On LS-Category of a Family of Rational Elliptic Spaces II

KHALID BOUTAHIR, YOUSSEF RAMI

Département de Mathématiques et Informatique, Faculté des Sciences, Université My Ismail, B. P. 11 201 Zitoune, Meknès, Morocco
khalid.boutahir@edu.umi.ac.ma y.rami@fs-umi.ac.ma

Abstract: Let X be a finite type simply connected rationally elliptic CW-complex with Sullivan minimal model \((\Lambda V, d)\) and let \(k \geq 2\) the biggest integer such that \(d = \sum_{i \geq k} d_i\) with \(d_i(V) \subseteq \Lambda^i V\). If \((\Lambda V, d_k)\) is moreover elliptic then \(\text{cat}(\Lambda V, d) = \text{cat}(\Lambda V, d_k) = \dim(V^\text{even})(k - 2) + \dim(V^\text{odd})\). Our work aims to give an almost explicit formula of LS-category of such spaces in the case when \(k \geq 3\) and when \((\Lambda V, d_k)\) is not necessarily elliptic.

Key words: Elliptic spaces, Lusternik-Schnirelman category, Toomer invariant.


1. Introduction

The Lusternik-Schnirelmann category (c.f. [7]), \(\text{cat}(X)\), of a topological space \(X\) is the least integer \(n\) such that \(X\) can be covered by \(n + 1\) open subsets of \(X\), each contractible in \(X\) (or infinity if no such \(n\) exists). It is an homotopy invariant (c.f. [3]). For \(X\) a simply connected CW complex, the rational L-S category, \(\text{cat}_0(X)\), introduced by Félix and Halperin in [2] is given by \(\text{cat}_0(X) = \text{cat}(X_Q) \leq \text{cat}(X)\).

In this paper, we assume that \(X\) is a simply connected topological space whose rational homology is finite dimensional in each degree. Such space has a Sullivan minimal model \((\Lambda V, d)\), i.e. a commutative differential graded algebra coding both its rational homology and homotopy (cf. §2).

By [1, Definition 5.22] the rational Toomer invariant of \(X\), or equivalently of its Sullivan minimal model, denoted by \(e_0(\Lambda V, d)\), is the largest integer \(s\) for which there is a non trivial cohomology class in \(H^*(\Lambda V, d)\) represented by a cocycle in \(\Lambda^{\text{wordlength}}\), this coincides in fact with the Toomer invariant of the fundamental class of \((\Lambda V, d)\). As usual, \(\Lambda^s V\) denotes the elements in \(\Lambda V\) of “wordlength” \(s\). For more details [1], [3] and [14] are standard references.

In [4] Y. Félix, S. Halperin and J. M. Lemaire showed that for Poincaré duality spaces, the rational L-S category coincides with the rational Toomer
invariant $e_0(X)$, and in [9] A. Murillo gave an expression of the fundamental class of $(AV, d)$ in the case where $(AV, d)$ is a pure model (cf. §2).

Let then $(AV, d)$ be a Sullivan minimal model. The differential $d$ is decomposable, that is, $d = \sum_{i \geq k} d_i$, with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$.

Recall first that in [8] the authors gave the explicit formula $\text{cat}(AV, d) = \dim V^{\text{odd}} + (k - 2) \dim V^{\text{even}}$ in the case when $(AV, d_k)$ is also elliptic.

The aim of this paper is to consider another class of elliptic spaces whose Sullivan minimal model $(AV, d)$ is such that $(AV, d_k)$ is not necessarily elliptic. To do this we filter this model by

$$F^p = \Lambda^{\geq (k-1)p} V = \bigoplus_{i = (k-1)p}^{\infty} \Lambda^i V. \quad (1)$$

This gives us the main tool in this work, that is the following convergent spectral sequence (cf. §3):

$$H^{p-q}(AV, \delta) \Rightarrow H^{p+q}(AV, d). \quad (2)$$

Notice first that, if $\dim(V) < \infty$ and $(AV, \delta)$ has finite dimensional cohomology, then $(AV, d)$ is elliptic. This gives a new family of rationally elliptic spaces.

In the first step, we shall treat the case under the hypothesis assuming that $H^N(AV, \delta)$ is one dimensional, being $N$ the formal dimension of $(AV, d)$ (cf. [5]). For this, we will combine the method used in [8] and a spectral sequence argument using (2). We then focus on the case where $\dim H^N(AV, \delta) \geq 2$. Our first result reads:

**Theorem 1.** If $(AV, d)$ is elliptic, $(AV, d_k)$ is not elliptic and $H^N(AV, \delta) = \mathbb{Q}, \alpha$ is one dimensional, then

$$\text{cat}_0(X) = \text{cat}(AV, d) = \sup\{s \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\}.$$

