = (\gamma, \chi) - D_{(A,u)}G^0D_{(A,u)}^*(\gamma, \chi)$ ,  $s = G^0D_{(A,u)}^*(\gamma, \chi)$  to get  $(\gamma, \chi) = (\beta, \varphi) + D_{(A,u)}(s)$ . Therefore  $Df_{(A,u)}|_{((0,0), id)}$  is surjective.

We then use an inverse mapping theorem for the Hilbert spaces (see e.g. [35, Chapter 6]) to deduce that around  $(A, u)$ ,  $\mathcal{C}_k^*(E)$  is locally diffeomorphic to a neighbourhood of  $((A, u), id)$  in  $S_{(A,u),\varepsilon} \times \hat{\mathcal{G}}_{k+1}(E)$ .

Step 2. We then prove that if for  $(\alpha_1, v_1), (\alpha_2, v_2) \in S_{(A,u),\varepsilon}$  there exists  $g \in \overline{\mathcal{G}}_{k+1}(E)$  such that

$$(A + \alpha_1, \tilde{u} + v_1) = g(A + \alpha_2, \tilde{u} + v_2), \quad (3.4)$$

then  $cg$  is close to  $id_E$  in  $L_{k+1}^2$  for some  $c \in U(1)$ .

Since we assume that  $(A, u)$  is irreducible, we can take  $c \in U(1)$  so that  $g' = cg - id_E \in \ker(D_{(A,u)})^\perp$ . From (3.4), we get

$$d_A g' = \alpha_1 g' - g' \alpha_2 + (\alpha_1 - \alpha_2), \quad [\tilde{u}, g'] = g' v_1 - v_2 g' + v_1 - v_2.$$

Hence,

$$D_{(A,u)} g' = (\alpha_1 g' - g' \alpha_2 + \alpha_{12}, g' v_1 - v_2 g' + v_{12}), \quad (3.5)$$

where  $\alpha_{12} = \alpha_1 - \alpha_2$ ,  $v_{12} = v_1 - v_2$ .

Since  $g'$  lies in  $(\ker D_{(A,u)})^\perp$ , there exists a constant  $C > 0$  independent of  $(A, u)$  and  $g'$  such that  $\|g'\|_{L_{k+1}^2} \leq C \|D_{(A,u)} g'\|_{L_k^2}$ . Thus, using (3.5), we obtain

$$\|g'\|_{L_{k+1}^2} \leq C \left( \|g'\|_{L_k^2} \left( \|\alpha_1\|_{L_k^2} + \|\alpha_2\|_{L_k^2} + \|v_2\|_{L_k^2} \right) + \|\alpha_{12}\|_{L_k^2} + \|v_{12}\|_{L_k^2} \right).$$

Hence,

$$\|g'\|_{L_{k+1}^2} \leq \frac{C}{1 - 3\varepsilon C} \left( \|\alpha_{12}\|_{L_k^2} + \|v_{12}\|_{L_k^2} \right)$$

for  $\varepsilon < 1/3C$ . Thus, we get  $\|cg - id_E\|_{L_{k+1}^2} < C'\varepsilon$  for  $\varepsilon$  small, where  $C'$  is a positive constant.

From this, the assertion of the lemma is reduced to Step 1.  $\blacksquare$

*Remark 3.3.* By modifying the proof of Lemma 3.2, one can prove that for  $(A, u) \in \mathcal{C}_k(E)$ , there exists  $\varepsilon > 0$  such that  $S_{(A,u),\varepsilon}/\hat{\Gamma}_{(A,u)}$  is diffeomorphic to  $P_{(A,u)} \left( S_{(A,u),\varepsilon}/\hat{\Gamma}_{(A,u)} \right)$  in  $\mathcal{C}_k(E)/\hat{\mathcal{G}}_{k+1}(E)$ , where  $\hat{\Gamma}_{(A,u)} = \Gamma_{(A,u)}/U(1)$ , following, for example, [22, Theorem 4.4].

