# Order statistics of diffusion on fractals

S. Bravo Yuste
*Departamento de Física, Universidad de Extremadura, 06071 Badajoz, Spain*
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When a large number $N$ of independent random walkers diffuse on a fractal substrate, what is the $m$th moment of the time spent $t_{j,N}$ by the first $j$ to cross a given distance from the starting place? The answer to this order-statistic problem is given in the form of an asymptotic expression (including the main and first two corrective terms) for large $N$. To first order, the $m$th moment and the variance of $t_{j,N}$ go as $(\ln N)^{m(1-d_w)}$ and $(\ln N)^{-2d_w}$, respectively, for $1 \leqslant j \ll N$, $d_w$ being the anomalous diffusion exponent of the fractal medium. It is shown that the first result can be obtained through less rigorous but more intuitive reasonings. Comparisons of asymptotic results with numerical calculations are provided for three fractal substrates and for the one-dimensional case. [S1063-651X(98)07605-3]

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## I. INTRODUCTION

Studies of diffusive processes are usually concerned with statistical properties associated with a *single* random walker wandering over an Euclidean substrate. There is a well-established large body of knowledge about this long-standing problem [1,2]. A more recent line of investigation is related to the diffusion process that takes place on a fractal substrate [3,4]. This process is characterized by its ''anomalous'' properties with respect to the ''normal'' (or Euclidean) process. Much less is known about the statistical properties associated with the diffusion of a *set* of random walkers [1–10], notwithstanding its interest [11]. For example, asymptotic expressions for the number of different sites visited by $N \gg 1$ random walkers that diffuse on Euclidean [5] and fractal [6] substrates have been obtained only recently.

In this paper we address a related (and also basic) problem: Given a set of $N$ independent random walkers, all starting at the same place on a fractal substrate, what is the time spent by the $j$th random walker to cross a given distance? Or, stated using the more colorful words coined by de Gennes [12], what is the escape time of the first $j$ ants (random walkers) of a battalion of $N$ members initially parachuted onto a site of a fractal labyrinth if the exits are placed at a distance $r$ from the origin? In more technical words, we address the problem of describing the order statistic of the diffusion process [8,13] on a fractal. For Euclidean substrates, rigorous results are known only for the one-dimensional case [8,9]. As will be seen, the results provided in this paper, which are also valid for the one-dimensional case, improve and extend those results (a short presentation of some of the ideas and results of this paper can be found in Ref. [10]).

The paper is organized as follows. In Sec. II we give and deduce relations about some statistical quantities that will be used later. Section III is the most technical section and is devoted to obtaining asymptotic expressions for the generating function of the $m$th moment of the first-passage time $t_{j,N}$ of the $j$th random walker out of a total of $N \gg j$. (This derivation was absent in Ref. [10].) The main and two first corrective asymptotic terms of these first-passage-time moments $\langle t_{j,N}^m \rangle$ and the main term of their variances $\sigma_{j,N}$ are explicitly given in Sec. IV. Some of these results are obtained with a less rigorous but more intuitive and simple reasoning in Sec. V. In Sec. VI results found from using these asymptotic expressions are compared with those obtained numerically. The paper ends in Sec. VII with a discussion and conclusions.

## II. FIRST-PASSAGE TIMES AND MORTALITY FUNCTION

The probability density $\psi_{j,N}(t)$ for the time spent by the $j$th out of $N$ noninteracting particles to first reach a given distance $r$ (i.e., the first-passage density of the $j$th particle of a set of $N$) is easily expressed [8,13] in terms of the first-passage-time density to this distance of a single particle $\psi(t) \equiv \psi_{1,1}(t)$, namely,

$$\psi_{j,N}(t) = N!/[(N-j)!(j-1)!]\psi(t)h^{j-1}(t)[1-h(t)]^{N-j}. \tag{2.1}$$

Here $h(t) = \int_0^t \psi(\tau)d\tau$ is the mortality function, i.e., the probability that a single diffusing particle has reached this distance $r$ during the time interval $[0,t]$. Thus, with the function $\psi(t)$ [or $h(t)$] at hand, one can evaluate the moments of $t_{j,N}$ (the ''$j$th-passage time''):

$$\langle t_{j,N}^m \rangle = \int_0^\infty t^m \psi_{j,N}(t)dt. \tag{2.2}$$

In what follows the mean time spent by a single random walker to reach the (arbitrary) distance $r$ will be taken as the time unit.

Van den Broeck in Ref. [15] has shown how to evaluate the Laplace transform of the first-passage-time density $\tilde{\psi}(s) \equiv 1/f(s)$ for finitely ramified fractals. For not too large values of $s, \tilde{\psi}(s)$ can be evaluated through the Taylor series expansion of $f(s)$. For large $s$, it is more suitable to use the asymptotic relation [14]

$$\tilde{\psi}(s) \approx \tilde{A}\exp(-\tilde{C}s^{1/d_w}). \tag{2.3}$$

The constant $\tilde{A}$ is related to the probability that one particle has gone from a site to any of its nearest neighbors on the $n$th generation (fractal) lattice via any of the shortest paths traced over the $(n+1)$th generation lattice [14]. Therefore, this constant can be known exactly. However, $\tilde{C}$ has to be estimated numerically. Away from the short-time regime the Laplace inversion of $\tilde{\psi}(s)=1/f(s)$ leads to

$$\psi(t)=\sum_{n=1}^{\infty}\frac{e^{x_n t}}{f'(x_n)}, \tag{2.4}$$

$x_n<0$ being the $n$th largest root of $f(x)=0$ [15]. The asymptotic behavior of $\psi(t)$ for small times can be obtained by Laplace inversion of Eq. (2.3) through the saddle-point method [16]:

$$\psi(t)\approx\hat{A}t^{-(1+\beta/2)}\exp(-C/t^{\beta})(1+\phi_1 t^{\beta}), \tag{2.5}$$

where $\hat{A}=\sqrt{\nu/(2\pi)}(\beta\tilde{C}\nu)^{\nu/2}\tilde{A}$, $C\equiv t_0^{\beta}=\beta^{\beta}(\tilde{C}\nu)^{\nu}$, $\phi_1=(1-2d_w)(d_w-2)/24C$, $\beta=1/(d_w-1)$, and $\nu=\beta+1$ (values of these constants can be found in Ref. [10]).