Let us explain in what follows, the algorithm that gives the first inequality,

$$\text{cat}(AV, d) \geq \sup\{s \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq s} V\} := r.$$

i) Initially we fix a representative $\omega_0 \in \Lambda^{\geq r} V$ of the fundamental class $\alpha$ with $r$ being the largest $s$ such that $\omega_0 \in \Lambda^{\geq s} V.$
ii) A straightforward calculation gives successively:
\[
\omega_0 = \omega_0^0 + \omega_0^1 + \cdots + \omega_0^l
\]
with
\[
\omega_0^i = (\omega_{i,0}^{0,0}, \omega_{i,1}^{0,0}, \ldots, \omega_{i,k}^{0,0} - 20) \in \Lambda^{(k-1)(p+i)}V \oplus \Lambda^{(k-1)(p+i)+1}V \\
\oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2}V.
\]

Using \(\delta(\omega_0) = 0\) we obtain
\[
d\omega_0 = a_0^2 + a_0^3 + \cdots + a_0^{t+l} \text{ with }
\]
\[
a_0^i = (a_{i,0}^{0,0}, a_{i,1}^{0,0}, \ldots, a_{i,k}^{0,0} - 20) \in \Lambda^{(k-1)(p+i)}V \oplus \Lambda^{(k-1)(p+i)+1}V \\
\oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2}V.
\]

iii) We take \(t\) the largest integer satisfying the inequality:
\[
t \leq \frac{1}{2(k-1)}(N - 2(k - 1)(p + l) - 2k + 5).
\]

Since \(d^2 = 0\), it follows that \(a_2^0 = \delta(b_2)\) for some
\[
b_2 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2)-(k-1)+j}V.
\]

iv) We continue with \(\omega_1 = \omega_0 - b_2\).

v) By the imposition iii), the algorithm leads to a representative \(\omega_{t+l-1} \in \Lambda^{\geq r}V\) of the fundamental class of \((\Lambda V, d)\) and then \(c_0(\Lambda V, d) \geq r\).

Now, \(\dim(V) < \infty\) imply \(\dim H^N(\Lambda V, \delta) < \infty\). Notice also that the filtration (1) induces on cohomology a gradation such that \(H^N(\Lambda V, \delta) = \bigoplus_{p+q=N} H^p_q(\Lambda V, \delta)\). There is then a basis \(\{\alpha_1, \ldots, \alpha_m\}\) of \(H^N(\Lambda V, \delta)\) with \(\alpha_j \in H^p_q(\Lambda V, \delta), (1 \leq j \leq m)\). Denote by \(\omega_{0j} \in \Lambda^{\geq r_j}V\) a representative of the generating class \(\alpha_j\) with \(r_j\) being the largest \(s_j\) such that \(\omega_{0j} \in \Lambda^{\geq s_j}V\). Here \(p_j\) and \(q_j\) are filtration degrees and \(r_j \in \{p_j(k-1), \ldots, p_j(k-1)+k-2\}\).

The second step in our program is given as follow:

**Theorem 2.** If \((\Lambda V, d)\) is elliptic and \(\dim H^N(\Lambda V, \delta) = m\) with basis \(\{\alpha_1, \ldots, \alpha_m\}\), then, there exists a unique \(p_j\), such that
\[
cat(X) = \sup\{s \geq 0, \alpha_j = [\omega_{0j}] \text{ with } \omega_{0j} \in \Lambda^{\geq s}V\} = r_j.
\]
Remark 1. The previous theorem gives us also an algorithm to determine LS-category of any elliptic Sullivan minimal model \((\Lambda V, d)\). Knowing the largest integer \(k \geq 2\) such that \(d = \sum_{i \geq k} d_i\) with \(d_i(V) \subseteq \Lambda^i V\) and the formal dimension \(N\) (this one is given in terms of degrees of any basis elements of \(V\)), one has to check for a basis \(\{\alpha_1, \ldots, \alpha_m\}\) of \(H^N(\Lambda V, \delta)\) (which is finite dimensional since \(\dim(V) < \infty\)). The NP-hard character of the problem into question, as it is proven by L. Lechuga and A. Murillo (cf [12]), sits in the determination of the unique \(j \in \{1, \ldots, m\}\) for which a represent cocycle \(\omega_0\) of \(\alpha_j\) survives to reach the \(E_\infty\) term in the spectral sequence (2).

2. Basic facts

We recall here some basic facts and notation we shall need.

A simply connected space \(X\) is called rationally elliptic if \(\dim H^*(X, \mathbb{Q}) < \infty\) and \(\dim(X) \otimes \mathbb{Q} < \infty\).

A commutative graded algebra \(H\) is said to have formal dimension \(N\) if \(H^p = 0\) for all \(p > N\), and \(H^N \neq 0\). An element \(0 \neq \omega \in H^N\) is called a fundamental class.

A Sullivan algebra ([3]) is a free commutative differential graded algebra (cdga for short) \((\Lambda V, d)\) (where \(\Lambda V = \text{Exterior}(V^{\text{odd}}) \otimes \text{Symmetric}(V^{\text{even}})\)) generated by the graded \(\mathbb{K}\)-vector space \(V = \bigoplus_{i=0}^{\infty} V^i\) which has a well ordered basis \(\{x_\alpha\}\) such that \(dx_\alpha \in \Lambda^i V\). Such algebra is said minimal if \(\deg(x_\alpha) < \deg(x_\beta)\) implies \(\alpha < \beta\). If \(V^0 = V^1 = 0\) this is equivalent to saying that \(d(V) \subseteq \bigoplus_{i=2}^{\infty} \Lambda^i V\).

