Representing Matrices, $M$-ideals and Tensor Products of $L_1$-predual Spaces

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Abstract: Motivated by Bratteli diagrams of Approximately Finite Dimensional (AF) $C^*$-algebras, we consider diagrammatic representations of separable $L_1$-predual spaces and show that, in analogy to a result in AF $C^*$-algebra theory, in such spaces, every $M$-ideal corresponds to directed sub diagram. This allows one, given a representing matrix of a $L_1$-predual space, to recover a representing matrix of an $M$-ideal in $X$. We give examples where the converse is true in the sense that given an $M$-ideal in a $L_1$-predual space $X$, there exists a diagrammatic representation of $X$ such that the $M$-ideal is given by a directed sub diagram and an algorithmic way to recover a representing matrix of $M$-ideals in these spaces. Given representing matrices of two $L_1$-predual spaces we construct a representing matrix of their injective tensor product.

Key words: representing matrix, generalized diagram, directed sub diagram, $M$-ideals, tensor products.


1. Introduction

In 1971 Lazar and Lindenstrauss (see [3]) introduced notion of representing matrices for separable $L_1$-predual spaces. The idea to construct representing matrix of a $L_1$-predual space depends on following result in [3, Theorem 3.2], which essentially says that any separable $L_1$-predual space is built up by putting together increasing union of $\ell^\infty_n$, $n = 1, 2, \ldots \infty$’s.

Theorem 1.1. Let $X$ be a separable infinite dimensional Banach space such that $X^*$ is isometric to $L_1(\mu)$ for some positive measure $\mu$. Let $F$ be a finite dimensional space whose unit ball is a polytope. Then there exists a sequence $\{E_n\}_{n=1}^\infty$ of finite dimensional subspaces of $X$ such that $E_1 \supset F$, $E_{n+1} \supset E_n$ and $E_n = \ell^\infty_n$ for every $n$ and $X = \overline{\bigcup_{n=1}^\infty E_n}$.

We now describe the notion of representing matrices. By Theorem 1.1 any separable $L_1$-predual space is $\overline{\bigcup_{n=1}^\infty F_n}$ and different such spaces are con-
structured depending on how one embeds $\ell^n_\infty \rightarrow \ell^{n+1}_\infty$.

Let $\{e_i\}_{i=1}^n$ denote the standard unit vector basis of $\ell^n_\infty$. By admissible basis of $\ell^n_\infty$ we mean a basis of the form $\{\theta_i e_{\pi(i)}\}_{i=1}^n$ where $\theta_i = \pm 1$ and $\pi$ is a permutation of $\{1, \ldots, n\}$.

It is easy to see that if $\{u_i\}$ is an admissible basis of $\ell^n_\infty$ then for any $m > n$ a linear operator $T : \ell^n_\infty \rightarrow \ell^m_\infty$ is an isometry if and only if there exists an admissible basis $\{v_i\}_{i=1}^m$ of $\ell^m_\infty$ such that

$$Tu_i = v_i + \sum_{j=n+1}^m a_{ij}^n v_j$$

with $\sum_{i=1}^n |a_{ij}^n| \leq 1$ for every $n+1 \leq j \leq m$.

Now for any separable $L_1$-predual space with the representation $X = \cup_{n \in \mathbb{N}} E_n$ where $E_n \subseteq E_{n+1}$ and each $E_n$ is isometric to $\ell^n_\infty$, we may choose admissible basis $\{e_i^n\}_{i=1}^n$ of $E_n$ such that, after relabelling,

$$T_n e_i^n = e_{i+1}^n + a_i^n e_{n+1}^n$$

with $\sum_{i=1}^n |a_{ij}^n| \leq 1$.

A triangular matrix $A = (a_{ij}^n)_{1 \leq i \leq j \leq n}$ associated with $X$ in this manner is called a representing matrix of $X$.

The construction of the representing matrix is best understood in the context of $C(K)$, $K$ is totally disconnected. For use in the later part of this paper, we illustrate this with an example by constructing of representing matrix for such a space.

Let $K$ be a totally disconnected compact metric space. Then there exists a sequence $\{\prod_n\}_{n=1}^\infty$ of partitions of $K$ into disjoint closed sets so that for every $n$, $\{\prod_n\}$ has $n$ elements, $\{\prod_{n+1}\}$ is a refinement of $\{\prod_n\}$ and

$$\varrho_n = \max_{A \in \prod_n} d(A) \rightarrow 0$$

where $d(A)$ denotes diameter of $A$.

Let $E_n$ be the linear span of the characteristic functions of the sets in $\prod_n$. Then it follows trivially that each $E_n$ is isometric to $\ell^n_\infty$, $E_n \subseteq E_{n+1}$ and $C(K) = \bigcup_{n=1}^\infty E_n$. Let us denote $\prod_n = \{K_1^n, K_2^n, \ldots, K_n^n\}$ for all $n \in \mathbb{N}$. We may write $1_{K_1^n} = 1_{K_1^n} + 1_{K_2^n}$. Now $\prod_3 = \{K_1^3, K_2^3, K_3^3\}$, $1_{K_1^3} = 1_{K_1^3} + 1_{K_3^3}$ and $1_{K_2^3} = 1_{K_3^3}$. We continue this procedure to get a representing matrix of $C(K)$ which is $0, 1$-valued [3, Theorem 5.1].

A $L_1$-predual space $X$ has a rich collection of structural subspaces of $X$, namely $M$-ideals. $M$-ideals in a $L_1$-predual space are themselves $L_1$-preduals and in some sense deterministic for the isometric properties of the
space, meaning, any isometric property of a $L_1$-predual space can be read off from some isometric properties of its $M$-ideals. On the other hand, representing matrices ‘encode’ every possible information of the structure of a $L_1$-predual space.

A separable predual $X$ of $L_1$ may be thought of as an isometric version (commutative, where $*$-isomorphism is replaced by linear isometry) of Approximately Finite Dimensional (AF) real $C^*$-algebras. Two sided norm closed ideals in an AF $C^*$-algebra are completely determined by hereditary directed sub diagrams of its Bratteli diagram (see [1]). The analogous notion of closed two sided ideals in a $C^*$-algebra in Banach space category is $M$-ideals. Here we present a representing diagram of a separable $L_1$-predual space, the diagram itself arise out of representing matrix of such a space. We show that every directed sub diagram of a representing diagram represents an $M$-ideal in the corresponding space. Since by definition of representing diagram, it is always hereditary, this is an exact analogy to the corresponding result for AF $C^*$-algebras. We believe the converse is also true and we establish it in some cases.