KURANISHI MODEL. This is also a familiar picture in gauge theory. We describe it for the Donaldson-Thomas instanton case, modifying known results in the ASD and Hermitian-Einstein connections (cf. [16, Proposition 8], [31, Chapter VII, §4, Theorem 4.20], and [36, Proposition 4.5.3]). We take  $(A, u) \in \mathcal{D}_k(E)$ , and consider a deformation  $(A + \alpha, u + v') \in \mathcal{D}_k(E)$ , where

$$(\alpha, v) \in L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3})).$$

Then,  $(\alpha, v)$  satisfies the following:

$$d_A^+ \alpha + \pi^+(\alpha \wedge \alpha) + B_u(v) + \Lambda^2[v, \bar{v}] + \bar{D}_A^* v + \bar{*} \alpha \bar{*} v = 0, \quad (3.6)$$

where  $B_u(v) := \Lambda^2([u, \bar{v}] + [v, \bar{u}])$ .

Associated to the operator

$$D_{(A,u)}^+(D_{(A,u)}^+)^* : L_k^2(\mathfrak{u}(E) \otimes \Lambda^+) \longrightarrow L_k^2(\mathfrak{u}(E) \otimes \Lambda^+),$$

there exist the Green operator  $G^2 : L_k^2(\mathfrak{u}(E) \otimes \Lambda^+) \rightarrow L_k^2(\mathfrak{u}(E) \otimes \Lambda^+)$  and the harmonic projection  $H : L_k^2(\mathfrak{u}(E) \otimes \Lambda^+) \rightarrow L_k^2(\mathfrak{u}(E) \otimes \Lambda^+)$  with the identity:

$$Id = H + D_{(A,u)}^+(D_{(A,u)}^+)^* \circ G^2$$

(see e.g. [52, Chapter IV, §5]). Using these, we define a map

$$K_{(A,u)} : L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3})) \longrightarrow L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3}))$$

by

$$K_{(A,u)}(\alpha, v) := \left( \alpha + (d_A^+)^* \circ G^2 \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*} \alpha \bar{*} v), \right. \\ \left. v + (\bar{D}_A' + (B_u^*)') \circ G^2 \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*} \alpha \bar{*} v) \right),$$

where  $(B_u^*)' = B_u^* \circ \pi^\omega$ ,  $B_u^* : \Omega^0 \omega \rightarrow A^{0,3}$  is the adjoint of  $B_u$ , and  $\pi^\omega$  is the orthogonal projection from  $\Omega^2$  to  $\Omega^0 \omega$ .

LEMMA 3.4. *A pair  $(\alpha, v) \in L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3}))$  satisfies (3.6) if and only if it satisfies  $D_{(A,u)}^+ K_{(A,u)}(\alpha, v) = 0$  and*

$$H(\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*} \alpha \bar{*} v) = 0.$$

*Proof.* Using the identity  $Id = H + D_{(A,u)}^+(D_{(A,u)}^+)^* \circ G^2$ , we rewrite the left-hand side of (3.6) as

$$\begin{aligned} & d_A^+(\alpha + (d_A^+)^* \circ G^2 \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v)) + B_u(v) \\ & + \bar{D}_A^*(v + (\bar{D}'_A + (B_u^*)') \circ G^2 \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v)) \\ & + H \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v) \\ & = D_{(A,u)}^+K_{(A,u)} + H \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v). \end{aligned} \quad (3.7)$$

Hence, if  $D_{(A,u)}^+K_{(A,u)} = 0$  and  $H \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v) = 0$ , then (3.6) holds.

