TRANSPORT COEFFICIENTS FOR A GRANULAR GAS AROUND SIMPLE SHEAR FLOW

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From gases to glasses in granular matter: Thermodynamic and hydrodynamic aspects.

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OUTLINE

- BOLTZMANN KINETIC EQUATION
- SIMPLE (UNIFORM) SHEAR FLOW (**USF**)
- SMALL PERTURBATIONS FROM USF: **TRANSPORT COEFFICIENTS**
- RESULTS FROM A **KINETIC MODEL**
- LINEAR STABILITY ANALYSIS
- CONCLUSIONS

BOLTZMANN KINETIC EQUATION

- Rapid granular flows $\longrightarrow$ Fluid of hard spheres with *inelastic* collisions

- Simplest model: *smooth* hard spheres. Inelasticity characterized by a constant coefficient of normal restitution

$$0 \leq \alpha \leq 1$$

- Kinetic level description: one-particle velocity distribution function

$$f(\mathbf{r}, \mathbf{v}, t)$$

Provides all the relevant information on the state of the system

- At low density: *inelastic* Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f(\mathbf{r},\mathbf{v},t) = J[\mathbf{v}|f(t),f(t)]$$

Boltzmann collision operator

$$J\left[\mathbf{v}_1|f,f\right] = \sigma^2 \int d\mathbf{v}_2 \int d\widehat{\boldsymbol{\sigma}}\, \Theta(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g}) \times \left[\alpha^{-2} f(\mathbf{r},\mathbf{v}_1')f(\mathbf{r},\mathbf{v}_2',t) - f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)\right]$$

Collision rules: $\mathbf{v}_1' = \mathbf{v}_1 - \frac{1}{2}\left(1+\alpha^{-1}\right)(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})\widehat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2' = \mathbf{v}_2 + \frac{1}{2}\left(1+\alpha^{-1}\right)(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})\widehat{\boldsymbol{\sigma}}$

$$\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$$

Difficult task to get **exact** solutions of the Boltzmann equation, especially under extreme conditions.

- Hydrodynamic fields:

Number density $$n(\mathbf{r},t)=\int d\mathbf{v}f(\mathbf{r},\mathbf{v},t)$$

Flow velocity $$\mathbf{u}(\mathbf{r},t)=\frac{1}{n(\mathbf{r},t)}\int d\mathbf{v}\mathbf{v}f(\mathbf{r},\mathbf{v},t)$$

Granular temperature $$T(\mathbf{r},t)=\frac{m}{3n(\mathbf{r},t)}\int d\mathbf{v}V^2(\mathbf{r},t)f(\mathbf{r},\mathbf{v},t)$$

$$\mathbf{V}(\mathbf{r},t)\equiv\mathbf{v}-\mathbf{u}(\mathbf{r},t)$$

- J[f,f] conserves the particle number and momentum but not the energy: **Macroscopic** balance equations

$$D_t n + n \nabla \cdot \mathbf{u} = 0 \ ,$$

$$D_t u_i + (mn)^{-1} \nabla_j P_{ij} = 0 \ ,$$

$$D_t T + \frac{2}{3n} \left(\nabla \cdot \mathbf{q} + P_{ij} \nabla_j u_i \right) = -\zeta T$$

Heat flux

Pressure tensor

Cooling rate (fractional energy changes per unit time)

$$\mathsf{P}(\mathbf{r},t) = \int d\mathbf{v}\, m\mathbf{V}\mathbf{V}\, f(\mathbf{r},\mathbf{v},t),$$

$$\mathbf{q}(\mathbf{r},t) = \int d\mathbf{v}\, \frac{1}{2} m V^2 \mathbf{V}\, f(\mathbf{r},\mathbf{v},t),$$

$$\zeta(\mathbf{r},t) = -\frac{1}{3n(\mathbf{r},t)T(\mathbf{r},t)} \int d\mathbf{v} m V^2 J[\mathbf{r},\mathbf{v}|f(t)]$$

Balance equations

↓

Closed set of equations

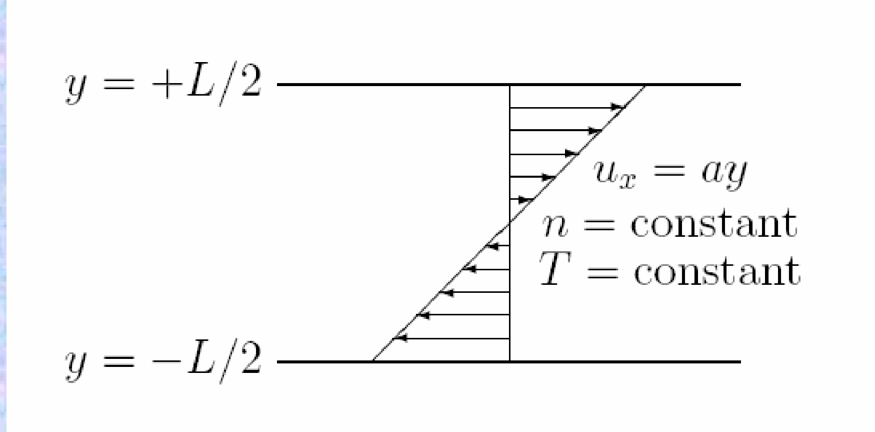
↓

Fluxes and cooling rate in terms of the fields and their gradients

UNIFORM SHEAR FLOW (USF)

- Due to the kinetic energy dissipation in collisions, energy must be externally injected to achieve stationary conditions.

- Mechanism of energy input: Simple shear flow



Time evolution of the *granular temperature* arises from the balance of two opposite effects: viscous heating and collisional cooling. When the shearing work is balanced by the dissipation in collisions, a *steady* state is reached.

Steady state condition → $aP_{xy} = -\frac{3}{2}\zeta p$ $p = nT$

- Intrinsic connection between the velocity gradient (*nonequilibrium* parameter) and dissipation (*coefficient of restitution*). Both parameters are **not** independent.

$$a^*(\alpha) = \frac{a}{\nu(T)}, \quad \nu(T) \propto \sqrt{T}$$

Collision frequency for hard spheres

- USF becomes *spatially homogeneous* in the local Lagrangian frame moving with the flow velocity:

$$f(\mathbf{r}, \mathbf{v}) \to f(\mathbf{V})$$

$$V_i = v_i - a_{ij} r_j, \quad a_{ij} = a\delta_{ix}\delta_{jy}$$

$$-aV_y \frac{\partial}{\partial V_x} f(\mathbf{V}) = J\left[\mathbf{V}|f,f\right]$$

- Rheological properties: Pressure tensor elements

$$\begin{aligned} a_{i\ell}P_{j\ell} + a_{j\ell}P_{i\ell} &= m\int d\mathbf{V} V_i V_j J[\mathbf{V}|f,f] \\ &\equiv \Lambda_{ij}. \end{aligned}$$

Closed problem once the Boltzmann collisional moment is known. This requires the explicit knowledge of the distribution function. Formidable task!!