From the above equation for $\psi(t)$ it is not difficult to find a similar asymptotic expression for the mortality function $h(t)=\int_0^t\psi(\tau)d\tau$. Thus, from Eq. (2.4) and taking into consideration that $1\equiv\langle t\rangle\equiv\int_0^\infty\psi(t)dt=-\Sigma_{n=1}^{\infty}[x_n f'(x_n)]^{-1}$, one finds

$$h(t)=1+\sum_{n=1}^{\infty}\frac{e^{x_n t}}{x_n f'(x_n)}. \tag{2.6}$$

This expression is well known and especially simple for the one-dimensional lattice:

$$h(t)=1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\exp[-\tfrac{1}{8}(2n-1)^2\pi^2 t]. \tag{2.7}$$

Given that $\int_0^t\tau^{\alpha}\exp(-b/t^{\beta})=b^{(\alpha+1)/\beta}\Gamma(-(\alpha+1)/\beta,b/t^{\beta})/\beta$ and that the incomplete Gamma function satisfies [17] $\Gamma(a+1,z)\approx z^a e^{-z}[1+a/z+O(1/z^2)]$ for $z\to\infty$, one obtains from Eq. (2.5) that

$$h(t)\approx\tilde{h}(t)\equiv At^{\beta/2}\exp[-(t_0/t)^{\beta}](1+h_1 t^{\beta}) \tag{2.8}$$

for small $t$, with $A=\hat{A}/\beta C$ and $h_1=\phi_1-1/2C$ (values of these constants for certain substrates can be found in Ref. [10]).

Several cases of first-passage-time densities of the $j$th particle $\psi_{j,N}(t)$ for the two-dimensional Sierpinsky gasket are shown in Fig. 1. (For other substrates, such as those to be considered below in Fig. 4, these densities are very similar.) When $j\ll N$, we see that these functions take significant values for small $t$ only. This means that only the behavior of $\psi(t)$ and $h(t)$ for small times are relevant in order to calculate $\langle t_{j,N}^m\rangle$ when $j\ll N$. We shall return to this fact in the next section.

## III. ASYMPTOTIC EVALUATION OF THE GENERATING FUNCTION

The generating function of the $m$th moment of the $j$th-passage time $\mathcal{U}_{N,m}(z)=\Sigma_{j=1}^{N}\langle t_{j,N}^m\rangle z^{j-1}$ can be written as [8,9]

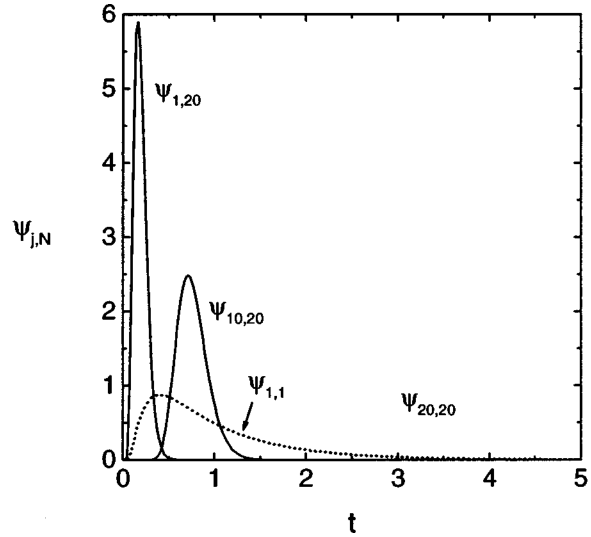


FIG. 1. First-passage-time distribution of the $j$th random walker out of $N$ for the two-dimensional Sierpinsky lattice. Here and in the rest of the figures times are expressed in units of the mean first-passage time of a *single* random walker.

$$\mathcal{U}_{N,m}(z)=\frac{m}{1-z}\int_0^\infty t^{m-1}\{(1-h+hz)^N-z^N\}dt. \tag{3.1}$$

Dropping the $z^N$ term, we get

$$U_{N,m}(z)\equiv\frac{m}{1-z}\int_0^\infty t^{m-1}\exp\{N\ln[1-h(t)(1-z)]\}dt. \tag{3.2}$$

The point is that $\mathcal{U}_{N,m}(z)$ and $U_{N,m}(z)$ have the same Taylor series expansion up to the term of order $z^{N-1}$, so that $\langle t_{j,N}^m\rangle$ can also be estimated through the evaluation of this pseudogenerating function $U_{N,m}(z)$. Therefore, by means of the asymptotic evaluation of this function one can also find asymptotic expressions for $\langle t_{j,N}^m\rangle$ when $j\ll N$. To this end we proceed as in Ref. [9]. We start by splitting the interval of integration

$$U_{N,m}=U_{N,m}^{(\tau)}+U_{N,m}^{(\infty)} \tag{3.3}$$

where

$$U_{N,m}^{(\tau)}=\frac{m}{1-z}\int_0^\tau dt t^{m-1}\exp\{N\ln[1-h(t)(1-z)]\}dt, \tag{3.4}$$

$$U_{N,m}^{(\infty)}=\frac{m}{1-z}\int_\tau^\infty dt t^{m-1}\exp\{N\ln[1-h(t)(1-z)]\}dt. \tag{3.5}$$