We now briefly describe the plan of this paper. In section 2 we present our main idea of diagrammatic representation of a separable $L_1$-predual space $X$ and directed sub diagram. We show any directed sub diagram corresponds to an $M$-ideal in $X$ and the residual diagram corresponds to $X/M$. If $M$ is an $M$-summand then we show the diagram for $X$ splits into two directed sub diagram. This recovers the result in [7]. We believe that the converse, that any $M$-ideal in a $L_1$-predual space $X$ is represented by a directed sub diagram of some diagram is true. However there is a problem here. There are $M$-ideals which have empty sub diagram. Nevertheless we present converse for $C(K)$ spaces (with extra assumption for general $K$). We also observe that for $A(K)$ -the space of affine continuous function on $K$, where $K$ is a separable Poulsen simplex (note that $A(K)$ is isometric to the Gurariy space in this case) given any $M$-ideal, there exists a diagrammatic representation of corresponding space such that the given $M$-ideal is represented by a directed sub diagram.

In Section 3 we describe a ‘Fill in the Gaps’ algorithm for construction of representing matrix from information that $X = \bigcup_{n=1}^{\infty} \ell^m$. This in one hand provides way to construct representing matrix for an $M$-ideal given by directed sub diagram and on the other, allows one to write down representing matrix of $X \otimes Y$, $X$, $Y$ $L_1$-preduals, knowing the representing matrix of $X$ and $Y$. We also show that for $C[0, 1]$, given an $M$-ideal, there exists a diagram-
matic representation of $C[0, 1]$ such that the given $M$-ideal is represented by a directed sub diagram.

Throughout this work we only consider separable $L_1$-predual spaces. Recall that a subspace $M$ of Banach space is called an $M$-ideal if there exists a projection (called $L$-projection) $P : X^* \rightarrow X^*$ such that $\ker P = M^\perp$ and $X^* = \text{Range } P \oplus_1 \ker P$, where $\oplus_1$ denote the $\ell_1$-sum. In this case $\text{Range } P$ is isometric to $M^*$. $M$ is said to be an $M$-summand in $X$ if $X = M \oplus_\infty N$. Trivially any $M$-summand is an $M$-ideal.

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2. DIRECTED DIAGRAMS AND $M$-IDEALS

For a $L_1$-predual space $X$ with a representing matrix $A = (a_{i n})_{1 \leq i \leq n}^{n \geq 1}$ we will consider the following diagrammatic representation of $X$.

A diagram $D$ of a $L_1$-predual space $X = \bigcup_{n=1}^{\infty} \ell_n^\infty$, and representing matrix $A = (a_{i n})_{1 \leq i \leq n}^{n \geq 1}$ consists of nodes and weighted arrows. The nodes at the $n$-th level of the diagram are $\{e_n^i : 1 \leq i \leq n\}$ where $\text{span}\{e_n^i : 1 \leq i \leq n\}$ is isometric to $\ell_n^\infty$, $n \in \mathbb{N}$. For a node $e_n^i$, there can be at most two arrows from $e_n^i$, one reaching to $e_n^{i+1}$ and another to $e_{n+1}^{i+1}$. Any arrow from $e_n^i$ to $e_{n+1}^i$ has weight 1 and there is an arrow from $e_n^i$ to $e_{n+1}^{i+1}$, then it has a weight $a_{n+1}^i$. For example if all $a_{n+1}^i \neq 0$ then we have the following diagram:
In case some \( a_n \)'s are zero we do not put arrows from \( e_n \) to \( e_{n+1} \). For example diagram for a space with \( a_1^1, a_2^2, a_3^3 = 0 \), will look like the following:

![Diagram](image)

In the following we describe the diagram for the space \( c \) with representing matrix \( A \) such that \( a_1^n = 1, n \geq 1 \) and \( a_j^n = 0, j \neq 1 \) (see [3]):

![Diagram](image)

Note that every representing matrix of a \( L_1 \)-predual space corresponds to a unique diagram \( D \) and vice-versa. For a given diagram \( D \) we will denote the corresponding space by \( X_D \).

Now we will introduce the notion of generalized diagram for a \( L_1 \)-predual space \( X \), where \( X = \bigcup X_n \) and \( X_n \) is isometric to \( \ell_\infty^m \) for an increasing
sequence \((m_n)\). Let \(\{e^1_{m_n}, \ldots, e^{m_n}_{m_n}\}\) be the admissible basis of \(X_n\). Any isometry \(T_{m_n} : \ell^m_{\infty} \to \ell^m_{\infty+1}\) is uniquely specified by scalars \((a^i_{m_n+j})\), \(1 \leq j \leq m_{n+1} - m_n\), \(1 \leq i \leq m_n\) such that

\[
T_{m_n} e^i_{m_n} = e^i_{m_{n+1}} + a^i_{m_{n+1}} e^{m_{n+1}}_{m_{n+1}} + \cdots + a^i_{m_{n+1}} e^{m_{n+1}}_{m_{n+1}}, \quad i = 1, 2, \ldots, m_n.
\]

For a node \(e^i_{m_n}\), there will be one arrow from \(e^i_{m_n}\) to \(e^{m_{n+1}}_{m_{n+1}}\). If \(a^i_{m_{n+1}} \neq 0\), then there will be a weighted arrow from \(e^i_{m_n}\) to \(e^{m_{n+1}+j}_{m_{n+1}}\), \(1 \leq j \leq m_{n+1} - m_n\) with weight \(a^{i}_{m_{n+1}+j}\).

**Definition 2.1.** A sub diagram \(S\) of \(D\) will be called a directed sub diagram if whenever \(e^i_n \in S\) for some \(n, i \in \mathbb{N}\), \(i \leq n\) then

(a) \(e^i_{n+1} \in S\),

(b) if \(a^i_n \neq 0\), \(e^{n+1}_n \in S\).

A sub diagram \(S \subseteq D\) is directed if whenever \(e^i_n \in S\) for some \(n, i \in \mathbb{N}\), \(i \leq n\) and there is an arrow from \(e^i_n\) to \(e^j_{n+1}\) then \(e^j_{n+1} \in S\).

We define directed sub diagram of a generalized diagram similarly.

If we take \(S \subseteq D\), and, \(S\) is directed then the original isometric embedding of \(X_n\) into \(X_{n+1}\) is preserved (see introduction). Hence \(X_S\) will be an isometric subspace of \(X_D\). Moreover there exists a norm one projection \(P : X^* \to X^*\) with \(\ker P = X_S^\perp\). To see this observe that \(X_S = \bigcup_{\ell \in \mathbb{N}} \ell^m_{\infty}\), hence \(X_S\) is itself a \(L_1\)-predual space which is an isometric subspace of \(X_D\). We prove that for any directed sub diagram \(S\) the space \(X_S\) is an \(M\)-ideal in \(X_D\) and the diagram \(D \setminus S\) represents the space \(X_D/X_S\).