- Good estimate of the low velocity moments of J[f,f] by using Grad's approximation:

$$f(\mathbf{V}) \to f_0(\mathbf{V})\left[1 + \frac{m}{2T}\left(\frac{P_{ij}}{p} - \delta_{ij}\right)V_i V_j\right]$$

- Jenkins&Richman,1985

Gaussian distribution

$$\Lambda_{ij} = -\nu\left[\beta\left(P_{ij} - p\delta_{ij}\right) + \zeta^* P_{ij}\right]$$

$$\nu(T) = \frac{16}{5} n\sigma^2 \sqrt{\frac{\pi T}{m}},$$

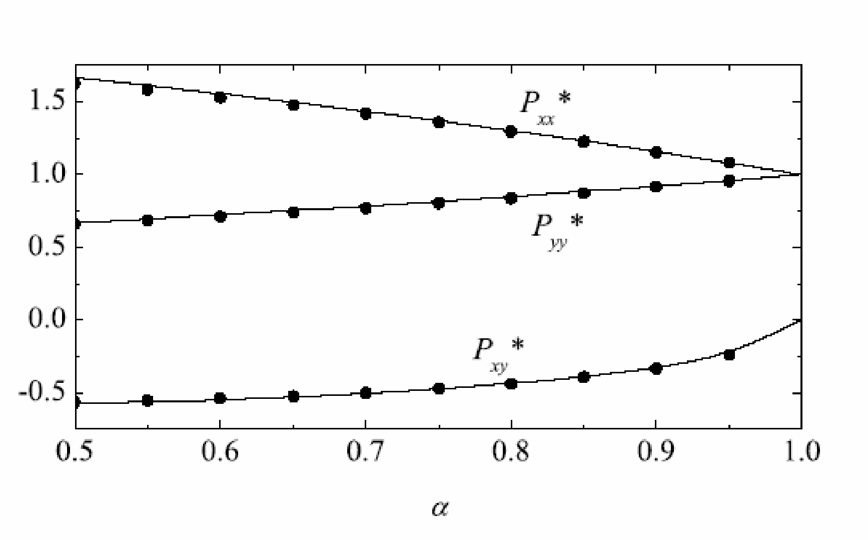
$$\zeta^* = \frac{\zeta}{\nu} = \frac{5}{12}(1-\alpha^2)$$

$$\beta = \frac{1+\alpha}{2}\left(1 - \frac{1-\alpha}{3}\right)$$

The set of coupled equations for $P_{ij}^* = P_{ij}/p$ can be exactly solved

$$P_{xx}^* = 3 - 2P_{yy}^*, \quad P_{yy}^* = P_{zz}^* = \frac{\beta}{\beta+\zeta^*}, \quad P_{xy}^* = -\frac{\beta}{(\beta+\zeta^*)^2} a^*$$

$$a^* = \sqrt{\frac{3}{2}\frac{\zeta^*}{\beta}}\,(\beta+\zeta^*)$$



Development of *inhomogeneities* and formation of clusters as the flow progresses. Time-dependent *microstructures*.

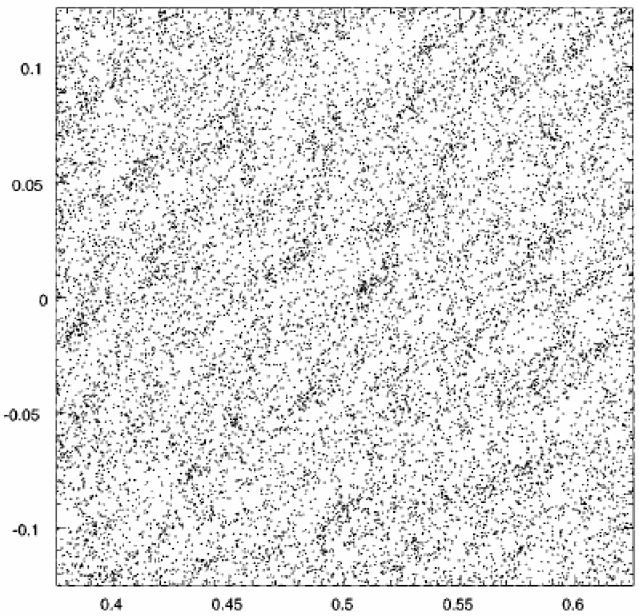


FIG. 3. The particle configuration for System II. The portion of the flow domain shown here corresponds to a square region in the center of System II whose area is 1/16 the area of System II. The time here corresponds to 100 collisions per particle following the initial condition.

MD simulations: Walton&Braun, JR1986; Hopkins&Louge, JFM 1991; Goldhirsch&Tan, PF 1996, 1997....; and more....

USF is unstable for long enough wavelength spatial perturbations

Many analytical attempts to understand this instability

1. Most of the stability analysis based on the Navier-Stokes equations: small velocity gradient, i.e., small dissipation. Savage, JFM 1992; Babic, JFM 1993; Alam&Nott, JFM 1997, JFM 1998; and more....

2. Solution of the Boltzmann equation in the quasielastic limit. Kumaran, Physica A 2000, PF 2001. *Anomalous* behavior of hydrodynamic modes.

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Inherent rheology of a granular fluid in uniform shear flow

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• **My alternative**: Linear stability analysis of the hydrodynamic equations with respect to the USF state. Transport coefficients of the *perturbed* USF (with a finite shear rate) instead of the Navier-Stokes coefficients. No restriction a priori to low-dissipation since the reference state goes beyond this range of values of dissipation.

• *First* step of this issue → *Transport coefficients* for a granular gas around USF

Gas is in a state that deviates from the USF by **small** spatial gradients. A new set of generalized transport coefficients that present a complex dependence on dissipation.