A Sullivan model ([3]) for a commutative differential graded algebra \((A, d)\) is a quasi-isomorphism (morphism inducing isomorphism in cohomology) \((\Lambda V, d) \rightarrow (A, d)\) with source, a Sullivan algebra. If \(H^0(A) = K, H^1(A) = 0\) and \(\dim(H^i(A, d)) < \infty\) for all \(i \geq 0\), then, [6, Th. 7.1], this minimal model exists. If \(X\) is a topological space any minimal model of the polynomial differential forms on \(X, A_{PL}(X)\), is said a Sullivan minimal model of \(X\).

\((\Lambda V, d)\) (or \(X\)) is said elliptic, if both \(V \) and \(H^*(\Lambda V, d)\) are finite dimensional graded vector spaces (see for example [3]).

A Sullivan minimal model \((\Lambda V, d)\) is said to be pure if \(d(V^{\text{even}}) = 0\) and \(d(V^{\text{odd}}) \subseteq \Lambda V^{\text{even}}\). For such one, A. Murillo [9] gave an expression of a cocycle representing the fundamental class of \(H(\Lambda V, d)\) in the case where \((\Lambda V, d)\) is elliptic. We recall this expression here:

Assume \(\dim V < \infty\), choose homogeneous basis \(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_m\}\)
of $V_{even}$ and $V_{odd}$ respectively, and write

$$dy_j = a_1^j x_1 + a_2^j x_2 + \cdots + a_{n-1}^j x_{n-1} + a_n^j x_n, \quad j = 1, 2, \ldots, m,$$

where each $a_i^j$ is a polynomial in the variables $x_i, x_{i+1}, \ldots, x_n$, and consider the matrix,

$$A = \begin{pmatrix}
    a_1^1 & a_2^1 & \cdots & a_n^1 \\
    a_1^2 & a_2^2 & \cdots & a_n^2 \\
    \vdots & \vdots & \ddots & \vdots \\
    a_1^m & a_2^m & \cdots & a_n^m
\end{pmatrix}.$$

For any $1 \leq j_1 < \cdots < j_n \leq m$, denote by $P_{j_1 \ldots j_n}$ the determinant of the matrix of order $n$ formed by the columns $i_1, i_2, \ldots, i_n$ of $A$:

$$\begin{pmatrix}
    a_{j_1}^1 & \cdots & a_{j_n}^n \\
    \vdots & \ddots & \vdots \\
    a_{j_n}^1 & \cdots & a_{j_1}^n
\end{pmatrix}.$$

Then (see [9]) if $\dim H^*(\Lambda V, d) < \infty$, the element $\omega \in \Lambda V$,

$$\omega = \sum_{1 \leq j_1 < \cdots < j_n \leq m} (-1)^{j_1 + \cdots + j_n} P_{j_1 \ldots j_n} y_1 \cdots \hat{y}_{j_1} \cdots \hat{y}_{j_n} \cdots y_m, \quad (3)$$

is a cocycle representing the fundamental class of the cohomology algebra.

3. Our spectral sequence

Let $(\Lambda V, d)$ be a Sullivan minimal model, where $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$. We first recall the filtration given in the introduction:

$$F^p = \bigoplus_{i=0}^{k-1} \Lambda^{i+p} V = \bigoplus_{i=0}^{k-1} \Lambda^i V. \quad (4)$$

$F^p$ is preserved by the differential $d$ and satisfies $F^p(\Lambda V) \otimes F^q(\Lambda V) \subseteq F^{p+q}(\Lambda V)$,

$\forall p, q \geq 0$, so it is a filtration of differential graded algebras. Also, since
$F^0 = \Lambda V$ and $F^{p+1} \subseteq F^p$ this filtration is decreasing and bounded, so it induces a convergent spectral sequence. Its $0^{th}$-term is

$$E_{0}^{p,q} = \left( \frac{F^p}{F^{p+1}} \right)^{p+q} = \left( \Lambda^{\geq p(k-1)} V \Lambda^{\geq (p+1)(k-1)} V \right)^{p+q}.$$  

Hence, we have the identification:

$$E_0^{p,q} = \left( \Lambda^{p(k-1)} V \oplus \Lambda^{p(k-1)+1} V \oplus \cdots \oplus \Lambda^{p(k-1)+k-2} V \right)^{p+q},$$  

with the product given by:

$$(u_0, u_1, \ldots, u_{k-2}) \otimes (u'_0, u'_1, \ldots, u'_{k-2}) = (v_0, v_1, \ldots, v_{k-2})$$

for all $(u_0, u_1, \ldots, u_{k-2})$, $(u'_0, u'_1, \ldots, u'_{k-2}) \in E_0^{p,q}$ with $v_m = \sum_{i+j=m} u_i u'_j$ and $m = 0, \ldots, k-2$.