Conversely, if (3.6) holds, then from (3.7) we get

$$D_{(A,u)}^+K_{(A,u)} + H \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v) = 0.$$

Thus,  $(D_{(A,u)}^+)^*D_{(A,u)}^+K_{(A,u)} = 0$ . This implies  $\|D_{(A,u)}^+K_{(A,u)}\|_{L^2_{k-1}(\mathfrak{u}(E) \otimes \Lambda^+)} = 0$ , hence,  $D_{(A,u)}^+K_{(A,u)} = 0$  and  $H \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v) = 0$ . ■

We put  $S_{(A,u),\varepsilon}^d := \{(\alpha, v) \in S_{(A,u),\varepsilon} : (\alpha, v) \text{ satisfies (3.6)}\}$ , and denote by  $\mathbf{H}_{(A,u)}^i(Z, \mathfrak{u}(E))$  ( $i = 0, 1, 2$ ) the harmonic spaces of the complex (3.1).

LEMMA 3.5.  $K_{(A,u)}(S_{(A,u),\varepsilon}^d) \subset \mathbf{H}_{(A,u)}^1(Z, \mathfrak{u}(E))$ .

*Proof.* From the definition of the map  $K_{(A,u)}$ , we have

$$\begin{aligned} D_{(A,u)}^*K_{(A,u)}(\alpha, v) &= D_{(A,u)}^*(\alpha, v) \\ &+ D_{(A,u)}^*(D_{(A,u)}^+)^*(G^2 \circ (\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v)) \end{aligned}$$

for  $(\alpha, v) \in S_{(A,u),\varepsilon}^d$ . This is equal to 0, because  $D_{(A,u)}^*(\alpha, v) = 0$  for  $(\alpha, v) \in S_{(A,u),\varepsilon}^d$ , and  $D_{(A,u)}^*(D_{(A,u)}^+)^* = 0$  as  $D_{(A,u)}^+D_{(A,u)} = 0$ . From Lemma 3.4, we also have  $D_{(A,u)}^+K_{(A,u)} = 0$ . Thus Lemma 3.5 holds. ■

From Lemmas 3.4 and 3.5, we deduce the following.

LEMMA 3.6. A pair  $(\alpha, v) \in L_k^2(\mathfrak{u}(E) \otimes (\Lambda^1 \oplus A^{0,3}))$  lies in  $S_{(A,u),\varepsilon}^d$  if and only if  $K_{(A,u)}(\alpha, v) \in \mathbf{H}_{(A,u)}^1(Z, \mathfrak{u}(E))$  and

$$H(\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v) = 0.$$

We now prove the following.

**THEOREM 3.7.** *Let  $(A, u) \in \mathcal{D}^*(E)$ . Then there exists a neighbourhood  $U$  of 0 in  $\mathbf{H}_{(A,u)}^1(Z, \mathbf{u}(E))$  such that around  $[(A, u)]$  the moduli space  $\mathcal{M}^*(E) = \mathcal{D}^*(E)/\hat{\mathcal{G}}(E)$  is locally modeled on the zero set of a real analytic map  $\kappa_{(A,u)} : U \rightarrow \mathbf{H}_{(A,u)}^2(Z, \mathbf{u}(E))$  with  $\kappa_{(A,u)}(0) = 0$ , and the first derivative of  $\kappa_{(A,u)}$  at 0 also vanishes.*

*Proof.* From the definition of the map  $K_{(A,u)}$ , we have  $K_{(A,u)}(0) = 0$ . Since the differential of  $K_{(A,u)}$  at 0 is identity, we can deduce, from the inverse mapping theorem on the Hilbert spaces (see e.g. [35, Chapter 6]), that there exist a neighbourhood  $U$  of 0 in  $\mathbf{H}_{(A,u)}^1(Z, \mathbf{u}(E))$  and a map

$$K_{(A,u)}^{-1} : U \longrightarrow L_k^2(\mathbf{u}(E)) \otimes (\Lambda^1 \oplus A^{0,3})$$

such that  $K_{(A,u)}^{-1}$  is a diffeomorphism between  $U$  and  $K_{(A,u)}^{-1}(U)$ . We then define a map  $\kappa_{(A,u)} : U \rightarrow \mathbf{H}_{(A,u)}^2$  by  $\kappa_{(A,u)} = \psi \circ K_{(A,u)}^{-1}$ , where  $\psi : \mathbf{H}_{(A,u)}^1 \rightarrow \mathbf{H}_{(A,u)}^2$  is defined by

$$\psi(\alpha, v) = H(\pi^+(\alpha \wedge \alpha) + \Lambda^2[v, \bar{v}] + \bar{*}\alpha\bar{*}v).$$