SMALL PERTURBATIONS FROM USF

USF can be *disturbed* by small spatial perturbations. Response of the system to these perturbations: additional contributions to the momentum and heat fluxes which are characterized by generalized transport coefficients.

$$\frac{\partial}{\partial t} f - aV_y \frac{\partial}{\partial V_x} f + (\mathbf{V} + \mathbf{u}_0) \cdot \nabla f = J[\mathbf{V}|f, f]$$

$$\mathbf{V} = \mathbf{v} - \mathbf{u}_0$$

Flow velocity of the undisturbed USF

$$u_{0,i} = a_{ij} r_j$$

The true velocity is now $\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$,

The true peculiar velocity $\mathbf{c} \equiv \mathbf{v} - \mathbf{u} = \mathbf{V} - \delta\mathbf{u}$

Small perturbation

- Hydrodynamic equations

$$\partial_t n + \mathbf{u}_0 \cdot \nabla n = -\nabla \cdot (n\delta\mathbf{u}),$$

$$\partial_t \delta\mathbf{u} + \mathsf{a} \cdot \delta\mathbf{u} + (\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla \delta\mathbf{u} = -(mn)^{-1} \nabla \cdot \mathsf{P},$$

$$\frac{3}{2} n \partial_t T + \frac{3}{2} n (\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla T + a P_{xy} + \nabla \cdot \mathbf{q} + \mathsf{P} : \nabla \delta\mathbf{u} = -\frac{3}{2} p \zeta$$

To solve them, one needs to express the fluxes and the cooling rate in terms of the hydrodynamic fields and their gradients:

$$A(\mathbf{r}, t) \equiv \{ n(\mathbf{r}, t), T(\mathbf{r}, t), \delta\mathbf{u}(\mathbf{r}, t) \}$$

- Chapman-Enskog expansion → Hydrodynamic regime

New ingredient: expansion around a local reference shear flow state (Lee&Dufty, PRE 1997).

$$f(\mathbf{r}, \mathbf{V}, t) \equiv f(A(\mathbf{r}, t), \mathbf{V})$$

Normal solution

$$f(\mathbf{r}, \mathbf{V}, t) = f^{(0)}(A(\mathbf{r}, t), \mathbf{V}) + \epsilon f^{(1)}(A(\mathbf{r}, t), \mathbf{V}) + \cdots$$

$$\epsilon \sim \mathcal{O}(\nabla) : \frac{\text{mean free path}}{\text{hydrodynamic length}}$$

The succesive approximations retain *all* the hydrodynamic orders in the shear rate.

$$\mathsf{P} = \mathsf{P}^{(0)} + \mathsf{P}^{(1)} + \cdots, \quad \mathbf{q} = \mathbf{q}^{(0)} + \mathbf{q}^{(1)} + \cdots, \quad \zeta = \zeta^{(0)} + \zeta^{(1)} + \cdots$$

$$\partial_t = \partial_t^{(0)} + \partial_t^{(1)} + \partial_t^{(2)} + \cdots$$

The action of these operators given by the hydrodynamic equations.

- Zeroth-order solution

$$\partial_t^{(0)} f^{(0)} - aV_y \frac{\partial}{\partial V_x} f^{(0)} = J[\mathbf{V}|f^{(0)}, f^{(0)}]$$

Normal solution

$$\partial_t^{(0)} f^{(0)} = \frac{\partial f^{(0)}}{\partial n} \partial_t^{(0)} n + \frac{\partial f^{(0)}}{\partial T} \partial_t^{(0)} T + \frac{\partial f^{(0)}}{\partial \delta u_i} \partial_t^{(0)} \delta u_i$$

$$\partial_t^{(0)} n = 0, \quad \partial_t^{(0)} T = -\frac{2}{3n} a P_{xy}^{(0)} - T\zeta^{(0)}$$

$$\partial_t^{(0)} \delta u_i + a_{ij} \delta u_j = 0.$$

Note that now the density and temperature are **local** quantities !!

$$\zeta^{(0)} = \frac{5}{12}(1-\alpha^2)\nu$$

$$\nu(T) = \frac{16}{5} n\sigma^2 \sqrt{\frac{\pi T}{m}}$$

Exact balance between viscous heating and cooling does not *locally* apply.

$$-\left(\frac{2}{3n}aP_{xy}^{(0)}+T\zeta^{(0)}\right)\partial_T f^{(0)}-ac_x\frac{\partial}{\partial c_y}f^{(0)}=J[\mathbf{c}|f^{(0)},f^{(0}]$$

$f^{(0)}(\mathbf{r},\mathbf{c},t)$ is not in general a stationary distribution since it depends on time through its dependence on temperature. Complex problem!!

In particular, if one uses Grad's approximation for $f^{(0)}(\mathbf{r},\mathbf{c},t)$

$$-\left(\frac{2}{3n}aP_{xy}^{(0)}+T\zeta^{(0)}\right)\partial_T P_{ij}^{(0)}+a_{i\ell}P_{j\ell}^{(0)}+a_{j\ell}P_{i\ell}^{(0)}=-\nu\left[\beta\left(P_{ij}^{(0)}-p\delta_{ij}\right)+\zeta^* P_{ij}^{(0)}\right]$$

In the hydrodynamic regime:

$$T\partial_T P_{ij}^{(0)}=T\partial_T pP_{ij}^*=p\left(P_{ij}^*-\frac{1}{2}a^*\partial_{a^*}P_{ij}^*\right)$$

$$-\left(\frac{2}{3n}a^*P^*_{xy}+\zeta^*\right)\left(P^*_{ij}-\frac{1}{2}a^*\partial_{a^*}P^*_{ij}\right)+a^*_{i\ell}P^*_{j\ell}+a^*_{j\ell}P^*_{i\ell}+(\beta+\zeta^*)P^*_{ij}+\beta\delta_{ij}=0$$

Behavior of the pressure tensor *near* the steady solution

$$P^*_{ij}(a^*)=P^*_{ij}(a^*_s)+\left(\frac{\partial P^*_{ij}}{\partial a^*}\right)_s(a^*-a^*_s)+\cdots$$

Can be determined **analytically** in terms of the real root of a cubic equation

First order solution

$$\left(\partial_t^{(0)} - aV_y\frac{\partial}{\partial V_x} + \mathcal{L}\right) f^{(1)} = -\left[\partial_t^{(1)} + (\mathbf{V}+\mathbf{u}_0)\cdot\nabla\right] f^{(0)}$$

$$\mathcal{L}X \equiv -\left(J[f^{(0)}, X] + J[X, f^{(0)}]\right)$$

Balance equations:

$$\partial_t^{(1)} n + \mathbf{u}_0\cdot\nabla n = -\nabla\cdot(n\delta\mathbf{u}),$$

$$\partial_t^{(1)}\delta\mathbf{u} + (\mathbf{u}_0+\delta\mathbf{u})\cdot\nabla\delta\mathbf{u} = -\frac{1}{\rho}\nabla\cdot\mathsf{P}^{(0)}$$

$$\frac{3}{2}n\partial_t^{(1)}T + \frac{3}{2}n(\mathbf{u}_0+\delta\mathbf{u})\cdot\nabla T + aP_{xy}^{(1)} + \mathsf{P}^{(0)}:\nabla\delta\mathbf{u} = -\frac{3}{2}p\zeta^{(1)}$$

After some algebra….

