

A Note on a Paper of S.G. Kim

DIANA M. SERRANO-RODRÍGUEZ

Departamento de Matemáticas, Universidad Nacional de Colombia
111321 - Bogotá, Colombia
dmserrano0@gmail.com, diserranor@unal.edu.co

Presented by Ricardo García

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Abstract: We answer a question posed by S.G. Kim in [3] and show that some of the results of his paper are immediate consequences of known results.

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The recent paper [3] deals with extreme multilinear forms and polynomials and the constants of the Bohnenblust-Hille inequalities. In this note we answer a question posed in [3] and show that two theorems stated in [3] are immediate corollaries of well known results of this field.

Let \mathbb{K} be \mathbb{R} or \mathbb{C} . The multilinear Bohnenblust-Hille inequality asserts that, given a positive integer m , there is an optimal constant $C(m : \mathbb{K}) \geq 1$ such that

$$\left(\sum_{i_1, \dots, i_m=1}^{\infty} |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C(m : \mathbb{K}) \|U\|,$$

for all bounded m -linear forms $U : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$. The case of complex scalars was first investigated in [1] and the case of real scalars seems to have been just explored more recently. It is well known that the exponent $2m/(m+1)$ is sharp, so one of the main goals of the research in this field is to investigate the constants involved. The following result was proved in [4]:

THEOREM 1. ([4, COROLLARY 5.4], 2018) *Let $m \geq 2$ be a positive integer. If the optimal constant $C(m : \mathbb{R})$ is attained in a certain $T : c_0 \times \dots \times c_0 \rightarrow \mathbb{R}$, then the quantity of non zero monomials of T is bigger than $4^{m-1} - 1$.*

As an immediate corollary we conclude that if $N_1, \dots, N_m \geq 1$ are positive integers such that

$$\prod_{j=1}^m N_j \leq 4^{m-1} - 1,$$

then

$$\sup \left(\sum_{i_1, \dots, i_m=1}^{N_1, \dots, N_m} |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} < C(m : \mathbb{R}),$$

where the sup runs over all norm one m -linear forms $T : \ell_\infty^{N_1} \times \dots \times \ell_\infty^{N_m} \rightarrow \mathbb{R}$. In particular,

$$\sup \left(\sum_{i,j,k=1}^2 |T(e_i, e_j, e_k)|^{\frac{6}{4}} \right)^{\frac{4}{6}} < C(3 : \mathbb{R}),$$

where the sup runs over all norm one m -linear forms $T : \ell_\infty^2 \times \ell_\infty^2 \times \ell_\infty^2 \rightarrow \mathbb{R}$, and this is the content of [3, Theorem 4.9].

The polynomial Bohnenblust-Hille inequality for real scalars asserts that, given a positive integer m , there is an optimal constant $C_p(m : \mathbb{R}) \geq 1$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_p(m : \mathbb{R}) \|Q\|,$$

for all $N \geq 1$ and for all m -homogeneous polynomials $Q : \ell_\infty^N(\mathbb{R}) \rightarrow \mathbb{R}$ given by

$$Q(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha.$$

To the best of our knowledge, the case of real scalars became unexplored until the publication of the paper [2] in 2015, where it is proved that the constants $C_p(m : \mathbb{R})$ cannot be chosen with a sub-exponential growth. More precisely,

THEOREM 2. ([2, THEOREM 2.2], 2015)

$$C_p(m : \mathbb{R}) > \left(\frac{2\sqrt[4]{3}}{\sqrt{5}} \right)^m > (1.177)^m,$$

for all positive integers $m \geq 2$.

The above result is, obviously, by far, rather precise than [3, Theorem 4.5], which states that

$$C_p(m : \mathbb{R}) \geq 2^{\frac{m+1}{2m}},$$

for all positive integers $m \geq 2$. The only case that deserves a little bit more of attention is the case $m = 2$, since

$$\left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^2 < 2^{\frac{3}{4}},$$

but in the case $m = 2$ a quick look at the proof of [2, Proof of Theorem 2.2] shows that

$$C_p(2 : \mathbb{R}) \geq \frac{3^{\frac{3}{4}}}{\frac{5}{4}} \approx 1.8236 > 2^{\frac{3}{4}},$$

and this answers in the negative the Question (2) posed by the author in [3, Question (2)].

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