The differential on $E_0$ is zero, hence $E_1^{p,q} = E_0^{p,q}$ and so the identification above gives the following diagram:

$$
\begin{array}{ccc}
E_1^{p,q} & \xrightarrow{\delta} & \left( \Lambda^{(k-1)p} V \oplus \Lambda^{(k-1)p+1} V \oplus \cdots \oplus \Lambda^{(k-1)p+k-2} V \right)^{p+q} \\
\end{array}
$$

with $\delta$ defined as follows,

$$\delta(u_0, u_1, \ldots, u_{k-2}) = (w_k, w_{k+1}, \ldots, w_{2k-2}) \quad \text{with} \quad w_{k+j} = \sum_{i+j' = j} d_k u_i u'_{j'}.$$

Let $E_1^p = E_1^{p,*} = \bigoplus_{q \geq 0} E_1^{p,q}$ and $E_1^* = \bigoplus_{p \geq 0} E_1^{p,*} = \Lambda V$ as a graded vector space. In this general situation, the $1^{st}$-term is the graded algebra $\Lambda V$ provided with a differential $\delta$, which is not necessarily a derivation on the set $V$ of generators. That is, $(\Lambda V, \delta)$ is a commutative differential graded algebra, but it is not a Sullivan algebra. This gives, consequently, our spectral sequence:

$$E_2^{p,q} = H^{p,q}(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d).$$  

Once more, using this spectral sequence, the algorithm completed by proves of claims that will appear, will give the appropriate generating class of $H^N(\Lambda V, \delta)$ that survives to the $\infty$ term. Accordingly, the explicit formula of LS category for this general case, is expressed in terms of the greater filtering degree of a represent of this class.
4. Proof of the main results

4.1. Proof of Theorem 1. Recall that \((\Lambda V, d)\) is assumed elliptic, so that, \(\text{cat}(\Lambda V, d) = e_0(\Lambda V, d)\) [4]. Notice also that the subsequent notations imposed us sometimes to replace a sum by some tuple and vice-versa.

4.1.1. The first inequality. In what follows, we put:

\[ r = \sup \{ s \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{2^s} V \} \]

Denote by \(p\) the least integer such that \(p(k - 1) \leq r < (p + 1)(k - 1)\) and let then \(\omega_0 \in \Lambda^{2^p} V\). We have

\[ \omega_0 \in (\Lambda^{(k-1)p} V \oplus \cdots \oplus \Lambda^{(k-1)p+k-2} V) \oplus (\Lambda^{(k-1)p+k-1} V \oplus \cdots \oplus \Lambda^{(k-1)p+2k-3} V) \oplus \cdots \]

Since \(|\omega_0| = N\) and \(\dim V < \infty\), there is an integer \(l\) such that

\[ \omega_0 = \omega_0^0 + \omega_0^1 + \cdots + \omega_0^l \]

with \(\omega_0^0 \neq 0\) and \(\forall i = 0, \ldots, l, \omega_0^i = (\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}) \in \Lambda^{(k-1)(p+i)} V \oplus \cdots \oplus \Lambda^{(k-1)(p+i)+k-2} V\).

We have successively:

\[
\delta(\omega_0^i) = \delta \left( \omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2} \right) = \left( \sum_{i'+i''=1} d_k \omega_0^{i,0}, \sum_{i'+i''=2} d_k \omega_0^{i,1}, \ldots, \sum_{i'+i''=k-2} d_k \omega_0^{i,k-2} \right),
\]

\[
\delta(\omega_0) = \sum_{i=0}^l \delta(\omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2}) \quad = \sum_{i=0}^l \left( \sum_{i'+i''=1} d_k \omega_0^{i,0}, \sum_{i'+i''=2} d_k \omega_0^{i,1}, \ldots, \sum_{i'+i''=k-2} d_k \omega_0^{i,k-2} \right).
\]

Also, we have \( d\omega_0 = d\omega_0^0 + d\omega_0^1 + \cdots + d\omega_0^l \), with:

\[
d\omega_0^0 = d \left( \omega_0^{0,0}, \omega_0^{0,1}, \ldots, \omega_0^{0,k-2} \right) \\
= \left( d_k \omega_0^{0,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{0,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{0,i''} \right) + \cdots \\
\in \left( \bigoplus_{k'=k-1}^{2k-3} \Lambda^{(k-1)p+k'} V \right) \oplus \cdots
\]

\[
d\omega_0^1 = d \left( \omega_0^{1,0}, \omega_0^{1,1}, \ldots, \omega_0^{1,k-2} \right) \\
= \left( d_k \omega_0^{1,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{1,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{1,i''} \right) + \cdots \\
\in \left( \bigoplus_{k'=2k-2}^{3k-4} \Lambda^{(k-1)p+k'} V \right) \oplus \cdots
\]

\[
\vdots
\]

\[
d\omega_0^l = d \left( \omega_0^{i,0}, \omega_0^{i,1}, \ldots, \omega_0^{i,k-2} \right) \\
= \left( d_k \omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{i,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{i,i''} \right) + \cdots \\
\in \left( \bigoplus_{k'=(k-1)p+(i+2)k-(i+3)}^{(k-1)p+(i+1)k-(i+1)} \Lambda^{(k-1)p+k'} V \right) \oplus \cdots
\]