First order distribution

$$f^{(1)} = \mathbf{X}_n \cdot \nabla n + \mathbf{X}_T \cdot \nabla T + \mathsf{X}_u : \nabla \delta \mathbf{u}$$

Solutions of the linear integral equations

$$\left(-ac_y \frac{\partial}{\partial c_x} + \mathcal{L}\right) X_{n,i} + \frac{2a}{3}\frac{T}{n}(P^*_{xy} - a^*\partial^*_a P^*_{xy}) X_{T,i} = Y_{n,i}$$

$$\left(-ac_y \frac{\partial}{\partial c_x} + \frac{1}{2}\zeta^{(0)} - \frac{1}{3}\nu a^{*2}\partial^*_a P^*_{xy} + \mathcal{L}\right) X_{T,i} = Y_{T,i}$$

$$\left(-ac_y \frac{\partial}{\partial c_x} + \mathcal{L}\right) X_{u,k\ell} - a\delta_{ky} X_{u,x\ell} - \zeta_{u,k\ell} T \partial_T f^{(0)} = Y_{u,k\ell}$$

$$\zeta^{(1)} = \zeta_{u,ji} \nabla_i \delta u_j$$

First-order contribution to the cooling rate

- First-order corrections to the fluxes:

$$P_{ij}^{(1)} = -\eta_{ijk\ell}\frac{\partial \delta u_k}{\partial r_\ell}$$

$$q_i^{(1)} = -\kappa_{ij}\frac{\partial T}{\partial r_j} - \mu_{ij}\frac{\partial n}{\partial r_j}$$

$$\eta_{ijk\ell} = -\int d\mathbf{c}\, m c_i c_j X_{u,k\ell}(\mathbf{c}),$$

$$\kappa_{ij} = -\int d\mathbf{c}\, \frac{m}{2} c^2 c_i X_{T,j}(\mathbf{c}),$$

$$\mu_{ij} = -\int d\mathbf{c}\, \frac{m}{2} c^2 c_i X_{n,j}(\mathbf{c}).$$

Anisotropy induced by the shear flow gives rise to *new* transport coefficients, reflecting broken symmetry.

Transport in a granular gas subjected to strong shear rate is a quite complex problem.

- For elastic collisions $(a^* = 0)$

the usual Navier-Stokes transport coefficients for ordinary gases are recovered, i.e.,

$$\eta_{ijk\ell} \to \eta_0 \left(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell} - \frac{2}{3}\delta_{ij}\delta_{k\ell} \right), \quad \kappa_{ij} \to \kappa_0 \delta_{ij}, \quad \mu_{ij} \to 0$$

Shear viscosity

$$\eta_0 = p/\nu$$

Thermal conductivity

$$\kappa_0 = 15\eta_0/4m$$

Explicit form of the transport coefficients requires to solve the above integral equations. Two problems:

- Mathematical difficulties embodied in the Boltzmann collision operator: Expansion in a complete set of polynomials (Sonine). In particular,

$$\zeta_{u,ij} = -\frac{1}{15}\sigma^2\sqrt{\frac{\pi}{mT}}(1-\alpha^2)\left(\frac{P_{k\ell}^{(0)}}{nT} - \delta_{k\ell}\right)\eta_{k\ell ij}$$

- Fourth-degree moments of the zeroth-order distribution are needed to get the heat flux. They are not provided by Grad's solution. Formidable task!!

- A possible way to overcome such difficulties is to consider a *kinetic model* of the Boltzmann equation. The general idea is to replace the true Boltzmann operator with a simpler more tractable operator that retains its most relevant properties.

- For elastic collisions, the well-known **BGK** model kinetic equation has been shown to be very reliable to address complex states not accesible via the Boltzmann equation.

- For *granulares* gases, several models have been proposed. They reduce to the BGK equation in the elastic case.

Brey&Dufty&Santos, JSP 1999:

$$J[f,f] \to -\beta\nu(f-f_0)+\frac{\zeta}{2}\frac{\partial}{\partial \mathbf{c}}\cdot(\mathbf{c}f)$$

$$\beta=\frac{1+\alpha}{2}\left(1-\frac{1-\alpha}{3}\right)$$

Adjustable parameter

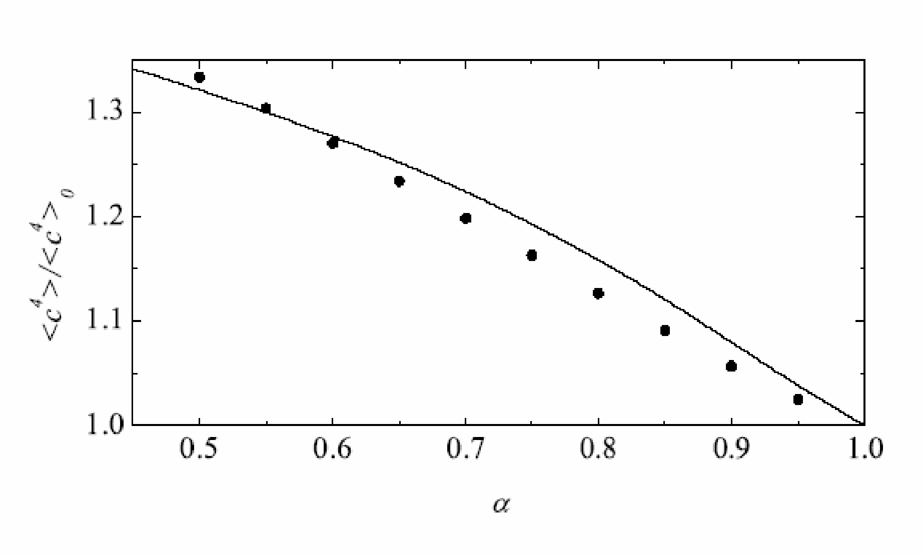
Local eq. distribution

Good points:

- Model yields the same macroscopic balance equations as the Boltzmann equation.
- In the USF problem, model gives the same expressions for the rheological properties as the Boltzmann equation.

What happens beyond the second-degree velocity moments?