We now take  $\varepsilon$  sufficiently small so that all the following hold. Firstly, from Lemma 3.6, the zero set of  $\kappa_{(A,u)}$  is mapped by  $K_{(A,u)}^{-1}$  diffeomorphically to an open subset in  $S_{(A,u),\varepsilon}^d$ . Next, from Proposition 3.2,  $S_{(A,u),\varepsilon}^d$  is diffeomorphic to  $p_{(A,u),\varepsilon}(S_{(A,u),\varepsilon}^d)$  in  $\mathcal{D}_k^*(E)/\hat{\mathcal{G}}_{k+1}(E)$ . Hence, the zero set of  $\kappa_{(A,u)}$  is diffeomorphic to a neighbourhood of  $[(A, u)]$  in  $\mathcal{D}_k^*(E)/\hat{\mathcal{G}}_{k+1}(E)$ . Moreover, from the elliptic regularity, the harmonic elements are actually smooth, therefore the neighbourhood of  $[(A, u)]$  in  $\mathcal{D}_k^*(E)/\hat{\mathcal{G}}_{k+1}(E)$  is isomorphic to a neighbourhood of  $[(A, u)]$  in  $\mathcal{M}^*(E)$ .

The assertions that  $\kappa_{(A,u)} = 0$  and the derivative of  $\kappa_{(A,u)}$  at 0 is zero just follow from the definition  $\kappa_{(A,u)} = \psi \circ K_{(A,u)}^{-1}$  and the fact that the differential of  $K_{(A,u)}$  at 0 is the identity. ■

From Theorem 3.7, one can deduce that  $\mathcal{M}^*(E)$  is smooth around  $[(A, u)]$  if  $\mathbf{H}_{(A,u)}^2(Z, \mathbf{u}(E)) = 0$ . But, as in the case of the Hermitian-Einstein connections (cf. [30], [31, Chapter VII, §4], [25, Chapter 2, §2.1], [36, Chapter 4, §4.5]), it can be improved in the following way. Firstly, we note that, corresponding to the decomposition of  $\mathbf{u}(r) = i\mathbb{R} \oplus \mathfrak{su}(r)$ , the bundle  $\mathbf{u}(E)$  naturally decomposes into  $\underline{\mathbb{R}}$  and  $\mathbf{u}(E)_0$  over  $Z$ , where  $\mathbf{u}(E)_0$  is the bundle of trace-free skew-Hermitian endmorphisms of  $E$ , and there is a subcomplex of the complex (3.1),

which is defined by using the bundle  $\mathbf{u}(E)_0$  instead of  $\mathbf{u}(E)$ . The decomposition is preserved by the operators of the complex, hence it induces a corresponding splitting of  $H_{(A,u)}^i(Z, \mathbf{u}(E))$  ( $i = 0, 1, 2$ ). For  $(\alpha_c, v_c) \in \Lambda^1(Z) \oplus A^{0,3}(Z)$ , it is always  $H(\pi^+(\alpha_c \wedge \alpha_c) + \Lambda^2[v_c, \bar{v}_c] + \bar{*}\alpha_c \bar{*}v_c) = 0$ , hence the map  $\kappa_{(A,u)}$  values in  $H^2(Z, \mathbf{u}(E)_0)$ . In particular, we obtain the following.