Therefore

\[
d\omega_0 = \sum_{i=0}^{l} \left( d_k \omega_0^{i,0}, \sum_{i'+i''=1} d_{k+i'} \omega_0^{i,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} \omega_0^{i,i''} \right) \\
+ \sum_{i=0}^{l} \left( d_{2k-2} \omega_0^{i,1} + (d_{2k-2} + d_{2k-3}) \omega_0^{i,2} + \cdots + (d_{2k-2} + d_{2k-3} + \cdots + d_{k+1}) \omega_0^{i,k-2} \right) + \sum_{k'>2k-2} d_{k'} \omega_0
\]
that is:

\[ d\omega_0 = \delta(\omega_0) + \sum_{i=0}^l \left( d_{2k-2}\omega_0^{i,1} + (d_{2k-2} + d_{2k-3})\omega_0^{i,2} + \cdots + (d_{2k-2} + \cdots + d_{k+1})\omega_0^{i,k-2} \right) + \sum_{k' > 2k-2} d_{k'}\omega_0. \]

As \( \delta(\omega_0) = 0 \), we can rewrite:

\[ d\omega_0 = a_0^0 + a_3^0 + \cdots + a_{t+l}^0 \quad \text{with} \quad a_i^0 = (a_i^{0,0}, a_i^{0,1}, \ldots, a_i^{0,k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+i)+j} V. \]

Note also that \( t \) is a fixed integer. Indeed, the degree of \( a_{t+l}^0 \) is greater than or equal to \( 2((k-1)(p+t+l) + k - 2) \) and it coincides with \( N + 1 \), \( N \) being the formal dimension of \( (\Lambda^* V, d) \).

Then

\[ N + 1 \geq 2((k-1)(p+t+l) + k - 2). \]

Hence

\[ t \leq \frac{1}{2(k-1)} \left( N - 2(k-1)(p+l) + 5 - 2k \right). \]

In what follows, we take \( t \) the largest integer satisfying this inequality.

Now, we have:

\[ d^2\omega_0 = da_2^0 + da_3^0 + \cdots + da_{t+l}^0 \]

\[ = d(a_2^{0,0}, a_2^{0,1}, \ldots, a_2^{0,k-2}) + d(a_3^{0,0}, a_3^{0,1}, \ldots, a_3^{0,k-2}) + \cdots + d(a_{t+l}^{0,0}, a_{t+l}^{0,1}, \ldots, a_{t+l}^{0,k-2}). \]

with

\[ d(a_2^{0,0}, a_2^{0,1}, \ldots, a_2^{0,k-2}) = d_k(a_2^{0,0}, a_2^{0,1}, \ldots, a_2^{0,k-2}) \]

\[ + d_{k+1}(a_2^{0,0}, a_2^{0,1}, \ldots, a_2^{0,k-2}) + \cdots \]

\[ = \left( d_k a_2^{0,0}, \sum_{i'+i''=1}^{k+1} d_{k+i'} a_2^{0,i''} , \ldots, \sum_{i'+i''=k-2} d_{k+i'} a_2^{0,i''} \right) \]

\[ + \left( d_{2k-1} a_2^{0,0} + d_{2k-2} a_2^{0,1} + \cdots, \right) + \cdots \]
\[ d(a_0^{0,0}, a_1^{0,1}, \ldots, a_{k-2}^{0,k-2}) = d_k(a_0^{0,0}, a_1^{0,1}, \ldots, a_{k-2}^{0,k-2}) + d_{k+1}(a_0^{0,0}, a_1^{0,1}, \ldots, a_{k-2}^{0,k-2}) + \cdots \]

\[ = \left( d_k a_0^{0,0}, \sum_{i' + i'' = 1} d_{k+i'} a_3^{0,i''}, \ldots, \sum_{i' + i'' = k-2} d_{k+i'} a_3^{0,i''} \right) \]

\[ + \left( d_{2k-1} a_3^{0,0} + d_{2k-2} a_3^{0,1} + \cdots, \ldots \right) + \cdots \]

It follows that:

\[ d^2 \omega_0 = \left( d_k a_0^{0,0}, \sum_{i' + i'' = 1} d_{k+i'} a_2^{0,i''}, \ldots, \sum_{i' + i'' = k-2} d_{k+i'} a_2^{0,i''} \right) \]

\[ + \left( d_{2k-1} a_2^{0,0} + d_{2k-2} a_2^{0,1} + \cdots, \ldots \right) + \cdots \]

Since \( d^2 \omega_0 = 0 \), we have

\[ \left( d_k a_0^{0,0}, \sum_{i' + i'' = 1} d_{k+i'} a_2^{0,i''}, \ldots, \sum_{i' + i'' = k-2} d_{k+i'} a_2^{0,i''} \right) = \delta(a_2^0) = 0 \]

with \( a_2^0 = (a_2^{0,0}, \ldots, a_2^{0,k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{k-1}(p+2+j) V \). Consequently \( a_2^0 \) is a \( \delta \)-cocycle.