• Fourth-degree moments (needed to compute transport coefficients in the *perturbed* USF problem)

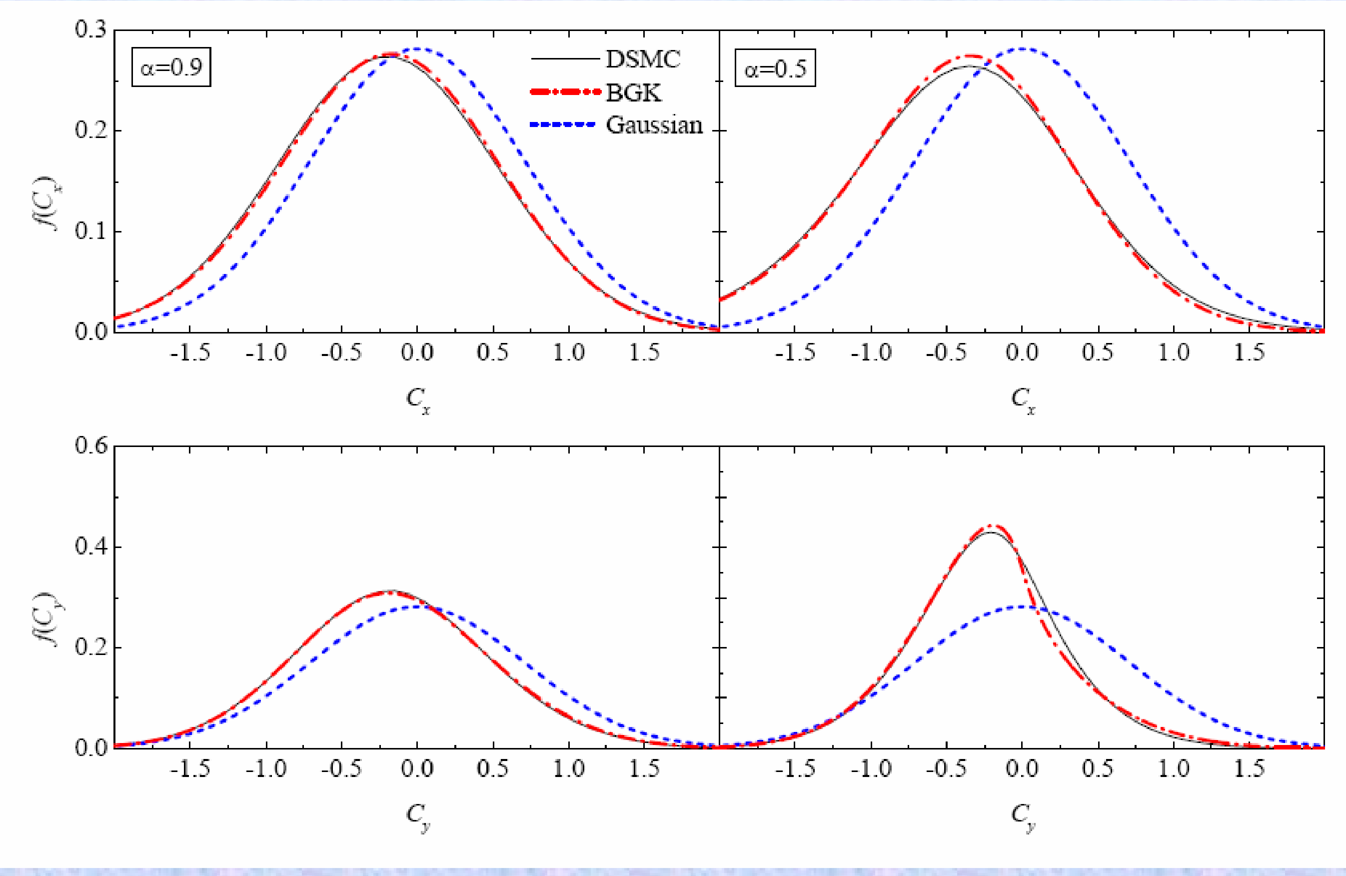


DSMC results
Astillero&Santos,2005

$$\langle c^4 \rangle = \int d\mathbf{c}\; c^4 f(\mathbf{c})$$

$$\langle c^4 \rangle_0 = 15nT^2/m^2$$

- Velocity distribution function



Astillero&Santos, 2005

$$f(\mathbf{V}) = n\left(\frac{m}{2T}\right)^{3/2} f^*(\boldsymbol{\xi}), \quad \boldsymbol{\xi} = \sqrt{\frac{m}{2T}}\mathbf{V}$$

$$f^*(\boldsymbol{\xi}) = \pi^{-3/2}\int_0^\infty ds\, e^{-(1-\frac{3}{2}\overline{\zeta})s} \exp\left[-e^{\overline{\zeta}s}\left(\boldsymbol{\xi} + s\overline{\mathsf{a}}\cdot\boldsymbol{\xi}\right)^2\right]$$

$$\overline{\mathsf{a}} = \mathsf{a}/(\nu\beta)$$

$$\overline{\zeta} = \zeta^{(0)}/(\nu\beta)$$

- Reliability of the kinetic model goes beyond the quasielastic limit

Perturbed USF problem: In the above integral equations

$$\mathcal{L}\mathbf{X}_{n,T} \rightarrow \nu\beta\mathbf{X}_{n,T} - \frac{\zeta^{(0)}}{2}\frac{\partial}{\partial \mathbf{c}}\cdot(\mathbf{c}\mathbf{X}_{n,T})$$

$$\mathcal{L}X_{u,ij} \rightarrow \nu\beta X_{u,ij} - \frac{\zeta^{(0)}}{2}\frac{\partial}{\partial \mathbf{c}}\cdot(\mathbf{c}X_{u,ij}) - \frac{\zeta_{u,ij}}{2}\frac{\partial}{\partial \mathbf{c}}\cdot\left(\mathbf{c}f^{(0)}\right)$$

The set of generalized transport coefficients can be *explicitly* determined as functions of dissipation.

- Some particular situations

  - A steady state in which temperature and density gradients along the y-direction coexist with the linear velocity field

$$u_{0,i} = a_{ij} r_j \quad \longrightarrow \quad \delta \mathbf{u} = \mathbf{0} \qquad \nabla p = 0$$