**COROLLARY 3.8.** *Around  $[(A, u)] \in \mathcal{M}^*(E)$  with  $H_{(A,u)}^2(Z, \mathbf{u}(E)_0) = 0$ , the moduli space  $\mathcal{M}^*(E)$  is smooth.*

*Remark 3.9.* Around  $(A, u) \in \mathcal{D}(E)$ , which is not irreducible, one can prove that  $\mathbf{H}_{(A,u)}^1(Z, \mathbf{u}(E))$  and  $\mathbf{H}_{(A,u)}^2(Z, \mathbf{u}(E))$  are  $\Gamma_{(A,u)}$ -invariant, and the map  $\kappa_{(A,u)}$  is  $\Gamma_{(A,u)}$ -equivariant. Hence, combining the claim in Remark 3.3, one can deduce that around  $[(A, u)]$  the moduli space  $\mathcal{M}(E)$  is locally modeled on  $\kappa_{(A,u)}^{-1}(0)/\Gamma_{(A,u)}$ .

4. THE HITCHIN-KOBAYASHI CORRESPONDENCE FOR THE DONALDSON-THOMAS INSTANTONS ON COMPACT KÄHLER THREEFOLDS

Perhaps one might ask what kind of a Hitchin-Kobayashi style correspondence would hold for the Donaldson-Thomas instanton on compact Kähler threefolds. In this section, we describe this, which actually follows from a result by Álvarez-Cónsul and García-Prada [1].

Let  $Z$  be a compact Kähler threefold, and  $E = (E, h)$  a Hermitian vector bundle over  $Z$  with Hermitian metric  $h$ . If  $(A, u)$  is a D-T instanton on  $E$ , then the connection  $A$  defines a holomorphic structure  $\bar{\partial}_A$  on  $E$  as  $F_A^{0,2} = 0$ , thus, we can think of  $E$  as a locally free sheaf  $\mathcal{O}(E, \bar{\partial}_A)$ . In addition, the  $\text{End}(E)$ -valued  $(0, 3)$ -form  $u$  is naturally identified with a section of the bundle  $\text{End}(E) \otimes K_Z^{-1}$ , so  $\bar{*}u$  is a section of the bundle  $\text{End}(E) \otimes K_Z$ . The equation  $\bar{\partial}_A^* u = 0$  implies  $\bar{\partial}_A \bar{*}u = 0$ , hence,  $\varphi := \bar{*}u$  is a holomorphic section of  $\text{End}(E) \otimes K_Z$ .

We then consider a pair  $(\mathcal{E}, \varphi)$  consisting of a torsion-free sheaf  $\mathcal{E}$  and a holomorphic section  $\varphi$  of  $\text{End}(\mathcal{E}) \otimes K_Z$ . A subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  is said to be a  $\varphi$ -invariant if  $\varphi(\mathcal{F}) \subset \mathcal{F} \otimes K_Z$ . We define a slope  $\mu(\mathcal{F})$  of a coherent subsheaf  $\mathcal{F}$  of  $\mathcal{E}$  by

$$\mu(\mathcal{F}) := \frac{1}{\text{rank}(\mathcal{F})} \int_Z c_1(\det \mathcal{F}) \wedge \omega^2.$$

**DEFINITION 4.1.** A pair  $(\mathcal{E}, \varphi)$  consisting of a torsion-free sheaf  $\mathcal{E}$  and a holomorphic section  $\varphi$  of  $\text{End}(\mathcal{E}) \otimes K_Z$  is called *semi-stable* if  $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$  for

any  $\varphi$ -invariant coherent subsheaf  $\mathcal{F}$  with  $\text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$ . A pair  $(\mathcal{E}, \varphi)$  is called *stable* if  $\mu(\mathcal{F}) < \mu(\mathcal{E})$  for any  $\varphi$ -invariant coherent subsheaf  $\mathcal{F}$  with  $\text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$ .

DEFINITION 4.2. A pair  $(\mathcal{E}, \varphi)$  consisting of a torsion-free sheaf  $\mathcal{E}$  and a holomorphic section  $\varphi$  of  $\text{End}(\mathcal{E}) \otimes K_Z$  is said to be *poly-stable* if it is a direct sum of stable sheaves with the same slopes in the sense of Definition 4.1.