**Claim 1.** \( a_2^0 \) is an \( \delta \)-coboundary.

**Proof.** Recall first that the general \( r \)-th term of the spectral sequence (6) is given by the formula:

\[ E_r^{p,q} = Z_r^{p,q} / Z_{r-1}^{p+1,q-1} + B_r^{p,q}, \]

where \( Z_r^{p,q} = \{ x \in [F^p(\Lambda V)]^{p+q} | dx \in [F^{p+r}(\Lambda V)]^{p+q+1} \} \) and

\[ B_r^{p,q} = d([F^{p-r}(\Lambda V)]^{p+q+1}) \cap F^p(\Lambda V) = d(Z_{r-1}^{p-r+1,q-r+2}). \]
Recall also that the differential $d_r : E^{p,q}_r \to E^{p+1,q-r+1}_r$ in $E^*_r$ is induced from the differential $d$ of $(\Lambda V,d)$ by the formula $d_r([v])_r = [dv]_r$, $v$ being any representative in $Z^p_q$ of the class $[v]_r$ in $E^{p,q}_r$.

We still assume that $\dim H^N(\Lambda V, \delta) = 1$ and adopt notations of § 4.1.1. Notice then $\omega_0 \in Z^2_{p,q}$ and it represents a non-zero class $[\omega_0]_2$ in $E^{p,q}_2$. Otherwise $\omega_0 = \omega'_0 + d(\omega''_0)$, where $\omega'_0 \in Z_{p+1,q-1}^1$ and $\omega''_0 \in B_{1}^{p,q}$, so that $\alpha = [\omega_0] = [\omega'_0 - (d - \delta)(\omega''_0)]$. But $\omega'_0 - (d - \delta)(\omega''_0) \in \Lambda^{p+1}$ is a contradiction to the definition of $\omega_0$. Now, using the isomorphism $E^*_r \cong H^*_\ast(\Lambda V, \delta)$, we deduce that, $[\omega_0]_2 \in E^{p,q}_2$ (being the only generating element) must survive to $E^3_{p,q}$, otherwise, the spectral sequence fails to converge. Whence $d_2([\omega_0]_2) = [a_0^{02}]_2 = 0$ in $E^{p+2,q-1}_2$, i.e., $a_0^{0} \in Z^0_{p+3,q-2} + B^1_{k+2,q-1}$. However $a_0^{0} = \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2) + j}V$, so $a_0^{0} \in B^{1}_{k+2,q-1}$, that is $a_0^{0} = d(x)$, $x \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2) + j}V$. By wordlength argument, we have necessary $a_0^{0} = \delta(x)$, which finishes the proof of Claim 1.

Notice that this is the first obstruction to $[\omega_0]$ to represent a non zero class in the term $E^3_{p,q}$ of (6). The others will appear progressively as one advances in the algorithm.

Let then $b_2 \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2) + (k-1) + j}V$ such that $a_0^{0} = \delta(b_2)$ and put $\omega_1 = \omega_0 - b_2$. Reconsider the previous calculation for it:

\[
d_2 \omega_1 = d \omega_0 - db_2 = (a_2^0 + a_3^0 + \ldots + a_{k-1}^0) - (d_k b_2 + d_{k+1} b_2 + \ldots),
\]

with

\[
d_k b_2 = d_k (b_2^0, b_2^1, \ldots, b_2^{k-2}) = (d_k b_2^0, d_k b_2^1, \ldots, d_k b_2^{k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2) + j}V,
\]

\[
d_{k+1} b_2 = d_{k+1} (b_2^0, b_2^1, \ldots, b_2^{k-2})
\]

\[
= (d_{k+1} b_2^0, d_{k+1} b_2^1, \ldots, d_{k+1} b_2^{k-2}) \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+2) + j+1}V,
\]

\[
\ldots
\]
This implies that
\[ d\omega_1 = a_{2}^0 + a_{3}^0 + \cdots + a_{t+l}^0 - \left( d_k b_2^0, \sum_{i'+i''=1} d_{k+i'} b_2^{i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} b_2^{i''} \right) \]

\[ - \left( d_{2k-1} b_2^0 + \cdots, \ldots \right) \]

\[ = a_{2}^0 - \delta(b_2) + a_{3}^0 - \left( d_{2k-1} b_2^0 + \cdots, \ldots \right) + \cdots \]

\[ = a_{3}^0 - \left( d_{2k-1} b_2^0 + \cdots, \ldots \right) + \cdots, \]

and then:
\[ d\omega_1 = a_{3}^1 + a_{4}^1 + \cdots + a_{t+l}^1, \quad \text{with} \quad a_{i}^1 \in k-2 \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+i)+j} V. \]