Generalized Fourier law

$$q_x^{(1)} = -\Delta_{xy} \frac{\partial T}{\partial y}, \quad q_y^{(1)} = -\Delta_{yy} \frac{\partial T}{\partial y}$$

$$\Delta_{ij} = \kappa_{ij} - \frac{n}{T} \mu_{ij}.$$

Previous results are recovered: Tij et al., JSP 2001

Self-consistency of our results

$$\eta^*_{ijk\ell} = \eta_{ijk\ell}/\eta_0$$

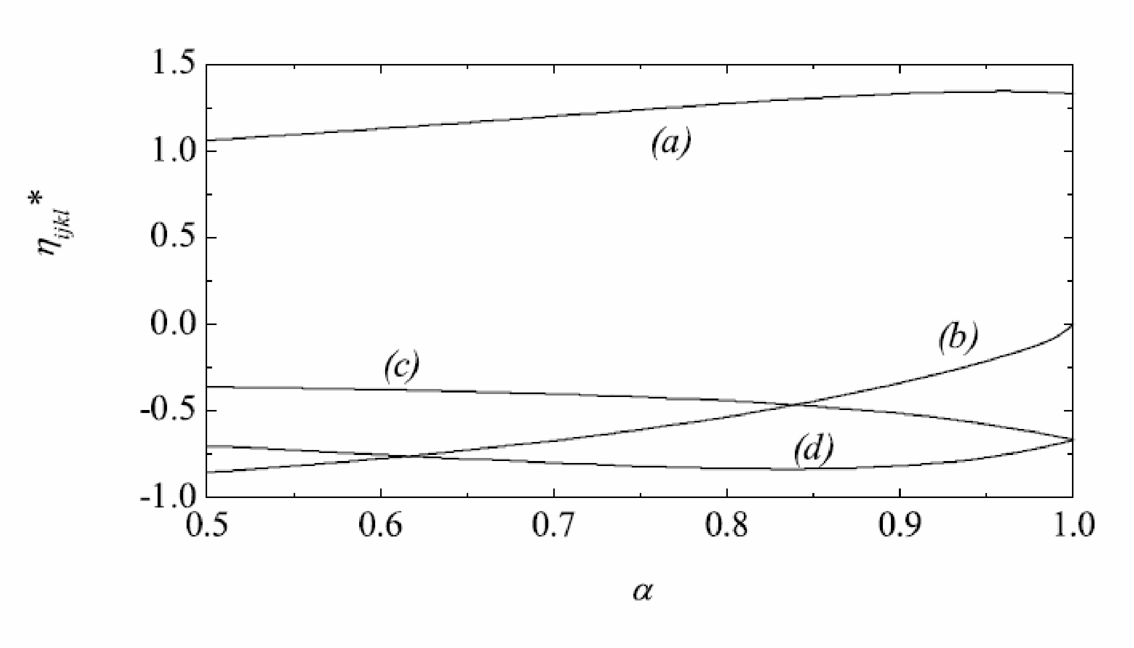


FIG. 4: Plot of the reduced coefficients (a) η^*_{yyyy}, (b) η^*_{xyyy}, (c) η^*_{zzyy}, and (d) η^*_{xxyy} as a function of the coefficient of restitution α.

$$\mu_{ij}^* = n\mu_{ij}/T\kappa_0$$

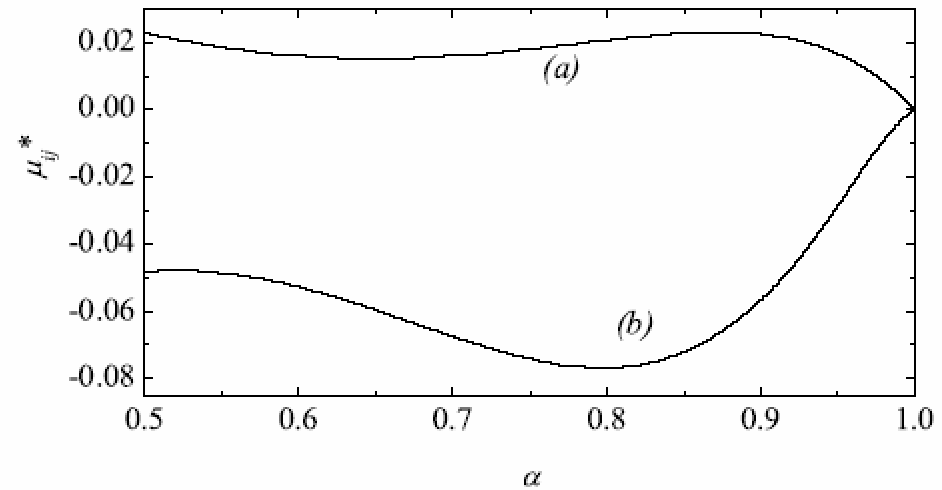


FIG. 2: Plot of the reduced coefficients μ_{yy}^* (a) and μ_{xy}^* (b) as a function of the coefficient of restitution α.

$$\kappa_{ij}^* = \kappa_{ij}/\kappa_0$$

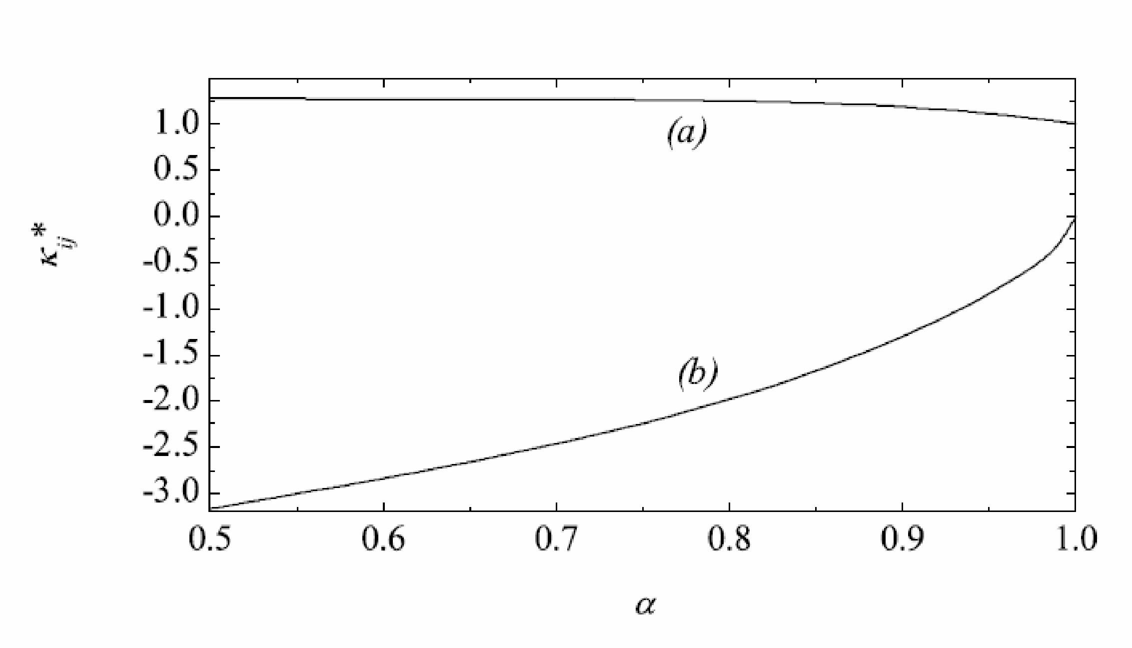


FIG. 3: Plot of the reduced coefficients (a) κ_{yy}^* and (b) κ_{xy}^* as a function of the coefficient of restitution α.

