HYDRODYNAMICS FOR INELASTIC MAXWELL MIXTURES

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OUTLINE

1. INELASTIC MAXWELL MODELS (IMM) FOR MIXTURES
2. HOMOGENEOUS COOLING STATE (HCS)
3. CHAPMAN-ENSKOG SOLUTION
4. TRANSPORT COEFFICIENTS
5. ONSAGER'S RECIPROCAL RELATIONS
6. CONCLUSIONS

Granular fluids → Smooth hard spheres with inelastic collisions (IHS)

Transport coefficients (low-density gas) → Linear integral equations (Sonine polynomial approximation)

Difficulties increase for multicomponent systems

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Hydrodynamics for a granular binary mixture at low density

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Alternative: **Inelastic Maxwell models** (IMM)

Its collision rate is velocity independent but their collision rules are the same as for IHS

↓ Potential interaction ??

Perhaps, the cost of sacrificing physical realism can be in part compensated by getting **exact** results

Many results for homogeneous states, but much less is known for ***inhomogeneous*** situations (**transport coefficients**)

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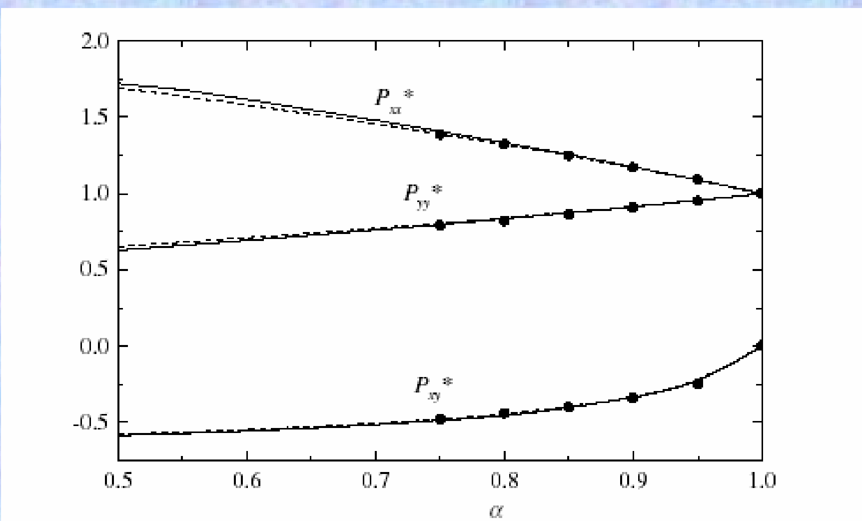
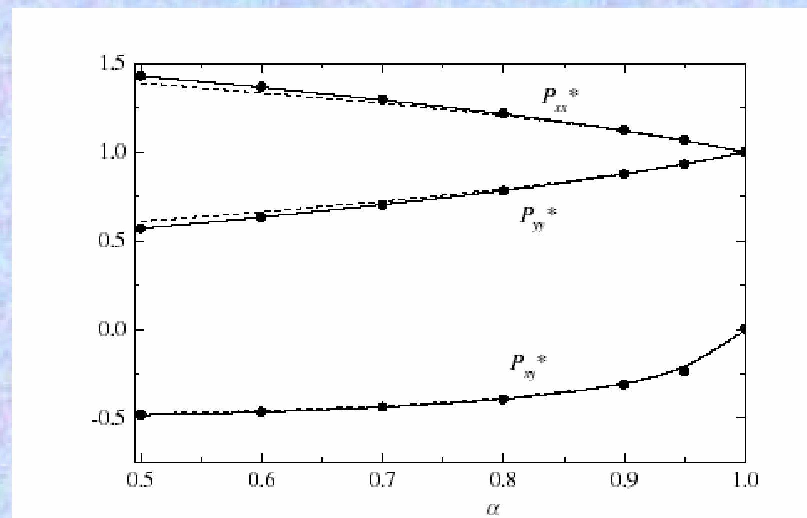
Nonlinear Transport in Inelastic Maxwell Mixtures Under Simple Shear Flow

Vicente Garzó[1]

Goog agreement with IHS





BOLTZMANN EQUATION FOR INELASTIC MAXWELL MIXTURES

$$\left(\partial_t + \mathbf{v}\cdot\nabla\right) f_r(\mathbf{r},\mathbf{v};t) = \sum_s J_{rs}\left[\mathbf{v}|f_r(t), f_s(t)\right]$$

$$J_{rs}\left[\mathbf{v}_1|f_r, f_s\right] = \frac{\omega_{rs}(\mathbf{r},t;\alpha_{rs})}{n_s(\mathbf{r},t)\Omega_d}\int d\mathbf{v}_2\int d\widehat{\boldsymbol{\sigma}}\left[\alpha_{rs}^{-1} f_r(\mathbf{r},\mathbf{v}_1',t) f_s(\mathbf{r},\mathbf{v}_2',t) - f_r(\mathbf{r},\mathbf{v}_1,t) f_s(\mathbf{r},\mathbf{v}_2,t)\right]$$