So,
\[ d^2\omega_1 = da_{3}^1 + da_{4}^1 + \cdots + da_{t+l}^1 \]

\[ = \left( d_k a_{3}^1, \sum_{i'+i''=1} d_{k+i'} a_{3}^{1,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} a_{3}^{1,i''} \right) \]

\[ + \left( d_{2k-1} a_{3}^1 + \cdots, \ldots \right) + \cdots \]

Since \( d^2\omega_1 = 0 \), by wordlength reasons,
\[ \left( d_k a_{3}^1, \sum_{i'+i''=1} d_{k+i'} a_{3}^{1,i''}, \ldots, \sum_{i'+i''=k-2} d_{k+i'} a_{3}^{1,i''} \right) = \delta(a_{3}^1) = 0. \]

We claim that \( a_{3}^1 = \delta(b_3) \) and consider \( \omega_2 = \omega_1 - b_3 \).

We continue this process defining inductively \( \omega_j = \omega_{j-1} - b_{j+1}, \ j \leq t+l-2 \) such that:
\[ d\omega_j = a_{j+2}^j + a_{j+3}^j + \cdots + a_{t+l}^j, \quad \text{with} \quad a_{j}^j \in \bigoplus_{h=0}^{k-2} \Lambda^{(k-1)(p+i)+h} V \]

and \( a_{j+2}^j \) a \( \delta \)-cocycle.
Claim 2. \( a^j_{j+2} \) is a \( \delta \)-coboundary, i.e., there is
\[
b_{j+2} \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)(p+j+2)-(k-1)+j} V
\]
such that \( \delta(b_{j+2}) = a^j_{j+2} \); \( 1 \leq j \leq t + l - 2 \).

Proof. We proceed in the same manner as for the first claim. Indeed, we have clearly for any \( 1 \leq j \leq t + l - 2 \), \( \omega_j = \omega_{j-1} - b_{j+1} = \omega_0 - b_2 - b_3 - \cdots - b_{j+1} \in Z^{p,q}_{j+2} \) and it represents a non zero class \( [\omega_j]_{j+2} \) in \( E^{p,q}_{j+2} \) which is also one dimensional. Whence as in Claim 1, we conclude that, \( a^j_{j+2} \) is a \( \delta \)-coboundary for all \( 1 \leq j \leq t + l - 2 \).

Consider \( \omega_{t+l-1} = \omega_{t+l-2} - b_{t+l} \), where \( \delta(b_{t+l}) = a^{t+l-2}_{t+l} \). Notice that
\[
|d(\omega_{t+l-1})| = |d(\omega_{t+l-2})| = N + 1, \text{ but by the hypothesis on } t, \text{ we have } d(\omega_{t+l-2}) = a^{t+l-2}_{t+l} \text{ and then }
\]
\[
|d(\omega_{t+l-2} - b_{t+l})| = |a^{t+l-2}_{t+l} - \delta(b_{t+l}) - (d - \delta)b_{t+l}| = |(d - \delta)b_{t+l}| > N + 1.
\]
It follows that \( d(\omega_{t+l-1}) = 0 \), that is \( \omega_{t+l-1} \) is a \( d \)-cocycle. But it can’t be a \( \delta \)-coboundary. Indeed suppose that \( \omega_{t+l-1} = (\omega^0 + \omega^1 + \cdots + \omega_t^0) - (b_2 + b_3 + \cdots + b_{t+l}) \), were a \( \delta \)-coboundary, by wordlength reasons, \( \omega^0 \) would be a \( \delta \)-coboundary, i.e., there is \( x \in \bigoplus_{j=0}^{k-2} \Lambda^{(k-1)p-(k-1)+j} V \) such that \( \delta(x) = \omega^0 \).
Then
\[
\omega_0 = \delta(x) + \omega^0_1 + \cdots + \omega^0_l.
\]
Since \( \delta(\omega^0) = 0 \), we would have \( \delta(\omega^0_1 + \cdots + \omega^0_l) = 0 \) and then \( [\omega_0] = [\omega^0_1 + \cdots + \omega^0_l] \). But \( \omega^0_1 + \cdots + \omega^0_l \in \Lambda^x \tau V \), contradicts the property of \( \omega^0 \). Consequently \( \omega_{t+l-1} \) represents the fundamental class of \( (AV, d) \).
Finally, since \( \omega_{t+l-1} \in \Lambda^x \tau V \) we have
\[
e_0(AV, d) \geq r.
\]

4.1.2. For the second inequality. Denote \( s = e_0(AV, d) \) and let \( \omega \in \Lambda^x \tau V \) be a cocycle representing the generating class \( \alpha \) of \( H^N(AV, d) \).