The time $\tau$ is chosen (small enough) so that $h(t)\approx\tilde{h}(t)$ for $0\leqslant t\leqslant\tau$ or, being more precise, so that

$$|e^{-Nh(\tau)}-e^{-N\tilde{h}(\tau)}|\simeq|h(\tau)-\tilde{h}(\tau)|Ne^{-N\tilde{h}(t)}=\frac{1}{k}, \tag{3.6}$$

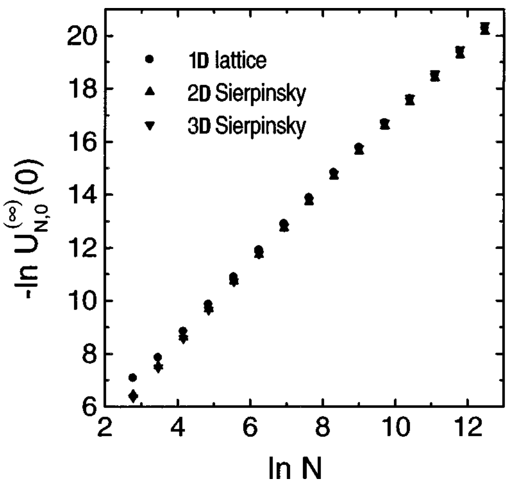


FIG. 2. Dependence on $N$ of $U^{(\infty)}_{N,1}(0)$ [the contribution to $\langle t_{1,N}\rangle = U_{N,1}(0)$ neglected in our asymptotic expansion] for the one-dimensional lattice and the two- and three-dimensional Sierpinsky lattices.

where $k \gg 1$ is an arbitrary large positive constant that, for convenience, we take as $1/|h(t)-\tilde{h}(t)|$. This means that the time $\tau$ is simply the solution of

$$e^{N\tilde{h}(\tau)} = N. \tag{3.7}$$

Using the above relations and because $h(t)$ is small for $t \lesssim \tau$, one can write

$$\exp\{N\ln[1-h(t)(1-z)]\} = \exp[-N(1-z)\tilde{h}(t)] \times\{1+O(N\tilde{h}^2(t))\}. \tag{3.8}$$

Neglecting terms of order $N\tilde{h}^2(t)$, one finds

$$U^{(\tau)}_{N,m} \approx \frac{m}{1-z}\int_0^\tau dt\, t^{m-1}\exp[-N(1-z)\tilde{h}(t)]. \tag{3.9}$$

Notice that these neglected terms decay essentially as $1/N$ because, from Eq. (3.7), $N\tilde{h}^2(\tau) = (\ln^2 N)/N$. Therefore, their algebraically decaying contribution to $U_{N,m}$ is negligible against that of the retained term, which, as we will see later, decays in a mildly logarithmic way for large $N$. Also, an inspection of $U_{N,m}$ as given by Eq. (3.2) shows that for $N \gg 1$ the integrand is only non-negligible when $h(t)$ is very small, i.e., for small times. This is in agreement with the remarks at the end of Sec. II (see Fig. 1). In other words, one expects a negligible contribution of $U^{(\infty)}_{N,m}$ to $U_{N,m}$. This is explicitly confirmed in Fig. 2, which shows the dependence on $N$ of $U^{(\infty)}_{N,1}(z=0)$, i.e., the dependence on $N$ of the contribution to $\langle t_{1,N}\rangle = U_{N,1}(0)$ of those times longer than $\tau$. We see that this contribution essentially decays as a power of $1/N$. This justifies calculating $U^{(\tau)}_{N,m}$ by Eq. (3.9) for obtaining the asymptotic dependence of $U_{N,m}(z)$. To this end we shall follow closely the procedure used in [8,9]. Inserting $\tilde{h}(t)$ as given by Eq. (2.8) into Eq. (3.9) one finds

$$U^{(\tau)}_{N,m} \approx \frac{m}{1-z} t_0^m [f_m^{(0)}(\lambda) + h_1 t_0^\beta f_m^{(1)}(\lambda)], \tag{3.10}$$

where

$$f_m^{(n)}(\lambda) = \lambda^n \int_0^\epsilon \frac{d\rho}{\rho^{1-n}} e^{-\lambda\rho} \frac{x^{m+(n+1)\beta}(\rho)}{\beta + \frac{\beta}{2}x^\beta(\rho)}, \tag{3.11}$$

$$\rho = x^{\beta/2}\exp(-x^{-\beta}), \tag{3.12}$$

with $\epsilon = (\tau/t_0)^{\beta/2}\exp[(-t_0/\tau)^\beta]$ and $\lambda = Nat_0^{\beta/2}(1-z)$. Notice that $\epsilon \approx t_0^{\beta/2}\tilde{h}(\tau)/A = \ln(N)/N \ll 1$ when $N \gg 1$. The function $x(\rho)$ can be estimated as follows. We take logarithms of both sides of Eq. (3.12), obtaining

$$-\ln\rho \equiv v^\beta = x^{-\beta} - \frac{\beta}{2}\ln x. \tag{3.13}$$

It is clear that $(xv)^{-1} \approx 1$ for small $x$ (note that $t < \tau \ll 1$ implies $x \ll 1$) because $\beta \ln x/2v^\beta$ is a minor correction of order $\ln v/v^\beta$. We can be more precise. Defining $\xi \equiv x^{-1} - v$ and using Eq. (3.13), it is not difficult to see that $\xi/v$ satisfies the relation

$$\frac{\beta}{2}\ln v + \beta\left(\frac{1}{2}+v^\beta\right)\frac{\xi}{v} + \frac{\beta}{2}\left[(\beta-1)v^\beta - \frac{1}{2}\right]\left(\frac{\xi}{v}\right)^2 = O\left(\frac{\xi^3}{v^{3-\beta}}\right), \tag{3.14}$$

with the solution

$$\frac{\xi}{v} = -\frac{1}{2}\frac{\ln v}{v^\beta} + \frac{1}{8}(1-\beta)\frac{\ln^2 v}{v^{2\beta}} + \frac{1}{4}\frac{\ln v}{v^{2\beta}} + O\left(\frac{\ln v}{v^\beta}\right)^3. \tag{3.15}$$