Then the correspondence can be stated as a one-to-one correspondence between a pair  $(\mathcal{E}, \varphi)$ , where  $\mathcal{E}$  is a locally-free sheaf on a Kähler threefold  $Z$  and a holomorphic section  $\varphi$  of  $\text{End}(\mathcal{E}) \otimes K_Z$ , which is stable in the sense of Definition 4.1; and the existence of a solution to the Donaldson-Thomas equations on  $\mathcal{E}$ . This fits into a setting studied by Álvarez-Cónsul and García-Prada [1] (see also [6]), and it is stated as a special case of their results as the case of a twisted quiver bundle with one vertex and one arrow, whose head and tail coincide, and with twisting sheaf the anti-canonical bundle. We state it in our setting as follows.

THEOREM 4.3. ([1]) *Let  $Z$  be a compact Kähler threefold with Kähler form  $\omega$ . Let  $(\mathcal{E}, \varphi)$  be a pair consisting of a locally-free sheaf  $\mathcal{E}$  on  $Z$  and a holomorphic section  $\varphi$  of  $\text{End}(\mathcal{E}) \otimes K_Z$ . Then,  $(\mathcal{E}, \varphi)$  is poly-stable if and only if  $\mathcal{E}$  admits a unique Hermitian metric  $h$  satisfying*

$$\Lambda F_h + \Lambda^3[\varphi, \bar{\varphi}^h] + 6\pi i \mu(\mathcal{E}) \text{Id}_{\mathcal{E}} = 0,$$

where  $F_h$  is the curvature form of  $h$ , and  $\Lambda := (\wedge \omega)^*$ .

Note that the equation  $\bar{\partial}_A^* u = 0$  in the Donaldson-Thomas equations on a compact Kähler threefold is implicitly addressed in Theorem 4.3 by saying that  $\varphi = \bar{*}u$  is a holomorphic section of  $\text{End}(\mathcal{E}) \otimes K_Z$ . One more remark is that a proof of the Hitchin-Kobayashi correspondence using the Mehta-Ramanathan argument for the Vafa-Witten equations in [46] could also apply to the Donaldson-Thomas instanton on smooth projective threefold as mentioned in [46].

#### ACKNOWLEDGEMENTS

I would like to thank Mikio Furuta, Ryushi Goto, Ryoichi Kobayashi, Hiroshi Ohta for valuable comments, and referees for many useful advice. I am also grateful to Katrin Wehrheim for wonderful encouragement. A part of this article was written when I had visited Beijing International Center for Mathematical Research, Peking University in 2008–2009,

I am very grateful to Gang Tian and the institute for their support and hospitality. A part of revision was made during my visit to Institut des Hautes Études Scientifiques in February to March of 2012. I would like to thank the institute for the support and giving me an excellent research environment. Last but not least, I would like to thank Dominic Joyce for enlightening me on these subjects over the years. This work was partially supported by JSPS Grant-in-Aid for Scientific Research No. 15H02054.