Write \( \omega = \omega_0 + \omega_1 + \cdots + \omega_t, \omega_i \in \Lambda^{s+i} V \). We deduce that:
\[
d\omega = \left( d_k \omega_0 + \sum_{i+i' = 1} d_k \omega_{i'} + \cdots + \sum_{i+i' = k-2} d_k \omega_{i'} \right) + d_k \omega_{k-1} + d_2 \omega_0 + \cdots
\]
\[
= \delta(\omega_0, \omega_1, \ldots, \omega_{k-2}) + \cdots
\]
Since \( d\omega = 0 \), by wordlength reasons, it follows that \( \delta(\omega_0, \omega_1, \ldots, \omega_{k-2}) = 0 \). If \((\omega_0, \omega_1, \ldots, \omega_{k-2})\), were a \( \delta \)-boundary, i.e., \((\omega_0, \omega_1, \ldots, \omega_{k-2}) = \delta(x)\), then
\[
\omega - dx = (\omega_0, \omega_1, \ldots, \omega_{k-2}) - \delta(x) + (\omega_{k-1} + \cdots + \omega_l) - (d - \delta)(x),
\]
so, \( \omega - dx \in \Lambda^{2s+k-1}V \), which contradicts the fact \( s = e_0(\Lambda V, d) \). Hence \((\omega_0, \omega_1, \ldots, \omega_{k-2})\) represents the generating class of \( H^N(\Lambda V, \delta) \). But \((\omega_0, \omega_1, \ldots, \omega_{k-2}) \in \Lambda^{2s}V \) implies that \( s \leq r \). Consequently, \( e_0(\Lambda V, d) \leq r \).

Thus, we conclude that
\[
e_0(\Lambda V, d) = r.
\]

4.2. Proof of Theorem 2. It suffices to remark that since \((\Lambda V, d)\) is elliptic, it has Poincaré duality property and then \( \dim H^N(\Lambda V, d) = 1 \). The convergence of (6) implies that \( \dim E^{\ast, \ast}_\infty = 1 \). Hence there is a unique \((p, q)\) such that \( p + q = N \) and \( E^{\ast, \ast}_\infty = E^{p,q}_\infty \). Consequently only one of the generating classes \( \alpha_1, \ldots, \alpha_m \) had to survive to \( E_\infty \). Let \( \alpha_j \) this representative class and \((p_j, q_j)\) its pair of degrees.

**Example 1.** Let \( d = \sum_{i \geq 3} d_i \) and \((\Lambda V, d)\) be the model defined by \( V^{\text{even}} = \langle x_2, x_2 \rangle, V^{\text{odd}} = \langle y_5, y_7, y_7 \rangle \), \( dx_2 = dx_2 = 0 \), \( dy_5 = x_3^3 \), \( dy_7 = x_2^4 \) and \( dy_7 = x_2^2 y_7 \), in which subscripts denote degrees.

For \( k \geq 3, l \geq 0 \), we have
\[
x_{2}^{k} x_2^{l} = x_2^{k-3} x_2^{3} x_2^{l} = d(y_{5} x_{2}^{k-3} x_2^{l}).
\]

For \( k \geq 4, l \geq 0 \),
\[
x_{2}^{k} x_2^{l} = x_2^{l} x_2^{k-4} x_2^{4} = d(x_2^{l} x_2^{k-4} y_7).
\]

Clearly we have
\[
\dim H(\Lambda V, d) < \infty \quad \text{and} \quad \dim H(\Lambda V, d_3) = \infty.
\]

Using A. Murillo’s algorithm (cf. §2) the matrix determining the fundamental class is:
\[
A = \begin{pmatrix}
  x_2^{3} & 0 \\
  0 & x_2^{3}
\end{pmatrix},
\]
so, \( \omega = -x_2^2x_2^3y_7^2 + x_2x_2^5y_5 \in \Lambda^{\geq 6}V \) is a generator of this fundamental cohomology class.

It follows that \( e_0(\Lambda V, d) = 6 \neq m + n(k - 2) \).

**Example 2.** Let \( d = \sum_{i \geq 3} d_i \) and \((\Lambda V, d)\) be the model defined by \( V^{\text{even}} = \langle x_2, x_3 \rangle, \ V^{\text{odd}} = \langle y_5, y_9, y_9 \rangle \), \( dx_2 = dx_2 = 0 \), \( dy_5 = x_2^3 \), \( dy_9 = x_2^5 \) and \( dy_9 = x_2^3x_2^2 \).

Clearly we have

\[
\dim H(\Lambda V, d) < \infty \text{ and } \dim H(\Lambda V, d_3) = \infty.
\]

Using A. Murillo’s algorithm (cf. §2) the matrix determining the fundamental class is:

\[
A = \begin{pmatrix}
  x_2^2 & 0 \\
  0 & x_2^4 \\
  x_2^2x_2^2 & 0
\end{pmatrix},
\]

so, \( \omega = -x_2^2x_2^3y_7^2 + x_2x_2^5y_5 \in \Lambda^{\geq 7}V \) is a generator of this fundamental cohomology class.

It follows that \( e_0(\Lambda V, d) = 7 \neq m + n(k - 2) \).

**References**