Inserting this solution into $x = v^{-1}(1-\xi/v)^{-1}$, one finds

$$\frac{x^\mu}{1+x^\beta/2} = v^{-\mu}\left[1 - \frac{v^{-\beta}}{2} + \frac{\mu}{2}v^{-\beta}\ln v + \frac{v^{-2\beta}}{4} - \frac{2\mu+\beta}{4}v^{-2\beta}\ln v + \frac{\mu}{8}(\mu+\beta)v^{-2\beta}\ln^2 v + O(v^{-3\beta}\ln^3 v)\right]. \tag{3.16}$$

Thus Eq. (3.10) can be written as

$$U^{(\tau)}_{N,m} \approx \frac{t_0^m \alpha}{1-z}\left\{ I_{1+\alpha} + \frac{1+\alpha}{2}J_{2+\alpha} - \frac{1}{2}I_{2+\alpha} + \frac{(1+\alpha)(2+\alpha)}{8}K_{3+\alpha} - \frac{3+2\alpha}{4}J_{3+\alpha} + \left[\frac{1}{4} - h_1 t_0^\beta(2+\alpha)\right]I_{3+\alpha}\right\}, \tag{3.17}$$

where $\alpha \equiv m/\beta = m(d_w - 1)$ and [9]

$$I_\mu=\int_0^\epsilon \frac{d\rho}{\rho}e^{-\lambda\rho}\frac{1}{\ln^\mu(1/\rho)}\approx\frac{1}{\mu-1}\frac{1}{\ln^{\mu-1}\lambda}-\frac{\gamma}{\ln^\mu\lambda}+\frac{\mu}{2}\frac{\pi^2/6+\gamma^2}{\ln^{\mu+1}\lambda}, \tag{3.18}$$

$$J_\mu=\int_0^\epsilon \frac{d\rho}{\rho}e^{-\lambda\rho}\frac{\ln[\ln(1/\rho)]}{\ln^\mu(1/\rho)}\approx\frac{1/(\mu-1)+\ln\ln\lambda}{(\mu-1)\ln^{\mu-1}\lambda}-\frac{\gamma\ln\ln\lambda}{\ln^\mu\lambda}, \tag{3.19}$$

$$K_\mu=\int_0^\epsilon \frac{d\rho}{\rho}e^{-\lambda\rho}\frac{\ln^2[\ln(1/\rho)]}{\ln^\mu(1/\rho)}\approx\frac{1}{\ln^{\mu-1}\lambda}\left[\frac{2}{(\mu-1)^3}+\frac{2\ln\ln\lambda}{(\mu-1)^2}+\frac{\ln^2\ln\lambda}{\mu-1}\right], \tag{3.20}$$

$\gamma\simeq 0.577\,215$ being the Euler constant. Inserting Eqs. (3.18)–(3.20) into Eq. (3.17), one finally gets

$$U_{N,m}(z)=\frac{1}{1-z}\frac{t_0^m}{\ln^\alpha\lambda}\left\{1+\frac{\alpha}{\ln\lambda}\left(\frac{1}{2}\ln\ln\lambda-\gamma\right)+\frac{\alpha}{2\ln^2\lambda}\left[(1+\alpha)\left(\frac{\pi^2}{6}+\gamma^2\right)+\gamma-2h_1t_0^\beta-\left(\frac{1}{2}+(1+\alpha)\gamma\right)\ln\ln\lambda+\frac{1+\alpha}{4}\ln^2\ln\lambda\right]+O\left(\frac{\ln^3\ln\lambda}{\ln^3\lambda}\right)\right\}. \tag{3.21}$$

For the one-dimensional lattice this formula is a generalization of those obtained in Ref. [9] [in that reference only one-dimensional formulas of $U_{N,m}(z)$ for $m=1$ and $m=2$ were given].

## IV. ESCAPE TIMES

From the preceding section we know that $\langle t_{j,N}^m\rangle$ is the coefficient of $z^{j-1}$ in the Taylor series expansion of $U_{N,m}(z)$. Therefore, the $m$th moment of the first-passage time of the first out of $N\gg 1$ diffusing particles is equal to $U_{N,m}(0)$, i.e.,

$$\langle t_{1,N}^m\rangle=\frac{t_0^m}{\ln^\alpha\lambda_0N}\left\{1+\frac{\alpha}{\ln\lambda_0N}\left(\frac{1}{2}\ln\ln\lambda_0N-\gamma\right)+\frac{\alpha}{2\ln^2\lambda_0N}\left[(1+\alpha)\left(\frac{\pi^2}{6}+\gamma^2\right)+\gamma-2h_1t_0^\beta-\left(\frac{1}{2}+(1+\alpha)\gamma\right)\ln\ln\lambda_0N+\frac{1+\alpha}{4}\ln^2\ln\lambda_0N\right]+O\left(\frac{\ln^3\ln\lambda_0N}{\ln^3\lambda_0N}\right)\right\}, \tag{4.1}$$

where $\lambda_0\equiv\lambda(z=0)/N=At_0^{\beta/2}$. The calculation of $\langle t_{j,N}^m\rangle$ for $j>1$ is more involved, as we shall now see. Because $\ln^n(1-z)=n!\sum_{i=n}^\infty(-1)^iS_i(n)z^i/i!$, where $S_i(n)$ are the Stirling numbers of first kind [17], one finds that

$$\frac{1}{\ln^\alpha\lambda}=\frac{1}{\ln^\alpha\lambda_0N}\sum_{n=0}^\infty(-1)^n\frac{(\alpha+n-1)!}{(\alpha-1)!}\times\frac{1}{\ln^n\lambda_0N}\sum_{i=n}^\infty(-1)^i\frac{S_i(n)}{i!}z^i. \tag{4.2}$$