LINEAR STABLITY ANALYSIS

- Linear stability analysis of the nonlinear hydrodynamic equations with respect to the USF state for *small* initial excitations. Hydrodynamic modes for states near USF.

Let us assume that the deviations $\delta x_\mu(\mathbf{r},t) = x_\mu(\mathbf{r},t) - x_{0\mu}(\mathbf{r})$ are small

$\{n, \mathbf{u}, T\}$

$\{n_0, \mathbf{u}_0, T_0\}$ USF state

$$\nabla n_0 = \nabla T_0 = 0, \quad \mathbf{u}_0 = \mathsf{a} \cdot \mathbf{r}$$

We linearize with respect to $\{\delta x_\mu(\mathbf{r},t)\} \equiv \{\delta n(\mathbf{r},t), \delta T(\mathbf{r},t), \delta \mathbf{u}(\mathbf{r},t)\}$

If at least one of the modes grows in time, USF state is linearly unstable.

- Simplification: perturbations along the y-direction only

$$\partial_t \delta n + n_0 \frac{\partial \delta u_y}{\partial y} = 0,$$

$$\frac{3}{2} n_0 \partial_t \delta T + \left(P_{ky}^{(0)} - a\eta_{xyky} \right) \frac{\partial \delta u_k}{\partial y} - \kappa_{yy} \frac{\partial^2 \delta T}{\partial y^2} - \mu_{yy} \frac{\partial^2 \delta n}{\partial y^2} = -\frac{3}{2} n_0 T_0 \zeta_{u,ky} \frac{\partial \delta u_k}{\partial y},$$

$$\partial_t \delta u_k + a\delta_{kx} \delta u_y + \frac{1}{mn_0} \left(\frac{\partial P_{ky}^{(0)}}{\partial n_0} \frac{\partial \delta n}{\partial y} + \frac{\partial P_{ky}^{(0)}}{\partial T_0} \frac{\partial \delta T}{\partial y} - \eta_{ky\ell y} \frac{\partial^2 \delta u_\ell}{\partial y^2} \right) = 0.$$

Fourier representation: $\delta \tilde{x}_\mu(k,t) = \int_{-L/2}^{L/2} dy\; e^{iky} \delta x_\mu(y,t)$

Wavenumber restricted to $k = 2\pi/L, 4\pi/L, 6\pi/L, \ldots$

$\tau = \nu_0 t$ $\delta \tilde{x}_\mu^* \equiv \{\rho_k, \theta_k, \mathbf{w}_k\}$

$$\partial_\tau \delta \tilde{x}_\mu^* + F_{\mu\nu} \delta \tilde{x}_\nu^* = 0$$

$$\rho_k = \frac{\delta \tilde{n}}{n_0}, \quad \theta_k = \frac{\delta \tilde{T}}{T_0}, \quad \mathbf{w}_k = \frac{\delta \tilde{\mathbf{u}}}{\sqrt{m/T_0}}$$

Its eigenvalues $\lambda_\mu(k, \alpha)$ determine the time evolution of $\delta \tilde{x}_\mu^*(k, t)$

- If $\mathrm{Re}\, \lambda_\mu(k, \alpha) > 0$ → USF is linearly stable

One perturbation is decoupled from the other ones:

$$\lambda_5(k, \alpha) = \eta_{zyzy}^* k^{*2}, \quad \eta_{zyzy}^* = \frac{\beta}{(\beta + \zeta^*)^2} > 0$$

- Some special limits:

- Elastic case $(\alpha = 1)$ →

$$\lambda_\mu(k,1) \to \left\{ i\sqrt{\frac{5}{3}}k^* + k^{*2}, -i\sqrt{\frac{5}{3}}k^* + k^{*2}, k^{*2}, k^{*2}, k^{*2} \right\}$$

$$k^* = \ell_0 k, \quad \ell_0 = \sqrt{\frac{T_0 m}{\nu_0}}$$

Excitations around equilibrium are *damped*

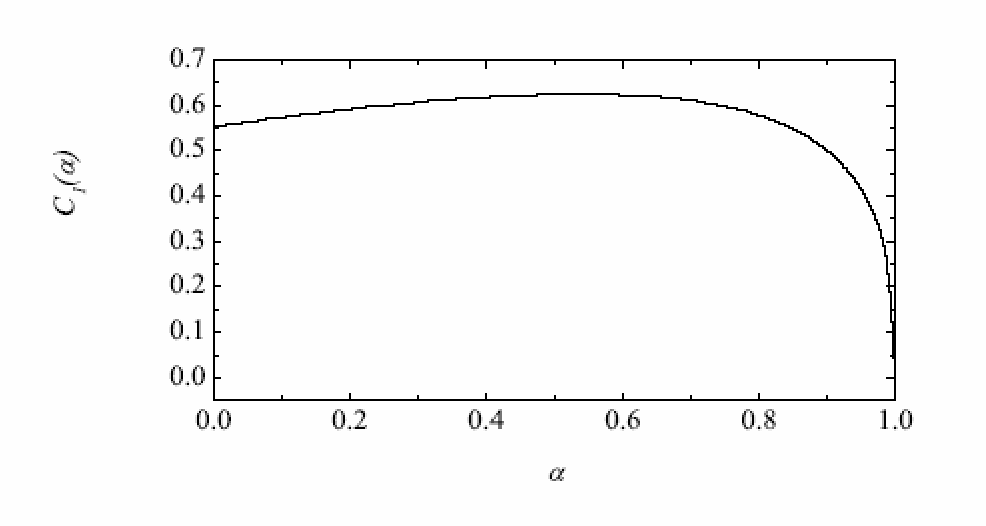
- Finite values of dissipation and small values of wavenumber: More complicated modes. Long-wavelength behavior is not *analytic* in the wavenumber

$$\lambda_\mu(k,\alpha) \sim k^{*2/3}$$

Anomalous behavior: Kumaran, PF 2001

$$\lambda_\mu(k,\alpha) \to \begin{pmatrix} C_0(\alpha)k^{*2} \\ -e^{i\pi/3}C_1(\alpha)k^{*2/3} + e^{-i\pi/3}C_2(\alpha)k^{*4/3} + C_3(\alpha)k^{*2} \\ -e^{-i\pi/3}C_1(\alpha)k^{*2/3} + e^{i\pi/3}C_2(\alpha)k^{*4/3} + C_3(\alpha)k^{*2} \\ C_1(\alpha)k^{*2/3} - C_2(\alpha)k^{*4/3} + C_3(\alpha)k^{*2} \\ \eta^*_{zyzy}k^{*2} \end{pmatrix}$$