**Theorem 2.2.** Let \(X\) be a \(L_1\)-predual space with a given diagram \(D\). Then for any directed sub diagram \(S\) of \(D\) the subspace \(X_S\) is an \(M\)-ideal in \(X\).

**Proof.** Let \(X = \bigcup X_n\), where \(X_n \subseteq X_{n+1}\), \(X_n\) is isometric to \(\ell^m_{\infty}\) for each \(n\). Let \(P : X^* \to X^*\) be a norm one projection with \(\ker P = X_S^\perp\), that is, \(X^* = X_S^\perp \oplus F\) where \(F = \text{Range } P\). We need to prove that \(X^* = X_S^\perp \oplus 1 F\).

Let \(M_n = \text{span}\{e^i_n : e^i_n \in S, 1 \leq i \leq n\}\) and \(F_n = \text{span}\{e^i_n : e^i_n \notin S, 1 \leq i \leq n\}\) for each \(n \in \mathbb{N}\).

Then \(X_n = M_n \oplus F_n\) and \(X_n^* = M_n^\perp \oplus F_n^\perp\). For any \(x^* \in X^*\) we can write \(x^* = x_1^* + x_2^*\) where \(x_1^* \in X_S^\perp\) and \(x_2^* \in F\). Then \(x^*|_{X_n} = x_1^*|_{X_n} + x_2^*|_{X_n}\) and \(\|x^*|_{X_n}\|^2 = \|x_1^*|_{X_n}\|^2 + \|x_2^*|_{X_n}\|^2\). For given \(\epsilon > 0\), we can choose some \(m \in \mathbb{N}\)
such that $||x^*_n|| \geq ||x^*|| - \epsilon$, $||x^*_n|| \geq ||x^*_1|| - \epsilon$ and $||x^*_n|| \geq ||x^*_2|| - \epsilon$ for all $n \geq m$. Now

$$||x^*_1|| + ||x^*_2|| \geq ||x^*|| \geq ||x^*_n|| = ||x^*_1|| + ||x^*_2|| \geq ||x^*_1|| + ||x^*_2|| - 2\epsilon.$$  

Thus it follows that $||x^*|| = ||x^*_1|| + ||x^*_2||$ for all $x^* \in X^*$. From this we can conclude that $X^* = X_{S^\perp} \oplus_F$. \[\square\]

**Remark 2.3.** Let $X$ be a $L_1$-predual space with a given generalized diagram $D$. Same proof as in Theorem 2.2 shows that directed sub diagram $S$ of $D$ represents the subspace $X_S$ which is an $M$-ideal in $X$.

Next Theorem is analogous to [1, Theorem III.4.4]) in the case of $L_1$-predual spaces.

**Theorem 2.4.** Let $X$ be a $L_1$-predual space with a given diagram $D$ and $S$ a directed sub diagram of $D$. Then the diagram $D \setminus S$ represents the space $X/X_S$.

**Proof.** Let $X = \overline{\bigcup X_n}$ where $X_n$ is isometric to $\ell_\infty^n$. As before, let $M_n = \text{span}\{e^n_i : e^n_i \in S, 1 \leq i \leq n\}$ and $F_n = \text{span}\{e^n_i : e^n_i \notin S, 1 \leq i \leq n\}$ for each $n \in \mathbb{N}$. Then $X_S = \overline{\bigcup M_n}$, $M_n = \ell_\infty^n$ for some $m \leq n$, is the $M$-ideal corresponding to the directed diagram $S$ and $X_n = M_n \oplus_\infty F_n$. Consider the norm one projection $P_n : X_n \to F_n$ where

$$P_n \left( \sum_{i=1}^n a_i e^n_i \right) = \sum_{e^n_i \notin S} a_i e^n_i.$$  

Let $i_n : F_n \to F_{n+1}$ be the isometry determined by arrows of the diagram $D \setminus S$, that is, for $e^n_i \in D \setminus S$,

$$i_n(e^n_i) = a_i e^{n+1}_i + a_i e^{n+1}_{n+1} \quad \text{if} \quad e^{n+1}_i, e^{n+1}_{n+1} \in D \setminus S,$$

$$i_n(e^n_i) = a_i e^{n+1}_i \quad \text{if} \quad e^{n+1}_i \in D \setminus S, e^{n+1}_{n+1} \notin D \setminus S,$$

$$i_n(e^n_i) = a_i e^{n+1}_i \quad \text{if} \quad e^{n+1}_i \in D \setminus S, e^{n+1}_{n+1} \notin D \setminus S,$$

$$i_n(e^n_i) = 0 \quad \text{if} \quad e^{n+1}_i, e^{n+1}_{n+1} \notin D \setminus S.$$  

It is straightforward to verify that $P_{n+1}X_n = i_n \circ P_n$.

We now define $P : \bigcup X_n \to \bigcup F_n$ by $Px = P_nx$ if $x \in X_n$. It follows that $P$ is well defined and extends as a quotient map from $X$ to the space determined by $\overline{\bigcup F_n}$ which is the space determined by the diagram $D \setminus S$. This completes the proof. \[\square\]
We now investigate the converse of Theorem 2.2. Explicitly stated the problem is the following.

**Problem 2.5.** Let $X$ be a $L_1$-predual space and $M$ an $M$-ideal in $X$. Then there exists a diagram $D$ representing $X$ and a directed sub diagram $S$ of $D$ such that $M = X_S$.

We believe the answer to Problem 2.5 is affirmative. We will present evidences towards this for $M$-summands in general and $M$-ideals in some class of $L_1$-predual spaces.

The following proposition shows that any $M$-summand in a $L_1$-predual space is represented by a directed sub diagram.

**Proposition 2.6.** Let $X$ be a $L_1$-predual space and $M$ be an $M$-summand in $X$. Then there exists a diagram $D$ representing $X$ such that $M$ corresponds to some directed sub diagram $S$ of $D$.