Interchanging the order of the sums and because $S_n(m)=0$ if $n<m$ and $S_{n+1}(1)=(-1)^nn!$, one gets

$$\frac{1}{\ln^\alpha\lambda}=\frac{1}{\ln^\alpha\lambda_0N}\left\{1+\sum_{n=1}^\infty z^n\left[\frac{\alpha}{n}\frac{1}{\ln\lambda_0N}+\frac{(-1)^n}{n!}\frac{(\alpha+1)!}{(\alpha-1)!}\frac{S_n(2)}{\ln^2\lambda_0N}+O\left(\frac{1}{\ln^3\lambda_0N}\right)\right]\right\}. \tag{4.3}$$

Also, it is not difficult to prove that

$$\frac{\ln\ln\lambda}{\ln^{\alpha+1}\lambda}=\frac{1}{\ln^{\alpha+1}\lambda_0N}\left[\ln\ln\lambda_0N+\sum_{n=1}^\infty\frac{(\alpha+1)\ln\ln\lambda_0N-1}{n\ln\lambda_0N}z^n+O\left(\frac{\ln\ln\lambda_0N}{\ln^2\lambda_0N}\right)\right]. \tag{4.4}$$

Inserting Eqs. (4.3) and (4.4) into Eq. (3.21), one finds

$$U_{N,m}(z)\approx\frac{1}{1-z}\left(\langle t_{1,N}^m\rangle+\frac{t_0^m\alpha}{\ln^{\alpha+1}\lambda_0N}\sum_{n=1}^\infty\frac{\Delta_n(\alpha)}{n}z^n\right), \tag{4.5}$$

where

$$\Delta_n(\alpha)=1+\frac{\alpha+1}{2\ln\lambda_0N}\left[(-1)^n\frac{2S_n(2)}{(n-1)!}+\ln\ln(\lambda_0N)-\frac{1}{\alpha+1}-2\gamma\right]+O\left(\frac{\ln^2\ln\lambda_0N}{\ln^2\lambda_0N}\right). \tag{4.6}$$

Expanding $1/(1-z)$ in powers of $z$ in Eq. (4.5) and extracting the coefficient of $z^{j-1}$, one finds

$$\langle t_{j,N}^m\rangle\approx\langle t_{1,N}^m\rangle+\frac{t_0^m\alpha}{\ln^{\alpha+1}\lambda_0N}\sum_{n=1}^{j-1}\frac{\Delta_n(\alpha)}{n}. \tag{4.7}$$

The variance $\sigma_{j,N}^2\equiv\langle t_{j,N}^2\rangle-\langle t_{j,N}\rangle^2$ can be obtained from Eqs. (4.7) and (4.1):

$$\sigma_{j,N}^2\approx\frac{t_0^2(d_w-1)^2}{\ln^{2d_w}\lambda_0N}\left[\frac{\pi^2}{6}-\left(\sum_{n=1}^{j-1}\frac{1}{n}\right)^2+\sum_{n=1}^{j-1}(-1)^n\frac{2S_n(2)}{n!}\right]. \tag{4.8}$$

Now it should be clear why we have calculated the main and *two* corrective terms of $U_{N,m}$: It turns out that the main and first corrective term of $\langle t_{j,N}^2\rangle$ are equal to those of $\langle t_{j,N}\rangle^2$, so that only the difference between their second corrective terms contributes to the *main* term of the variance. It is notable that to lowest order in $\ln N$ the coefficient of varia-

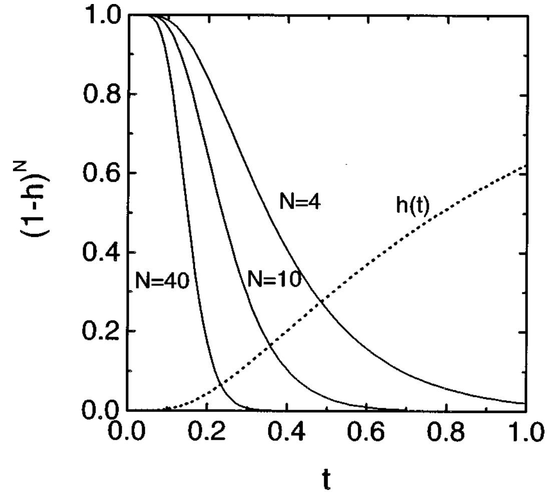


FIG. 3. Function $[1-h(t)]^N$ vs $t$ with $N=4,10,40$ for the two-dimensional Sierpinsky gasket. We see that the larger the value of $N$, the closer $[1-h(t)]^N$ is to a step function. The dotted line is simply the mortality function $h(t)$.

tion $t_{j,N}/\sigma_{j,N}$ goes as $\ln N$ independently of the substrate (this was checked numerically in Ref. [10]).