Free parameters

Coefficient of restitution for collisions of type r-s

$$\alpha_{rs} \le 1$$

$$\mathbf{v}_1' = \mathbf{v}_1 - \mu_{sr}\left(1+\alpha_{rs}^{-1}\right)(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})\widehat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2' = \mathbf{v}_2 + \mu_{rs}\left(1+\alpha_{rs}^{-1}\right)(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})\widehat{\boldsymbol{\sigma}}$$

$$\mu_{rs} = m_r/(m_r+m_s)$$

Hydrodynamic fields:

$$n_r = \int d\mathbf{v}\, f_r(\mathbf{v}),$$

$$\rho\mathbf{u} = \sum_r \rho_r \mathbf{u}_r = \sum_r \int d\mathbf{v} m_r \mathbf{v}\, f_r(\mathbf{v}),$$

$$nT = p = \sum_r n_r T_r = \sum_r \frac{m_r}{d}\int d\mathbf{v} V^2 f_r(\mathbf{v})$$

Collisional invariants

$$\int d\mathbf{v} J_{rs}[\mathbf{v}|f_r, f_s] = 0,$$

$$\sum_{r,s} \int d\mathbf{v} m_r \mathbf{v} J_{rs}[\mathbf{v}|f_r, f_s] = 0,$$

$$\sum_{r,s} \int d\mathbf{v} \frac{1}{2} m_r V^2 J_{rs}[\mathbf{v}|f_r, f_s] = -\frac{d}{2} n T \zeta$$

Cooling rate

$\zeta \longrightarrow$ Fractional energy changes per unit time

Total energy is ***not*** conserved

Macroscopic balance equations

$$D_t n_r + n_r \nabla \cdot \mathbf{u} + \frac{\nabla \cdot \mathbf{j}_r}{m_r} = 0 \ ,$$

$$D_t \mathbf{u} + \rho^{-1} \nabla \cdot \mathsf{P} = 0 \ ,$$

$$D_t T - \frac{T}{n} \sum_r \frac{\nabla \cdot \mathbf{j}_r}{m_r} + \frac{2}{dn} (\nabla \cdot \mathbf{q} + \mathsf{P} : \nabla \mathbf{u}) = -\zeta T$$

$$\mathbf{j}_r = m_r \int d\mathbf{v}\, \mathbf{V}\, f_r(\mathbf{v})$$

$$\mathsf{P} = \sum_r \int d\mathbf{v}\, m_r \mathbf{V}\mathbf{V}\, f_r(\mathbf{v})$$

$$\mathbf{q} = \sum_r \int d\mathbf{v}\, \frac{1}{2} m_r V^2 \mathbf{V}\, f_r(\mathbf{v})$$

Maxwell potential: nice mathematical properties of the Boltzmann collision operator

Velocity moment of order *k* of the Boltzmann collision operator

only involves moments of order less than or equal to *k*

Elastic fluids: Nonlinear transport properties

Inelastic Maxwell mixtures: Exact results

$$\int d\mathbf{v} m_r \mathbf{V} J_{rs}[f_r, f_s] = -\frac{w_{rs}}{\rho_s d}\mu_{sr}(1+\alpha_{rs})(\rho_s \mathbf{j}_r - \rho_r \mathbf{j}_s)$$

$$\begin{aligned}\int d\mathbf{v} m_r \mathbf{V}\mathbf{V} J_{rs}[f_r, f_s] =& -\frac{w_{rs}}{\rho_s d}\mu_{sr}(1+\alpha_{rs})\Big\{2\rho_s \mathsf{P}_r - (\mathbf{j}_r\mathbf{j}_s + \mathbf{j}_s\mathbf{j}_r)\\ & -\frac{2}{d+2}\mu_{sr}(1+\alpha_{rs})\left[\rho_s \mathsf{P}_r + \rho_r \mathsf{P}_s - (\mathbf{j}_r\mathbf{j}_s + \mathbf{j}_s\mathbf{j}_r)\right.\\ & \left.+\left[\frac{d}{2}(\rho_r p_s + \rho_s p_r) - \mathbf{j}_r\cdot\mathbf{j}_s\right]\mathbb{1}\right]\Big\}\end{aligned}$$