**Proof.** Let $N$ be the complement of $M$ in $X$, that is, $X = M \oplus \infty N$. Then by [7, Proposition 2.4] it follows that $X$ has a representing matrix of the form

$$A = \begin{bmatrix}
0 & a_2^1 & 0 & a_4^1 & 0 & a_6^1 & \ldots \\
0 & a_3^2 & 0 & a_5^2 & 0 & \ldots \\
0 & a_4^3 & 0 & a_6^3 & \ldots \\
0 & a_5^4 & 0 & \ldots \\
0 & a_6^5 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots
\end{bmatrix}$$

where $B_A = (b_n^i)$ with $b_n^i = a_{2n}^{2i-1}$, $C_A = (c_n^i)$ with $c_n^i = a_{2n}^{2i}$, $n \in \mathbb{N}$, $1 \leq i \leq n$, are the matrices for $M$ and $N$ respectively. Let $S_1$ and $S_2$ be the diagrams corresponding to matrices $B_A$ and $C_A$ respectively. Now it follows that $S_1$ and $S_2$ are directed sub diagrams of the diagram of $X$ corresponding to the representing matrix $A$. \[\blacksquare\]

**Remark 2.7.** Directed sub diagrams $S_1$ and $S_2$ considered in Proposition 2.6 are disjoint in the sense that no arrows of $S_1$ enters into $S_2$ and vice-versa.

We now consider $M$-ideals in $C(K)$-spaces. We need to recall few notation and a result from [7].
Let $X$ be a $L_1$-predual space with $X = \bigcup_{n=1}^{\infty} \ell^n_\infty$ and $\{e^i_n : 1 \leq i \leq n\}$ are admissible bases of $\ell^n_\infty$, $n \in \mathbb{N}$. Define $\phi_j \in X^*$, $j \in \mathbb{N}$, by
\[
\phi_j(e^i_n) = \begin{cases} 
0 & \text{if } i \neq j, \\
1 & \text{if } i = j; \ i = 1, \ldots, n; \ j \leq n; \ n \in \mathbb{N}.
\end{cases}
\]
By $\text{ext } B_{X^*}$ we will denote the extreme point of $B_{X^*}$.

**Lemma 2.8.** [7, Lemma 1.2] Let $X$ and $\{\phi_j\}$ be as above. Then
(a) $\phi_j \in \text{ext } B_{X^*}$ for all $j \in \mathbb{N}$, and
(b) $\{\pm \phi_i : i \in \mathbb{N}\} = \overline{\text{ext } B_{X^*}}$, where closure is taken in weak$^*$-topology of $B_{X^*}$.

**Remark 2.9.** For each $i$, $\ker \phi_i$ represents the space $X_{S_i}$ for some directed sub diagram $S_i$ of a given diagram $D$ of $X$ where the line passing through $e^i_1$ is a part of the diagram $D \setminus S_i$.

The idea of the proof for the following result is to use the flexibility provided by Lemma 2.8 for the choice of $\phi_i$ in a totally disconnected compact metric space $K$. Recall that for any $C(K)$ space where $K$ is a compact metric space, an $M$-ideal is given by $J_D = \{f \in C(K) : f|_D = 0\}$, where $D$ is some closed subset of $K$.

**Proposition 2.10.** Let $K$ be a totally disconnected compact metric space and $D$ a closed subset of $K$. Then there exists a diagram $D$ representing $C(K)$ and a directed sub diagram $D \subseteq S$ such that $J_D = X_S$.

**Proof.** Since $K$ is a totally disconnected we can get a sequence $\{\prod_n\}_{n=1}^{\infty}$, $\prod_n = \{K_{n1}^1, K_{n2}^2, \ldots, K_{nn}^n\}$ of partitions of $K$ into disjoint closed sets. $\prod_{n+1}$ is a refinement of $\prod_n$ and $\theta_n = \max_{A \in \prod_n} \text{diam} (A) \to 0$ (see introduction).

Let $D_0 = \{d_n : n \in \mathbb{N}\}$ be a countable dense set in $D$. Choose $\phi_1 = \delta_{d_1}$. For $n \geq 2$ by renaming the elements in $\prod_n$ we assume that $d_1 \in K_{n1}^1$.

For $n = 2$ if
(a) $D_0 \cap K_2^2 \neq \emptyset$, we find the least $n_0$ such that $d_{n_0} \in D_0 \cap K_2^2$ and choose $\phi_2 = \delta_{d_{n_0}}$. We will assume for all $n \geq 3$, $d_{n_0} \in K_{n0}^2$, again by possibly renaming the members of $\prod_n$,
(b) otherwise choose and fix any $k \in K_2^2$ and take $\phi_2 = \delta_k$. We will assume for all $n \geq 3$, $k \in K_{n0}^2$. 

We will follow the same procedure for \( n \geq 3 \).

We need to ensure that each \( d_n \) will be chosen. Let \( N \) be the least number among all \( k \)'s such that \( d_k \in K_i^n \) for some \( i, n \). Let \( m \neq N \) and \( d_m \in K_i^n \) as well. Since \( \text{diam}(K_i^n) \rightarrow 0 \), we can choose some suitable large \( M \in \mathbb{N} \) such that \( d_m \in K_i^M \) for some \( i, n \). Let \( m \neq N \) and \( d_m \in K_i^M \) as well. Since \( \text{diam}(K_i^n) \rightarrow 0 \), we can choose some suitable large \( M \in \mathbb{N} \) such that \( d_m \in K_i^M \) and \( m \) is the least among all \( k \)'s such that \( d_k \in K_i^M \).

So following the algorithm above we define \( \phi_M = \delta_{d_m} \).

Let \( D \) be the diagram representing \( C(K) \) given by the partition \( \{ \prod_n \} \) after renaming the elements of \( \{ \prod_n \} \) as considered above. Since \( D_0 \) is dense in \( D \), we have

\[
J_D = \bigcap_{d \in D} \ker \delta_d = \bigcap_{d \in D_0} \ker \delta_d.
\]

Thus \( J_D = X_S \), where \( S \) is the intersection of directed diagrams corresponding to kernel of \( \phi_i = \delta_{d_i} \), \( d_i \in D_0 \). \( \Box \)

Next result shows affirmative answer to Problem 2.5 for general \( C(K) \) space with additional assumption on an \( M \)-ideal. By \( \text{int} \ D \) we mean interior of a set \( D \).

**Proposition 2.11.** Let \( K \) be any compact metric space and \( D \) a closed subset of \( K \) such that \( D = \text{int} D \). Then the \( M \)-ideal \( J_D \) corresponds to the space \( X_S \) for some directed sub diagram \( S \) of given diagram \( D \) of \( C(K) \), provided, \( S \) is not an empty diagram.