### V. SIMPLER WAYS TO OBTAIN SOME OF THE PRESENT RESULTS

It is instructive to see how one can find some of the previous laboriously obtained asymptotic results by only resorting to simple and intuitive arguments. Let us start by showing how we can easily estimate $\langle t_{1,N}^m\rangle=\mathcal{U}_{N,m}(0)=m\int_0^\infty dt t^{m-1}[1-h(t)]^N$ in fairly good agreement with the rigorous result of Eq. (4.1). The following reasoning was already used in Ref. [9] in order to get $\langle t_{1,N}^m\rangle$, but only for the one-dimensional case. The crucial fact in our argument is that the function $[1-h(t)]^N$ approaches a unit step function $\Theta(t-\tau)$ when $N\to\infty$, where $\tau$ is the step's width [18] (see Fig. 3). Therefore, $\langle t_{1,N}^m\rangle\approx m\int_0^\tau dt t^{m-1}=\tau^m$. We shall estimate the width $\tau$ by solving the transcendental equation

$$[1-h(\tau)]^N=\kappa \tag{5.1}$$

with, say, $\kappa=1/2$. This value for $\kappa$ is arbitrary: It is clear that any other value between 0 and 1 would also be valid because the step's width is not very sensitive to the value of $\kappa$ if $N\gg 1$. For these values of $N$ the above equation $1/2=\exp[N\ln(1-h(t)]$ becomes $1/2\approx\exp[-Nh(\tau)]\approx\exp[-NAt_0^{\beta/2}\exp(-(t_0/\tau)^\beta)]$, so that

$$\tau^\beta\approx\frac{t_0^\beta}{\ln N+\dfrac{\beta}{2}\ln\tau+\ln A-\ln\ln 2}. \tag{5.2}$$

A first solution is given by $\tau^\beta=t_0^\beta/\ln N$. Therefore,

$$\langle t_{1,N}^m\rangle=\frac{t_0^m}{\ln^\alpha N}, \tag{5.3}$$

with $\alpha=m/\beta$, is a first approximation to $\langle t_{1,N}^m\rangle$. Notice that this expression fully agrees with the approximation of order zero (main term) of the rigorously obtained Eq. (4.1). Inserting the above first solution for $\tau^\beta$ into the right-hand side of Eq. (5.2), we get a better approximation

$$\tau^\beta\approx\frac{t_0^\beta}{\ln N-\frac{1}{2}\ln\ln N+\dfrac{\beta}{2}\ln t_0+\ln A-\ln\ln 2} \tag{5.4}$$

and therefore that

$$\langle t_{1,N}^m\rangle\approx\frac{t_0^m}{\ln^\alpha N}\left(1+\alpha\frac{\frac{1}{2}\ln\ln N-\dfrac{\beta}{2}\ln t_0-\ln A+\ln\ln 2}{\ln N}\right), \tag{5.5}$$

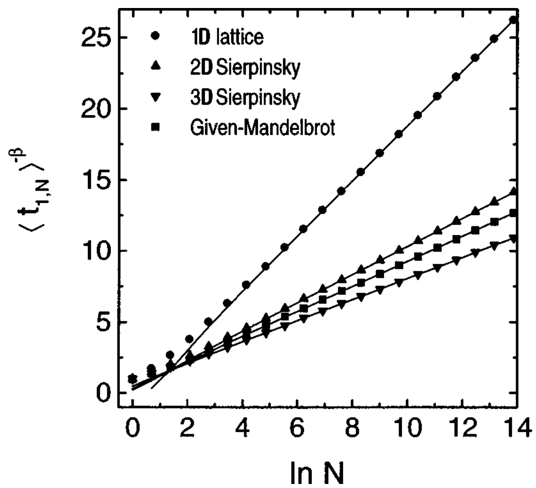


FIG. 4. Dependence on $N$ of the mean first-passage time of the first particle out of $N$, $\langle t_{1,N}\rangle$, for the one-dimensional lattice, the two- and three-dimensional Sierpinsky lattices, and the Given-Mandelbrot curve. Here $\beta=1/(d_w-1)$. The symbols correspond to the numerical estimates for $N=2^n$ with $n=0,1,\ldots,20$ and the solid lines to Eq. (4.1) with $m=1$.

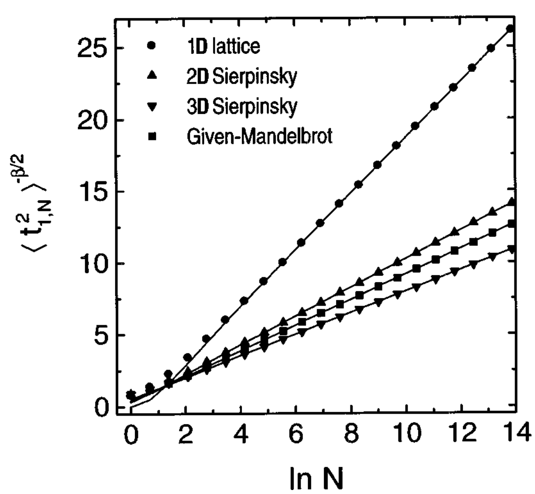


FIG. 5. Dependence on $N$ of the second moment of the first-passage time of the first particle out of $N$, $\langle t_{1,N}^2\rangle$, for the same substrates as in Fig. 4. The symbols correspond to the numerical estimates for $N=2^n$ with $n=0,1,\ldots,20$ and the solid lines to Eq. (4.1) with $m=2$.