$$\begin{aligned}\int d\mathbf{v}\frac{m_r}{2}V^2\mathbf{V} J_{rs}[f_r, f_s] =& \frac{\omega_{rs}}{\rho_s}\frac{\mu_{sr}}{d(d+2)}(1+\alpha_{sr}).\\ & \times\{[\mu_{sr}(1+\alpha_{rs})(d+8-3\mu_{sr}(1+\alpha_{rs})) - 3(d+2)]\rho_s \mathbf{q}_r\\ & +3\mu_{sr}^2(1+\alpha_{sr})^2\rho_r \mathbf{q}_s + \frac{d}{2}[\mu_{sr}(1+\alpha_{rs})(3\mu_{sr}(1+\alpha_{rs})-4)+d+2]p_r \mathbf{j}_s\\ & +\frac{d}{2}\mu_{sr}(1+\alpha_{rs})[d+4-3\mu_{sr}(1+\alpha_{rs})]p_s \mathbf{j}_r\\ & +[\mu_{sr}(1+\alpha_{rs})(3\mu_{sr}(1+\alpha_{rs})-(d+6))+d+2]\mathsf{P}_r\cdot\mathbf{j}_s\\ & +\mu_{sr}(1+\alpha_{rs})[2-3\mu_{sr}(1+\alpha_{rs})]\mathsf{P}_s\cdot\mathbf{j}_r\}\end{aligned}$$

Cooling rate can be evaluated exactly

$$\zeta = T^{-1}\sum_r x_r T_r \zeta_r$$

$$\zeta_r = \sum_s \zeta_{rs} = -\sum_s \frac{1}{d n_r T_r}\int d\mathbf{v} m_r V^2 J_{rs}[\mathbf{v}|f_r, f_s]$$

Partial kinetic temperatures

$$\zeta_{rs} = \frac{2\omega_{rs}}{d}\mu_{sr}(1+\alpha_{rs})\left[1-\frac{\mu_{sr}}{2}(1+\alpha_{rs})\frac{\theta_r+\theta_s}{\theta_s}+\frac{\mu_{sr}(1+\alpha_{rs})-1}{d\rho_s p_r}\mathbf{j}_r\cdot\mathbf{j}_s\right]$$

$$\theta_r = \frac{m_r}{\gamma_r}\sum_s m_s^{-1}$$

$$\gamma_r \equiv T_r/T$$

$$p_r = n_r T_r$$

Balance equations become a closed set once the fluxes and the cooling rate are obtained in terms of the hydrodynamic fields and their gradients

Chapman-Enskog solution $\longrightarrow$ Hydrodynamic regime

Normal solution

$$f_r(\mathbf{r}, \mathbf{v}, t) = f_r[\mathbf{v}|x_1(t), p(t), T(t), \mathbf{u}(t)]$$

$$f_r = f_r^{(0)} + \epsilon\, f_r^{(1)} + \epsilon^2\, f_r^{(2)} + \cdots$$

$$\epsilon \sim \mathcal{O}(\nabla) : \frac{\text{mean free time}}{\text{hydrodynamic length}}$$

T(t) $\longrightarrow$ Reference state is **not** the local equilibrium distribution

Local homogeneous cooling state (HCS)

$$\partial_t f_r^{(0)}(V;t) = \sum_s J_{rs}[f_r^{(0)}, f_s^{(0)}]$$

Normal solution $\longrightarrow$

$$f_r^{(0)}(\mathbf{V},t) = n_r v_0^{-d}(t)\Phi_r(V/v_0(t))$$

$$v_0(t) = \sqrt{2T(t)(m_1+m_2)/m_1 m_2}$$

$$\partial_t T = -\zeta^{(0)} T, \quad \partial_t T_r = -\zeta_r^{(0)} T_r \longrightarrow \partial_t \ln T_1(t)/T_2(t) = \zeta_2^{(0)} - \zeta_1^{(0)}$$