**Proof.** Let \( \phi_j = \delta_{k_j} \), \( k_j \in K \). Since \( \{ \phi_j \} \) are weak*-dense in extreme points of the dual unit ball of \( C(K) \) and \( D = \text{int} D \), we have a sub collection \( \phi_j, \subseteq \text{int} D \) such that \( \phi_j = \delta_{k_j} \) and \( k_j \) is dense in \( D \). It follows \( J_D = \bigcap_{d \in D} \ker \phi_j \), and hence \( J_D \) is represented by the directed sub diagram \( S \) of \( D \) which is generated by intersection of directed sub diagram representing ker \( \phi_j \), where \( \phi_j = \delta_{k_j} \). \( \Box \)

**Remarks 2.12.**

1. If we assume \( K \) to be a ‘nice’ compact metric space, then given \( D \) a closed subset in \( K \), we can construct a diagrammatic representation of \( C(K) \) such that \( J_D \) corresponds to a directed sub diagram. We will do it in next section as we need algorithm to construct representing matrix of a \( L_1 \)-predual space \( X \) when it is given in the form \( X = \bigcup_{n \geq 1} \ell^*_\infty \).

2. Let \( K \) be (the) separable Poulsen simplex. Then the space \( A(K) \) - the space of real valued affine continuous functions on \( K \) is the separable
Gurariy space. It was proved in ([8]) that any infinite dimensional $M$-ideal in separable Gurariy space is isometric to itself. Thus any representing diagram of the Gurariy space represents $M$-ideals in it and Problem 2.5 has affirmative solution for the Gurariy space.

We note that an empty diagram is always a directed sub diagram of any given diagram $D$. It may be the case that an $M$-ideal in a $L_1$-predual space corresponds to an empty diagram. We give an easy example towards this.

**Example 2.13.** Consider the matrix $A$ such that $a^n_i = 1$ for all $n$ and $a^i_n = 0$ for all $i > 1$. It is proved in [3] that $A$ represents $c$. Consider the $M$-ideal $J = \{(x_n) \in c : x_n = 0, \ n \geq 2\}$. Then $J = \cap \phi_n$, $\phi_n = \delta_n$, $n \geq 2$.

In second figure on page 5, except the line segment starting from the node $n$ and the line segment that starts from the node $e^1_1$ and ends at $e^1_{n-1}$, all the diagram represents the space $\ker \phi_n$. It is straightforward to verify that $\cap_{n \geq 2} \ker \phi_n$ is empty.

Another difficulty in solving Problem 2.5 affirmatively in general is empty diagram may represent a space which is not an $M$-ideal. We give an example of this in a typical non $G$-space. Note that an empty diagram is always directed.

**Example 2.14.** Let $X = \left\{ f \in C[1,\omega_0] : f(\omega_0) = \frac{f(1) + f(2)}{2} \right\}$. Then $X$ is a $L_1$-predual space which is not a $G$-space (see [5]). We will consider the following admissible basis for $X$ (see [2]):

$$e^1_1 = (1,1,1,1,\ldots), \quad e^1_2 = (1,0,\frac{1}{2},\frac{1}{2},\ldots);$$
$$e^2_2 = (0,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots), \quad e^1_3 = (1,0,0,\frac{1}{2},\frac{1}{2},\ldots);$$
$$e^2_3 = (0,1,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots), \quad e^3_3 = (0,0,0,1,0,\ldots), \ldots.$$

For $n \in [1,\omega_0]$, we denote by $J_n$ the $M$-ideal \{ $f \in C[1,\omega_0] : f(n) = 0$ \}. Each $J_n$ is of codimension 1 in $C[1,\omega_0]$. We consider the $M$-ideal in $C[1,\omega_0]$, $J_{[3,\omega_0]} = \{ f \in C[1,\omega_0] : f(n) = 0, \ n \geq 3 \}$.

Now consider the subspace $J_{[3,\omega_0]} \cap X = \cap_{n \geq 3} J_n \cap X$ of $X$. As in Example 2.13 it is easy to check that the intersection of corresponding directed sub diagrams of $J_n \cap X$ for $n \geq 3$ is empty diagram.

However, $J_{[3,\omega_0]} \cap X$ is not an $M$-ideal in $X$. To see this we observe that $J_{[3,\omega_0]} \cap X$ is the range of norm one projection $P : X \to X$ given by $P(f) = (f(1),-f(1),0,0,\ldots)$. Thus if $J_{[3,\omega_0]} \cap X$ is an $M$-ideal then it is an $M$-summand as well. So for any $f \in X$, $\|f\| = \max\{\|Pf\|, \|(I-P)f\|\}$. 
However if we consider the element $f \in X$ where $f(1) = 1$, $f(2) = 0$ and $f(n) = 1/2$ for all $n \geq 3$, i.e., $f = (1, 0, 1/2, 1/2, 1/2, \ldots)$ then

$$(1, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots) = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, \ldots\right) + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\right)$$

and the norm of both side will not match. Thus $J_{[3, \omega_0]} \cap X$ is not an $M$-summand in $X$.

3. Fill in the Gaps

In this section we provide an algorithm to construct representing matrix of a $L_1$-predual space $X$ where $X$ is given by $X = \bigcup X_n$ and $X_n$ is isometric to $\ell^m_{\infty}$ for an increasing sequence $(m_n)$. This construction is implicit in the description of representing matrix given in [3]. However we fix an algorithm (there may be several as seen below) and use it for finding representing matrix of $X \hat{\otimes} Y$ - the injective tensor product of two separable $L_1$-predual spaces, knowing the representing matrices of $X$ and $Y$.

First we need to provide following justification to our construction.

**FACT:** Let $X$ be a $L_1$-predual space such that $X = \bigcup X_n$, where $X_n \subseteq X_{n+1}$ and $X_n$ is isometric to $\ell^m_{\infty}$ for some increasing sequence $(m_n)$. If $Z$ is a $L_1$-predual space with $Z = \bigcup Z_n$, where $Z_n \subseteq Z_{n+1}$, $Z_n$ is isometric to $\ell^n_{\infty}$, $Z_{m_n} = X_n$ and the isometry $T_n : X_n \to X_{n+1}$ is same as composition of isometries of $Z_{m_n}$ to $Z_{m_n+1}$, $Z_{m_n+1}$ to $Z_{m_n+2}$, $Z_{m_n+2}$ to $Z_{m_n+3}$ given by the representing matrix of $Z$, then $Z$ is isometric to $X$.

We now describe the proposed algorithm.