## REFERENCES

- [1] L. ÁLVAREZ-CÓNSUL, O. GARCÍA-PRADA, Hitchin-Kobayashi correspondence, quivers and vortices, *Comm. Math. Phys.* **238** (2003), 1–33.
- [2] M.F. ATIYAH, N.J. HITCHIN, I.M. SINGER, Self-duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A* **362** (1978), 425–461.
- [3] L. BAULIEU, H. KANNO, I.M. SINGER, Special quantum field theories in eight and other dimensions, *Comm. Math. Phys.* **194** (1998), 149–175.
- [4] O. BEN-BASSAT, C. BRAV, V. BUSSI, D. JOYCE, A 'Darboux Theorem' for shifted symplectic structures on derived Artin stacks, with applications, *Geom. Topol.* **19** (2015), 1287–1359.
- [5] D. BORISOV, D. JOYCE, Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds, arXiv:1504.00690.
- [6] S.B. BRADLOW, O. GARCÍA-PRADA, I. MUNDET I RIERA, Relative Hitchin-Kobayashi correspondences for principal pairs, *Q. J. Math.* **54** (2003), 171–208.
- [7] C. BRAV, V. BUSSI, D. DUPONT, D. JOYCE, B. SZENDROI, Symmetries and stabilization for sheaves of vanishing cycles, With an appendix by Jörg Schürmann, *J. Singul.* **11** (2015), 85–151.
- [8] C. BRAV, V. BUSSI, D. JOYCE, A Darboux theorem for derived schemes with shifted symplectic structure, arXiv:1305.6302.
- [9] V. BUSSI, D. JOYCE, S. MEINHARDT, On motivic vanishing cycles of critical loci, arXiv:1305.6428.
- [10] Y. CAO, Donaldson-Thomas theory for Calabi-Yau four-folds, arXiv:1309.4230.
- [11] Y. CAO, N.C. LEUNG, Donaldson-Thomas theory for Calabi-Yau fourfolds, arXiv:1407.7659.
- [12] Y. CAO, N.C. LEUNG, Orientability for gauge theories on Calabi-Yau manifolds, arXiv:1502.01141.
- [13] Y. CAO, N.C. LEUNG, Relative Donaldson-Thomas theory for Calabi-Yau four-folds, arXiv:1502.04417.
- [14] Y. CAO, N.C. LEUNG, Remarks on mirror symmetry of Donaldson-Thomas theory for Calabi-Yau 4-folds, arXiv:1506.04218.
- [15] R. DIJKGRAAF, S. GUKOV, A. NEITZKE, C. VAFA, Topological M-theory as unification of form theories of gravity, *Adv. Theor. Math. Phys.* **9** (2005), 603–665.

- [16] S.K. DONALDSON, An application of gauge theory to four dimensional topology, *J. Differential Geom.* **18** (1983), 279–315.
- [17] S.K. DONALDSON, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc. (3)* **50** (1985), 1–26.
- [18] S.K. DONALDSON, Infinite determinants, stable bundles and curvature, *Duke Math. J.* **54** (1987), 231–247.
- [19] S.K. DONALDSON, P.B. KRONHEIMER, “The Geometry of Four-Manifolds”, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1990.
- [20] S. DONALDSON, E. SEGAL, Gauge theory in higher dimensions, II, in “Surveys in Differential Geometry, Volume XVI”, Int. Press, Somerville, MA, 2011, 1–41.
- [21] S.K. DONALDSON, R.P. THOMAS, Gauge theory in higher dimensions, in “The Geometric Universe”, Oxford University Press, Oxford, 1998, 31–47.
- [22] D.S. FREED, K.K. UHLENBECK, “Instantons and Four-Manifolds”, Second edition, Springer-Verlag, New York, 1991.
- [23] P. GILKEY, “Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem”, Second edition, CRC Press, Boca Raton, FL, 1995.
- [24] N. HITCHIN, The self-duality equations on a Riemann surface, *Proc. London Math. Soc. (3)* **55** (1987), 59–126.
- [25] M. ITOH, H. NAKAJIMA, Yang-Mills connections and Einstein-Hermitian metrics, in “Kähler Metric and Moduli Spaces”, Academic Press, Boston, MA, 1990, 395–457.
- [26] A. IQBAL, N. NEKRASOV, A. OKOUNKOV, C. VAFA, Quantum foam and topological strings, *J. High Energy Phys.* **4** (2008), 011.
- [27] D. JOYCE, A classical model for derived critical loci, *J. Differential Geom.* **101** (2015), 289–367.
- [28] D. JOYCE, Y. SONG, “A Theory of Generalized Donaldson-Thomas Invariants”, *Mem. Amer. Math. Soc.* 217 (2012), no. 1020.
- [29] Y-H. KIEM, J. LI, Categorification of Donaldson-Thomas invariants via Perverse Sheaves, arXiv:1212.6444.
- [30] H. KIM, Moduli of Hermite-Einstein vector bundles, *Math. Z.* **195** (1987), 143–150.
- [31] S. KOBAYASHI, “Differential Geometry of Complex Vector Bundles”, Publications of the Mathematical Society of Japan, 15; Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo; 1987.
- [32] M. KONTSEVICH, Y. SOIBELMAN, Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, arXiv:0811.2435.
- [33] M. KONTSEVICH, Y. SOIBELMAN, Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants, *Commun. Number Theory Phys.* **5** (2011), 231–352.
- [34] M. KONTSEVICH, Y. SOIBELMAN, Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and mirror symmetry, in “Homological Mirror Symmetry and Tropical Geometry”, Lect. Notes