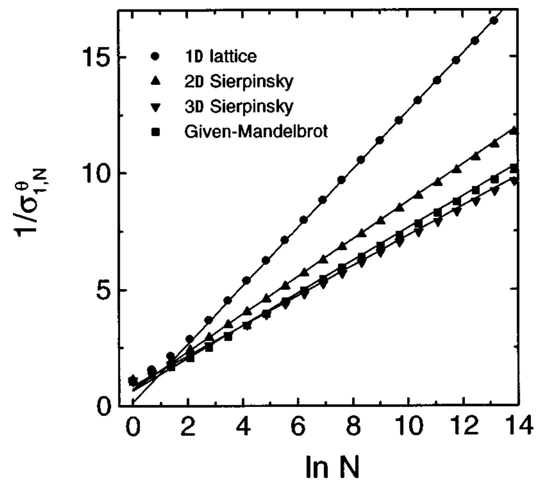


FIG. 6. Dependence on $N$ of the variance of the first-passage time of the first particle out of $N$, $\sigma^2_{1,N}$, for the same substrates as in Fig. 4. Here $\theta$ stands for $1/d_w$. The symbols correspond to the numerical estimates for $N=2^n$ with $n=0,1,\ldots,20$ and the solid lines to Eq. (4.8) with $j=1$.

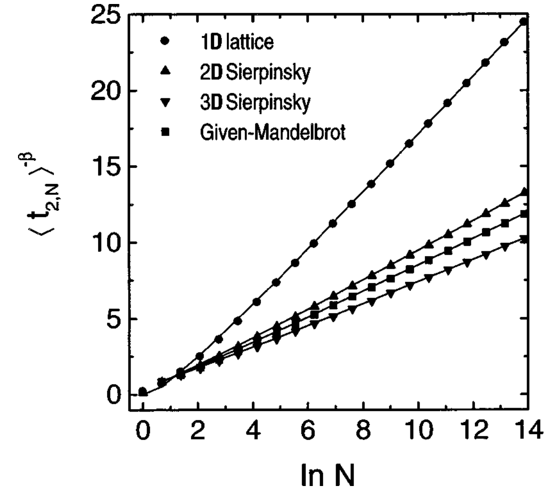


FIG. 7. Dependence on $N$ of the mean first-passage time of the *second* particle out of $N$, $\langle t_{2,N}\rangle$, for the same substrates as in Fig. 4. The symbols correspond to the numerical estimates for $N=2^n$ with $n=1,\ldots,20$ and the solid lines to Eq. (4.7) with $m=1$ and $j=2$.

with $\alpha=m/\beta$, in good agreement with the rigorous expression of Eq. (4.1).

Let us see now another intuitive and simple way of estimating the mean escape time of the $j$th particle $\langle t_{j,N}\rangle$. Our argument goes as follows. From the definition of the mortality function $h(t)$ and because the $N$ diffusing particles are independent, one would expect that, for example, the mean first-passage time of one-half of these particles should be approximately given by $t_{1/2}$, where $h(t_{1/2})=1/2$. In the same way, one expects that the mean first-passage time of $j$ out of these $N$ diffusing particles $\langle t_{j,N}\rangle$ can be approximated by $t_{j/N}$, where

$$h(t_{j/N})=j/N. \tag{5.6}$$

If $j\ll N$, then $t_{j/N}\ll 1$ and $h(t_{j/N})\approx\tilde{h}(t_{j/N})$, so that the equation for $t_{j/N}$ becomes $j/N\approx At_{j/N}^{\beta/2}\exp[-(t_0/t_{j/N})^{\beta}]$, i.e.,

$$t_{j/N}^{\beta}\approx\frac{t_0^{\beta}}{\ln N-\ln j+\dfrac{\beta}{2}\ln t_{j/N}+\ln A}. \tag{5.7}$$

We solve this equation in the same way as Eq. (5.2), finding a first approximated solution for $j\ll N$: $t_{j/N}^{\beta}\approx t_0^{\beta}/\ln N$. Inserting this solution into the right-hand side of Eq. (5.7), we get a better approximation

$$t_{j/N}\approx t_{1/N}+\frac{t_0\alpha}{\ln^{\alpha+1}N}\ln j, \tag{5.8}$$

where

$$t_{1/N}\approx\frac{t_0}{\ln^{\alpha}N}\left[1+\frac{\alpha}{\ln N}\left(\frac{1}{2}\ln\ln N-\frac{\beta}{2}\ln t_0-\ln A\right)\right] \tag{5.9}$$

and $\alpha=1/\beta=d_w-1$. These expressions are very close to those rigorously obtained in Sec. IV, namely, Eqs. (4.1) and (4.7) (notice that $\ln j\simeq\Sigma_{n=1}^{j}1/n-\gamma$ for large $j$ [17]).

In the next section we shall compare the results provided by these simply obtained formulas and the rigorous asymptotic expression of Eq. (4.7) with those obtained numerically. As we will see, the formulas of this section lead to surprisingly good results.

TABLE I. Mean first-passage time of the first random walker out of $N$, $\langle t_{1,N}\rangle$, for the two-dimensional Sierpinsky lattice as given by different approximations. Times are expressed in units of the mean first-passage time of a *single* random walker. The label ''zero'' refers to the main term (order zero) of Eq. (4.1), ''step'' refers to Eq. (5.5), ''frac.'' to Eq. (5.8), ''asym.'' to the full asymptotic expression of Eq. (4.1), and ''num.'' to the numerical results.

| Equation | $N=10$ | $N=10^2$ | $N=10^3$ | $N=10^4$ | $N=10^5$ | $N=10^6$ |
|---|---|---|---|---|---|---|
| zero | 0.3233 | 0.1293 | 0.0757 | 0.0517 | 0.0385 | 0.0303 |
| step | 0.1678 | 0.1111 | 0.0715 | 0.0507 | 0.0384 | 0.0304 |
| frac. | 0.2359 | 0.1247 | 0.0768 | 0.0534 | 0.0400 | 0.0315 |
| asym. | 0.2773 | 0.1200 | 0.0729 | 0.0508 | 0.0383 | 0.0303 |
| num. | 0.2549 | 0.1190 | 0.0728 | 0.0508 | 0.0383 | 0.0302 |