Let $X = \bigcup X_n$ where for each $n$, $X_n = \ell^m_{\infty}$ with admissible basis $\{e^i_{m_n}\}_{i=1}^{m_n}$. Any isometry $T_{m_n}$ from $\ell^m_{\infty}$ to $\ell^{m_{n+1}}_{\infty}$ in terms of admissible basis is given by

$$T_{m_n}e_{m_n} = e_{m_{n+1}}^i + a_{m_n+1}^i e_{m_{n+1}} + \ldots + a_{m_{n+1}}^i e_{m_{n+1}}, \quad i = 1, 2, \ldots, m_n. \quad (1)$$

Hence given $X$ as above and isometric embeddings $\ell^m_{\infty} \to \ell^{m_{n+1}}_{\infty}$ we know exactly $(m_{n+1} - m_n)m_n$ numbers of

$$\left(a_j^i\right), \quad i = 1, \ldots, m_n, \quad j = m_n + 1, \ldots, m_{n+1} - m_n.$$

Let us assume $C = (c^i_n)_{n \geq 1}$ is a representing matrix for $X$. We will write
\{e_{m_n}^i\} in terms of \{e_{m_{n+1}}^i\} according to isometries given by C:

\[
e_{m_n}^i = e_{m_{n+1}}^i + c_{m_n}^{m_{n+1}+1}
= e_{m_{n+2}}^i + c_{m_n}^{m_{n+2}} + c_{m_n} (e_{m_{n+2}}^{m_{n+1}+1} + e_{m_{n+2}}^{m_{n+1}+2})
= e_{m_{n+2}}^i + c_{m_n}^{m_{n+1}+1} + (c_{m_n}^{m_{n+1}} e_{m_{n+1}}^{m_{n+1}+1}) e_{m_{n+2}}^{m_{n+2}} = \ldots
\]

This way we will have \(\frac{(m_{n+1} - m_n)}{2}(m_{n+1} + m_n - 1)\) numbers of \(c_n^i\) unknowns. We will put

\[
c_{m_n+j}^{m_{n+1}+i} = 0, \quad i = 1, \ldots, m_{n+1} - m_n, \quad 1 \leq j \leq i.
\]

It is a straightforward verification that this way we will have \(\frac{(m_{n+1} - m_n)}{2}(m_{n+1} - m_n)\) of \(c_n^i\)'s zero. Thus remaining \(m_{n+1} - m_n\) of \(c_n^i\)'s equal the number of known variables \(a_n^i\)'s and can be expressed in terms of linear equations.

We emphasize that the above way of choosing \(c_n^i\) is not unique and different ways will give us different representing matrices. Note that here we can not recover first \(m_1 - 1\) columns of the representing matrix by the above algorithm so it can be chosen arbitrarily (see [6, Theorem 4.7]).

**Remark 3.1.** Let the admissible basis of \(X_n\) is \(\{e_{m_n}^i : 1 \leq i \leq m_n\}\). If we follow the above algorithm of ‘Fill in the Gaps’ from \(X_n\) to \(X_{n+1}\) where \(X_n\) is isometric to \(\ell^\infty_{m_n}\) and \(X_{n+1}\) is isometric to \(\ell^\infty_{m_{n+1}}\) then the basis elements \(e_{m_{n+1}}^{m_{n+1}}, e_{m_{n+2}}^{m_{n+2}}, \ldots, e_{m_{n+1}}^{m_{n+1}}\) are same as \(e_{m_n}^{m_{n+1}}\) for all \(i \geq 1\).

We illustrate this procedure by considering two special cases. First one is simple trial case with \(m_n = 2n\) and our second example provides us with representing matrix of \(C[0, 1]\) with entries 0 and \(\frac{1}{2}\).

**Example 3.2.** Let \(C = (c_n^i)_{1 \leq i \leq n}^{n \geq 1}\) be a representing matrix of \(X\) and \(X_n = \text{span} \{e_{2n}^1, \ldots, e_{2n}^n\}\) and \(T_n : X_n \rightarrow X_{n+1}\) is an isometric embedding with

\[
e_{2n}^i = c_{2(n+1)}^i + a_{2n+1} c_{2(n+1)}^{2n+1} + a_{2(n+1)} c_{2(n+1)}^{2(n+1)}, \quad 1 \leq i \leq 2n, \quad n \geq 1.
\]

If we write the expression for \(e_{2n}^i\) according to the matrix \(C\) then we get

\[
e_{2n}^i = c_{2(n+1)}^i + c_{2n} c_{2(n+1)}^{2n+1} + (c_{2n+1} c_{2n+1} c_{2(n+1)}^{2n+1}) c_{2(n+1)}^{2(n+1)}.
\]

From above two expressions for \(e_{2n}^i\) we have \(a_{2n+1}^i = c_{2n}^i\) and \(a_{2(n+1)}^i = c_{2n+1}^i + c_{2n} c_{2n+1}^{2n+1} + c_{2n} c_{2n+1}^{2(n+1)}\).
Now if we proceed by above algorithm and put \( c_{2n+1} = 0 \), \( n \in \mathbb{N} \) we get 
\[ a_{2n+1} = c_{2n} \]
\( 1 \leq i \leq 2n, \ n \geq 1 \), and, we have the following representing matrix for \( X \),
\[
C = \begin{bmatrix}
- a_3^1 & a_4^1 & a_5^1 & a_6^1 & \ldots \\
   a_3^2 & a_4^2 & a_5^2 & a_6^2 & \ldots \\
     a_3^3 & a_4^3 & a_5^3 & a_6^3 & \ldots \\
     :     & a_5^4 & a_6^4 & \ldots & \vdots \\
     :     & :            & 0 & \ldots & \vdots \\
     :     & :            & :            & \ldots & \vdots \\
     :     & :            & :            & \ldots & 0 \\
\end{bmatrix}.
\]

The Fact stated above indeed justifies that the resulting matrix is a representing matrix of \( X \).

**Example 3.3.** Consider the function \( \phi : \mathbb{R} \to \mathbb{R} \), \( \phi(t) = 1 + t \) for \( t \in [-1,0) \), \( \phi(t) = 1 - t \) for \( t \in [0,1] \), and \( \phi(t) = 0 \) for \( t \notin [-1,1] \). Define \( g_{k,2^n} = \phi(2^n t - k) \), \( t \in [0,1] \). We can write \( C[0,1] = \bigcup X_n \), \( X_n = \text{span}\{g_{k,2^n} : k = 0,1,\ldots,2^n\} \) where \( \{g_{k,2^n} : k = 0,1,\ldots,2^n\} \) is an admissible basis of \( X_n \).

Then for all \( n = 0,1,\ldots \) and \( k = 1,2,\ldots,2^n - 1 \) we have (see [4])
\[
g_{k,2^n} = \frac{1}{2} g_{2k-1,2^{n+1}} + g_{2k,2^n+1} + \frac{1}{2} g_{2k+1,2^n+1},
\]
\[
g_{0,2^n} = g_{0,2^n+1} + \frac{1}{2} g_{1,2^n+1},
\]
\[
g_{2^n,2^n} = \frac{1}{2} g_{2^n+1,2^n+1} + g_{2^n+1,2^{n+1}}.
\]

Let \( C = (c_{i j})_{1 \leq i \leq n, n \geq 1} \) be a representing matrix of \( C[0,1] \). First we have to write the expression for \( g_{k,2^n} \) according to \( C \). Now comparing the equations with the above and put \( c_{2^{n+1}+j} = 0 \), \( 1 \leq i \leq 2^{n+1} - 2^n - 1 \), \( 1 \leq j \leq i \), we will get a representing matrix of \( C[0,1] \) with entries 0 and \( \frac{1}{2} \) only.