Unstable if $C_1 \geq 0$



•USF becomes unstable for long enough wavelength perturbations

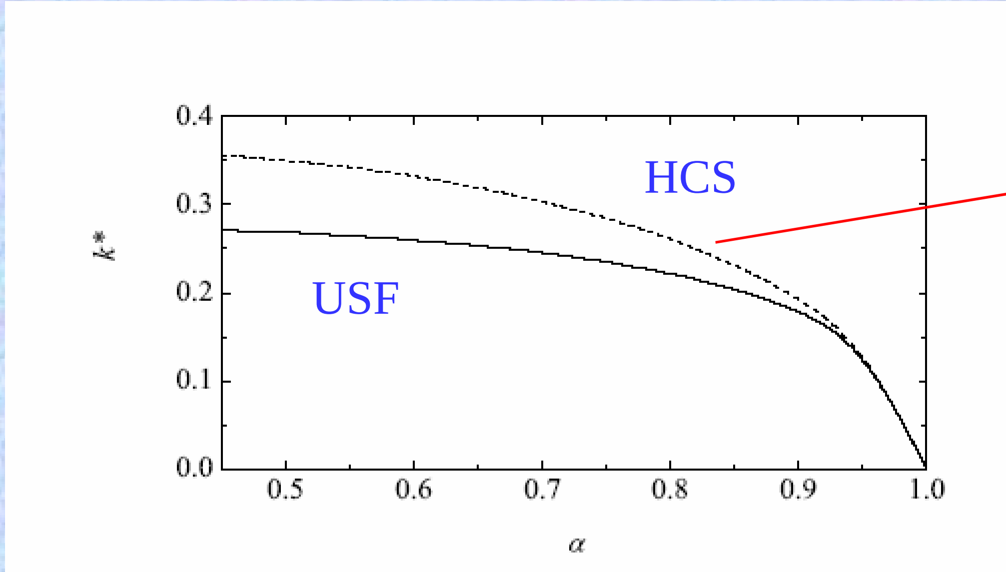
It is possible that the hydrodynamic modes are stable for wavenumbers larger than those considered in the limit $k^* \to 0$

One needs to get the eigenvalues with the full nonlinear dependence of the wavenumber

Results show that $\mathrm{Re}\ \lambda_\mu(k^*, \alpha) > 0$ if $k^* > k_s^*(\alpha)$

Critical value

• USF is linearly stable if $k^* > k_s^*(\alpha)$

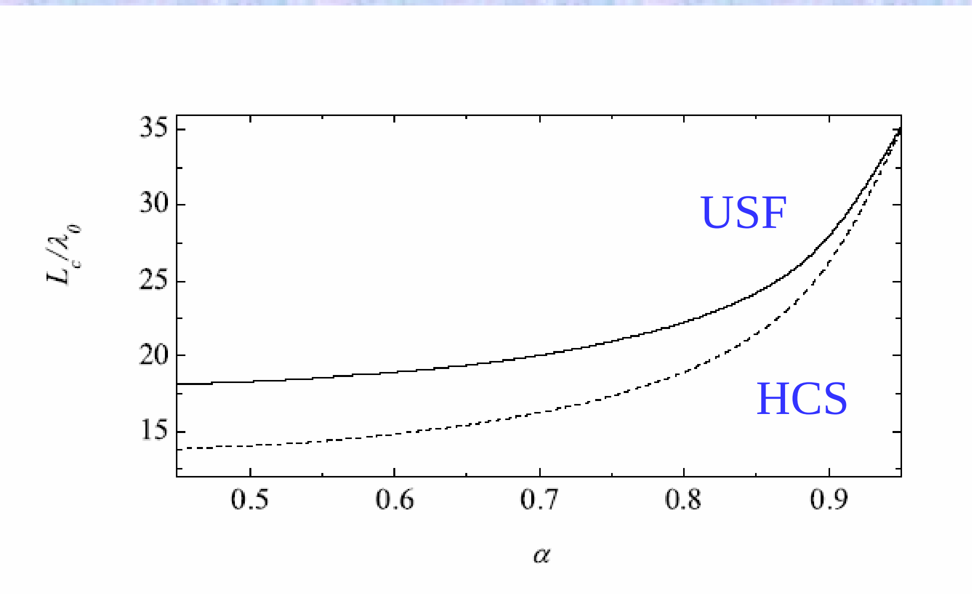


$$k^* = \sqrt{\frac{\zeta^*}{2\eta^*}}$$

In a system with periodic boundary conditions, the smallest allowed wavenumber is

$$k = \frac{2\pi}{L}$$

System becomes *unstable* when $L > L_c$



Mean free path $\lambda_0 = \left(\sqrt{2}\pi n\sigma^2\right)^{-1}$

$$\alpha = 0.8, \quad L_c^{\mathrm{USF}} = 22.17\lambda_0, \quad L_c^{\mathrm{HCS}} = 18.79\lambda_0$$

$$\alpha = 0.6, \quad L_c^{\mathrm{USF}} = 18.93\lambda_0, \quad L_c^{\mathrm{HCS}} = 14.82\lambda_0$$

At a given value of dissipation, *larger* systems are required in the USF to observe the instability.

CONCLUSIONS

- *Transport properties* of a granular gas for states near USF
  Due to the anisotropy induced by shear flow, tensorial quantities are required to describe momentum and heat fluxes.

- *Chapman-Enskog-like* expansion around the USF state rather than the HCS or the local equilibrium. Different approximations are nonlinear functions of the (reduced) shear rate (or equivalently, of the coefficient of restitution).

- In general, zeroth-order solution is a *time*-dependent function since there is no exact balance between viscous heating and collisional cooling at each point of the system.

- In the first order of the expansion, the set of generalized transport coefficients are given in terms of the solutions of linear integral equations. Many new transport coefficients.

- For practial purposes and to get explicit results, a *kinetic model* of the Boltzmann equation has been used. Its reliability has been clearly shown in the USF problem, even for strong dissipation. The results show that in general the deviation of the transport coefficients from their corresponding elastic values is quite significant.

- The knowledge of these coefficients allows one to perform a *linear stability* analysis of the hydrodynamic equations for states close to USF. As expected, our results show that the USF is *unstable* for any finite value of dissipation at sufficiently long wavelengths. Anomalous behavior of the hydrodynamic modes. Comparison of the analytical results with Monte Carlo simulations in the next future.