- Unione Mat. Ital., 15, Springer, Cham, 2014, 197–308.
- [35] S. LANG, “Real Analysis”, Second edition, Addison-Wesley Publishing Company, Reading, MA, 1983.
  - [36] M. LÜBKE, A. TELEMAN, “The Kobayashi-Hitchin Correspondence”, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.
  - [37] D. MAULIK, D.N. NEKRASOV, A. OKOUNKOV, R. PANDHARIPANDE, Gromov-Witten theory and Donaldson-Thomas theory, I, *Compos. Math.* **142** (2006), 1263–1285.
  - [38] D. MAULIK, D.N. NEKRASOV, A. OKOUNKOV, R. PANDHARIPANDE, Gromov-Witten theory and Donaldson-Thomas theory, II, *Compos. Math.* **142** (2006), 1286–1304.
  - [39] N. NEKRASOV,  $\mathbf{Z}$ -theory: chasing  $m/f$  theory, *C. R. Phys.* **6** (2005), 261–269.
  - [40] N. NEKRASOV, H. OOGURI, C. VAFA,  $S$ -duality and topological strings, *J. High Energy Phys.* **10** (2004), 009.
  - [41] R. REYES CARRIÓN, A generalization of the notion of instanton, *Differential Geom. Appl.* **8** (1998), 1–20.
  - [42] S.M. SALAMON, “Riemannian Geometry and Holonomy Groups”, Pitman Research Notes in Mathematics Series, 201; Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York; 1989.
  - [43] Y. TANAKA, A construction of  $Spin(7)$ -instantons, *Ann. Global Anal. Geom.* **42** (2012), 495–521.
  - [44] Y. TANAKA, A weak compactness theorem of the Donaldson-Thomas instantons on compact Kähler threefolds, *J. Math. Anal. Appl.* **408** (2013), 27–34.
  - [45] Y. TANAKA, A removal singularity theorem of the Donaldson-Thomas instanton on compact Kähler threefolds, *J. Math. Anal. Appl.* **411** (2014), 422–428.
  - [46] Y. TANAKA, Stable sheaves with twisted sections and the Vafa-Witten equations on smooth projective surfaces, *Manuscripta Math.* **146** (2015), 351–363.
  - [47] C.H. TAUBES, Casson’s invariant and gauge theory, *J. Differential Geom.* **31** (1990), 547–599.
  - [48] R.P. THOMAS, A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on  $K3$  fibrations, *J. Differential Geom.* **54** (2000), 367–438.
  - [49] G. TIAN, Gauge theory and calibrated geometry, I, *Ann. of Math. (2)* **151** (2000), 193–268.
  - [50] K. UHLENBECK, S-T. YAU, On the existence of Hermitian-Yang-Mills connections in stable vector bundles, *Comm. Pure Appl. Math.* **39** (suppl.) (1986), S257–S293.
  - [51] K. UHLENBECK, S-T. YAU, A note on our previous paper: On the existence of Hermitian-Yang-Mills connections in stable vector bundles, *Comm. Pure Appl. Math.* **42** (1989), 703–707.
  - [52] R.O. WELLS, JR., “Differential Analysis on Complex Manifolds”, Third edition, with a new appendix by Oscar Garcia-Prada, Graduate Texts in Mathematics, 65, Springer, New York, 2008.