We now answer Problem 2.5 in affirmative for \( C[0,1] \).

**Theorem 3.4.** Let \( D \) be a closed subset of \([0,1]\). Then there exists a diagram \( D \) representing \( C[0,1] \) such that the \( M \)-ideal \( J_D \) corresponds to the space \( X_S \) for some directed sub diagram \( S \) of \( D \), provided \( S \) is not an empty diagram.
Proof. Let $D_0 = \{d_n : n \in \mathbb{N}\}$ be a countable dense set in $D$. We can extend $D_0$ to a set $M = \{k_i : i \in \mathbb{N}\}$ such that $M = [0, 1]$. Consider $e^1_2 = 1 - t$, $t \in [0, 1]$ and $e^2_2 = t$, $t \in [0, 1]$. Without loss of generality choose an element $k_1 \in [0, 1]$ and consider

$$e^1_3 = 1 - \frac{1}{k_1}t \quad \text{if } t \in [0, k_1], \quad e^1_3 = 0 \quad \text{if } t \in [k_1, 1];$$

$$e^2_3 = 0 \quad \text{if } t \in [0, k_1], \quad e^2_3 = \frac{t - k_1}{1 - k_1} \quad \text{if } t \in [k_1, 1] \quad \text{and}$$

$$e^3_3 = \frac{1}{k_1}t \quad \text{if } t \in [0, k_1], \quad e^3_3 = \frac{1 - t}{1 - k_1} \quad \text{if } t \in [k_1, 1].$$

Here $e^1_2$, $e^2_2$, $e^3_2$, $e^3_3$ satisfy the following equations: $e^1_2 = e^3_2 + (1 - k_1)e^3_3$, $e^2_2 = e^3_2 + k_1e^3_3$. Now with out loss of generality choose $k_2 \in [0, k_1]$ and $k_3 \in [k_1, 1]$. Consider

$$e^1_5 = 1 - \frac{1}{k_2}t \quad \text{if } t \in [0, k_2], \quad e^1_5 = 0 \quad \text{if } t \in [k_2, 1];$$

$$e^2_5 = 0 \quad \text{if } t \in [0, k_3], \quad e^2_5 = \frac{t - k_3}{1 - k_3} \quad \text{if } t \in [k_3, 1];$$

$$e^3_5 = 0 \quad \text{if } t \in [0, k_2], \quad e^3_5 = \frac{t - k_2}{k_1 - k_2} \quad \text{if } t \in [k_2, k_1],$$

$$e^4_5 = \frac{k_3 - t}{k_3 - k_1} \quad \text{if } t \in [k_1, k_3], \quad e^3_5 = 0 \quad \text{if } t \in [k_3, 1];$$

$$e^4_5 = \frac{1}{k_2}t \quad \text{if } t \in [0, k_2], \quad e^4_5 = \frac{k_1 - t}{k_1 - k_2} \quad \text{if } t \in [k_2, k_1];$$

$$e^5_5 = 0 \quad \text{if } t \in [k_1, 1] \quad \text{and} \quad e^5_5 = 0 \quad \text{if } t \in [0, k_1],$$

$$e^5_5 = \frac{t - k_1}{k_3 - k_1} \quad \text{if } t \in [k_1, k_3], \quad e^5_5 = \frac{1 - t}{1 - k_3} \quad \text{if } t \in [k_3, 1].$$

By the construction $e^3_5$, $e^2_5$, $e^1_5$, $e^1_5$, $e^4_5$, $e^4_5$, $e^5_5$ satisfy the following equations:

$$e^1_3 = e^5_1 + \frac{k_1 - k_2}{k_2}e^4_5, \quad e^2_3 = e^5_2 + \frac{k_3 - k_1}{1 - k_1}e^5_5 \quad \text{and} \quad e^3_3 = e^5_3 + \frac{k_2}{k_1}e^4_5 + \frac{1 - k_3}{1 - k_1}e^5_5.$$

Similarly we can construct $e^i_5$, $1 \leq i \leq 2^n + 1$. Take an element $f \in C[0, 1]$. Define a sequence $(p_n)_{n=0}^\infty$ in the following way. Let $p_0 = f(0)e^1_2$,

$$p_1 = p_0 + (f(1) - p_0(1))e^2_2, \quad p_2 = p_1 + (f(k_1) - p_1(k_1))e^3_3,$$

$$p_3 = p_2 + (f(k_2) - p_2(k_2))e^4_4, \quad p_4 = p_3 + (f(k_3) - p_3(k_3))e^5_5,$$

and so on. Here $p_0$ and $f$ takes the same value at 0 while $p_1$ and $f$ takes the same value at 0 and 1 and interpolates linearly in between, $p_2$ and $f$ takes
same value at 0, 1 and \( k_1 \) and interpolates linearly in between, and so on. It is straightforward to check that \( \lim_n \|p_n - f\|_\infty = 0 \). Therefore we can write \( C[0, 1] = \bigcup E_n \), where 

\[
E_n = \text{span} \left\{ e_{2^n+1}^i : 1 \leq i \leq 2^n + 1 \right\}
\]

and \( E_n \) is isometric to \( \ell_\infty^{2^{n+1}} \). We know that the support of \( e_{2^n+1}^i \) is going to zero as \( n \) approaches to infinity and it consists a single element of \( \{k_i : i \in \mathbb{N}\} \). Each \( i \in \mathbb{N}, k_i \) will be in some \( T_{j_i} = \cap_{n=1}^{\infty} \text{supp}\{e_{2^n+1}^i\} \) and any two \( T_{j_i} \)‘s are disjoint. Here we consider the generalized diagram of \( C[0, 1] \) with respect to above basis and from \( n \)-th to \((n+1)\)-th step we choose \( 2^{n-1} \) of \( k_i \)‘s and these \( k_i \)‘s lie in the support of exactly one of the basis elements of \( e_{2^n+1}, \ldots, e_{2^n+1} \).