TABLE II. Mean first-passage time of the second random walker out of $N$, $\langle t_{2,N}\rangle$, for the two-dimensional Sierpinsky lattice as given by different approximations. Times are expressed in units of the mean first-passage time of a *single* random walker. The labels have the same meaning as in Table I, except for ''asym,'' which now refers to Eq. (4.7).

| Equation | $N=10$ | $N=10^2$ | $N=10^3$ | $N=10^4$ | $N=10^5$ | $N=10^6$ |
|---|---|---|---|---|---|---|
| frac. | 0.3645 | 0.1504 | 0.0868 | 0.0585 | 0.0431 | 0.0335 |
| asym. | 0.3509 | 0.1453 | 0.0846 | 0.0573 | 0.0429 | 0.0330 |
| num. | 0.3739 | 0.1497 | 0.0858 | 0.0578 | 0.0425 | 0.0331 |

## VI. COMPARISON WITH NUMERICAL RESULTS

In this section we shall compare the results provided by the analytical asymptotic expressions obtained in Secs. IV and V with those calculated numerically for different substrates. This can be accomplished by using the relation

$$\langle t_{j+1,N}^m\rangle=\langle t_{j,N}^m\rangle+m\frac{N!}{j!(N-j)!}\int_0^\infty t^{m-1}h^j(t)\times[1-h(t)]^{N-j}dt, \tag{6.1}$$

with

$$\langle t_{1,N}^m\rangle=m\int_0^\infty t^{m-1}[1-h(t)]^N dt. \tag{6.2}$$

This expression is readily obtained from Eqs. (2.1) and (2.2) integrating by parts. Analogously, in order to calculate numerically the variance, one can write it as

$$\sigma_{j+1,N}^2=\sigma_{j,N}^2+2\frac{N!}{j!(N-j)!}\int_0^\infty (t-\langle t_{j+1,N}\rangle)h(t)^j\times[1-h(t)]^{N-j}dt, \tag{6.3}$$

where

$$\sigma_{1,N}^2=\langle t_{1,N}\rangle^2+2\int_0^\infty (t-\langle t_{1,N}\rangle)[1-h(t)]^N dt. \tag{6.4}$$

The mortality function is evaluated through Eq. (2.8) for small times and by means of Eq. (2.6) otherwise.

In Figs. 4–6 we compare the first moment, the second moment, and the variance of the first-passage time of the first particle with the corresponding asymptotic analytical results for four substrates. The same comparison is made in Fig. 7, but for the first moment of the first-passage time of the second particle. We see that the agreement is very good even for a relatively small number of particles. The results for $\langle t_{j,N}^m\rangle$ with $N=10^n$ and $n=1,2,\ldots,6$ found by means of the simply obtained formulas of Eqs. (5.3), (5.5), and (5.8) and by means of the rigorous asymptotic expression of Eq. (4.7) are compared with the numerical results in Table I . In Table II we compare $\langle t_{j,N}^m\rangle$ for $N=10^n$ and $n=1,2,\ldots,6$ as given by Eq. (5.8) and by the rigorous asymptotic expression of Eq. (4.7) against numerical results. The goodness of the results provided by the simply obtained formulas is notable.

## VII. CONCLUSIONS

In this paper we have addressed a basic problem about the diffusion of a set of particles on a fractal substrate (including the one-dimensional lattice), namely, the order-statistic problem. That is, we have been interested in giving a statistical description of the time at which the $j$th random walker out of a total of $N$ reach, for the first time, a given distance. Our answer was given in terms of rigorously obtained and detailed asymptotic expressions (the main and first two corrective terms included) of the $m$th moment of this time $\langle t_{j,N}^m\rangle$ for $N\gg 1$. For example, it was found that, to first order, $\langle t_{j,N}^m\rangle$ goes as $(\ln N)^{m(1-d_w)}$, the $n$th corrective term being essentially of order $\ln^{-n}N$ with respect to the main asymptotic term. This means that the exit times decay in a mildly logarithmic way with $N$ and that the corrective terms are usually not negligible even for very large values of $N$. It is also worth pointing out that in order to get only the main asymptotic term of the variance of $t_{j,N}$ [which goes as $(\ln N)^{-2d_w}$] it was necessary to know $\langle t_{j,N}\rangle$ and $\langle t_{j,N}^2\rangle$ up to the second corrective term. It was also shown that the main term of $\langle t_{j,N}^m\rangle$ for either $j=1$ (and arbitrary $m$) or $m=1$ (and arbitrary $j$) can be obtained by means of very simple arguments and we found that these arguments also lead to a good estimate of the next corrective term.

It should be noticed that the results reported in this paper have only been proved for finitely ramified deterministic fractals because in our derivation it was crucial to know the form of the mortality function and we only know it [cf. Eq. (2.8)] for this class of fractals. Thus a natural question is whether these results are applicable to other classes of self-similar substrates (such as other kinds of fractals or even disordered media). I think that the answer is yes. First, there are precedents in which it is known that statistical properties related to the diffusion processes are described in the same way for deterministic fractals (finitely ramified fractals included) as for disordered media (the propagator for the short-time regime is a well-known example). Another (indirect) argument (already discussed in Ref. [10]) is as follows. In Ref. [10] it was proved that the number of different sites of a fractal visited by a set of $N\gg 1$ independent random walkers $S_N(t)$ can be derived from Eq. (4.1). Notice that this equation would in principle only be valid for finitely ramified fractals. However, it turned out that the expression for $S_N(t)$ so obtained was valid for disordered media too [6]. Therefore, it seems reasonable to expect that Eq. (4.1) and the other order-statistic results found in this paper could also be valid for any self-similar substrate.

## ACKNOWLEDGMENTS

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