Now if we follow the algorithm for ‘Fill in the Gaps’ from \( n \)-th to \((n+1)\)-th step and consider \( k_i \in \text{supp}(e_{2^n+1}^i) \) chosen above, then \( k_i \in \text{supp}(e_{2^{n+1}+1}^i) \) and \( k_i \notin \text{supp}(e_{2^n+1}^l), l \neq 2^n + j \) for \( 2^n + 1 \leq m \leq 2^{n+1} - 1, j \geq 1 \) (see Remark 3.1). So by following the same procedure of choosing \( \phi_i \) as in Proposition 2.10 we will get for any \( k_i \) there exists a \( \phi_m \) such that \( \phi_m = \delta_{k_i} \) and the set \( \{k_i\}_{i=1}^{\infty} \) is dense in \([0, 1]\) (see [2, Lemma 2]). Given that \( D_0 \) is dense in \( D \) so \( J_D = \cap_{d \in D} \ker \delta_d = \cap_{d \in D_0} \ker \delta_d \). Thus \( J_D = X_S \), where \( S \) is the intersection of directed diagrams corresponding to kernel of \( \phi_i = \delta_{d_i}, d_i \in D_0 \). This completes the proof.

**Representing matrix for \( X \otimes Y \):** If \( X \) and \( Y \) are separable \( L_1 \)-predual spaces, then it is known that \( X \otimes Y \) is also a separable \( L_1 \)-predual space. We adopt the above algorithm to find a representing matrix for \( X \otimes Y \). Let \( X \) and \( Y \) has representing matrices \( (a_n^i)_{n \geq 1}^{1 \leq i \leq n} \) and \( (b_n^j)_{n \geq 1}^{1 \leq j \leq n} \) respectively corresponding to the admissible basis \( \{e_n^i : 1 \leq i \leq n\} \) and \( \{f_n^j : 1 \leq j \leq n\} \). Then \( X \otimes Y = \bigcup_{n=1}^{\infty} E_{n^2} \), where \( E_{n^2} \) is isometric to \( \ell_\infty^{n^2} \) with admissible bases \( \{e_n^i \otimes f_n^j, i = 1, \ldots, n; j = 1, \ldots, n\} \). We will denote this collection as \( \{E_{n^2}^i : 1 \leq i \leq n^2\} \) with the following convention:

(a) First \( n^2 \) terms of the admissible basis of \( E_{(n+1)^2} \) is same as the admissible basis of \( E_{n^2} \). For example if \( E_{n^2}^i = e_{(n-1)^2}^k \otimes e_{(n-1)^2}^l \) then \( E_{(n+1)^2}^i = e_{n^2}^k \otimes e_{n^2}^l \).

(b) We will choose \( \{E_{(n+1)^2}^i : 1 \leq i \leq (n+1)^2 - n^2\} \) by the following way.

Take \( E_{(n+1)^2}^{n+1} = e_{n+1}^1 \otimes f_{n+1}^1 \), \( E_{(n+1)^2}^{n+2} = e_{n+1}^1 \otimes f_{n+1}^1 \). For \( i = 2k + 1, k \in \mathbb{N} \), \( E_{(n+1)^2}^{2k+1} = e_{n+1}^k \otimes f_{n+1}^1 \). For \( i = 2k + 2, k \in \mathbb{N} \), \( E_{(n+1)^2}^{2k+2} = e_{n+1}^1 \otimes f_{n+1}^k \).
We will now follow algorithm for ‘Fill in the Gaps’ described above. Let us illustrate this with first few steps.

Let $C = (c_{i,j})_{i,j \geq 1}$ be the representing matrix of $X \otimes Y$. According to the above convention $E^1_1 = e_1 \otimes f_1^1$ and $E^1_2 = e_2 \otimes f_2^1$, $E^3_1 = e_2 \otimes f_1^2$, $E^4_1 = e_2 \otimes f_2^1$, $E^4_2 = e_3 \otimes f_2^2$. By expanding $E^1_1$ in terms of $\{E^i_j\}_{i=1}^4$ according to the given representing matrix of $X$ and $Y$ we get

$$E^1_1 = E^1_1 + b_1^1 E^2_4 + a_1^1 E^3_4 + a_1^1 b_1^1 E^4_4.$$  

Similarly expansion of $E^1_1$ in terms of $\{E^i_j\}_{i=1}^4$ according to the representing matrix $C$ of $X \otimes Y$,

$$E^1_1 = E^1_1 + c_1^1 E^2_4 + (c_1^1 + c_1^2) E^3_4 + (c_1^3 + c_1^4) E^4_4.$$  

By following the algorithm we will get $c_1^1 = b_1^1$, $c_2^1 = a_1^1$, $c_3^1 = 0$, $c_3^2 = a_1^1 b_1^1$, $c_3^3 = 0$, $c_3^4 = 0$. By expanding $\{E^i_j\}_{i=1}^4$ in terms of $\{E^j_i\}_{i=1}^9$ according to given representing matrices for $X$, $Y$ and matrix $C$ we will get

$$c_1^1 = b_1^1, \quad c_2^2 = b_2^1, \quad c_3^3 = 0, \quad c_4^4 = 0, \quad c_5^5 = 0, \quad c_6^6 = 0, \quad c_7^7 = 0, \quad c_8^8 = 0.$$  

Proceeding as above we will get representing matrix of $X \otimes Y$ as

$$C = \begin{bmatrix}
    b_1^1 & a_1^1 & a_1^1 b_1^1 & b_2^2 & a_2^2 & 0 & 0 & a_1^1 b_1^1 & \ldots \\
    \vdots & 0 & 0 & b_2^2 & 0 & 0 & a_2^2 & a_1^1 b_2^1 & \ldots \\
    \vdots & \vdots & 0 & 0 & a_2^2 & b_1^1 & 0 & a_2^2 b_1^1 & \ldots \\
    \vdots & \vdots & \vdots & 0 & b_1^1 & 0 & a_2^2 b_1^1 & a_2^2 b_1^1 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ldots \\
\end{bmatrix}.$$
Remark 3.5. From above description of representing matrix for $X \otimes Y$ we can actually read off representing matrices $(a_i^n)_{n \geq 1}^{1 \leq i \leq n}$ and $(b_i^n)_{n \geq 1}^{1 \leq i \leq n}$ for $X$ and $Y$ respectively. For example representing matrix of $Y$ is given by

$$B = \begin{bmatrix} c_1^1 & c_1^4 & c_1^9 & c_1^{42} & \cdots & c_1^n & \cdots \\ c_2^1 & c_2^4 & c_2^9 & c_2^{42} & \cdots & c_2^n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ c_9^1 & c_9^4 & c_9^9 & c_9^{42} & \cdots & c_9^n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix} .$$

Thus if a $L_1$-predual space has a representing matrix like $C$, it is actually tensor product of two $L_1$-predual spaces with representing matrices $A$ and $B$.

References