Since $f_r^{(0)}$ depends on time only through *T(t)*

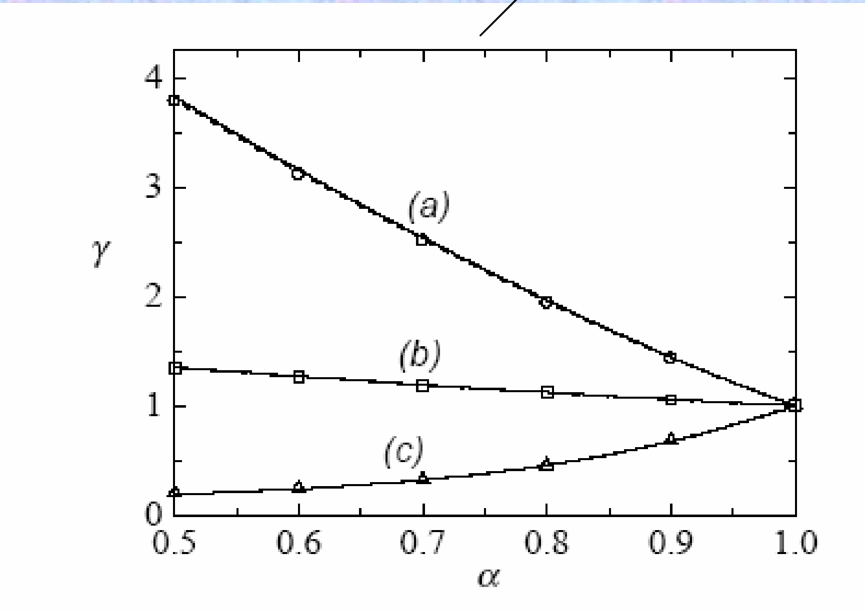
↓ HCS condition

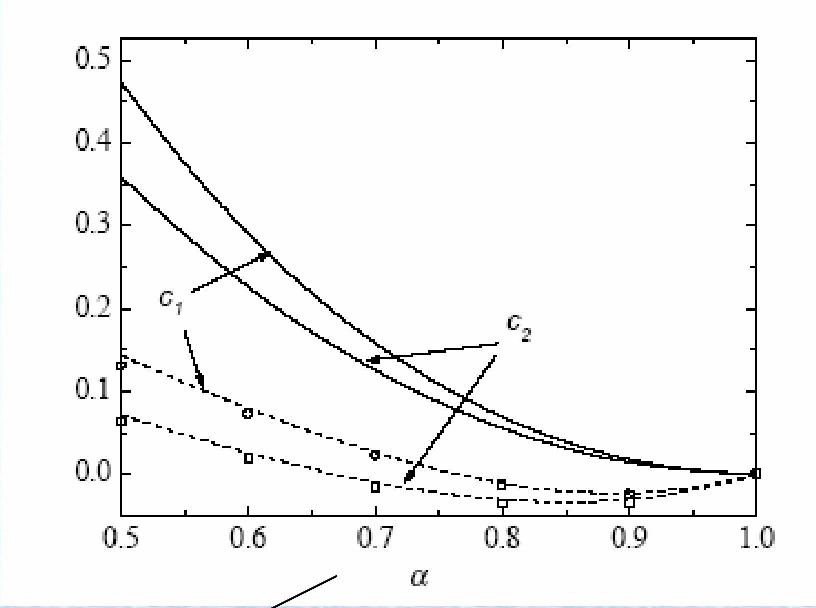
$$\zeta_2^{(0)} = \zeta_1^{(0)}$$

We adjust free paramaters of the model to get the same cooling rates as for IHS

$$\omega_{rs} = 4x_s \left(\frac{\sigma_{rs}}{\sigma_{12}}\right)^{d-1} \left(\frac{\theta_r + \theta_s}{\theta_r \theta_s}\right)^{1/2} \nu_0$$

Breakdown of energy equipartition





Fourth-cumulant (Non-Gaussianity)

First-order solution (Navier-Stokes hydrodynamic equations)

$$\left(\partial_t^{(0)} + \mathcal{L}_r\right) f_r^{(1)} + \mathcal{M}_r f_s^{(1)} = \mathbf{A}_r \cdot \nabla x_1 + \mathbf{B}_r \cdot \nabla p + \mathbf{C}_r \cdot \nabla T + D_{r,ij} \nabla_i u_j$$

Linearized Boltzmann collision operators

$$\mathbf{A}_r(\mathbf{V}) = -\left(\frac{\partial}{\partial x_1} f_r^{(0)}\right)_{p,T} \mathbf{V},$$

$$\mathbf{B}_r(\mathbf{V}) = -\frac{1}{p}\left[f_r^{(0)}\mathbf{V} + \frac{p}{\rho}\left(\frac{\partial}{\partial \mathbf{V}} f_r^{(0)}\right)\right],$$

$$\mathbf{C}_r(\mathbf{V}) = \frac{1}{T}\left[f_r^{(0)} + \frac{1}{2}\frac{\partial}{\partial \mathbf{V}} \cdot \left(\mathbf{V} f_r^{(0)}\right)\right]\mathbf{V},$$

$$D_{r,ij}(\mathbf{V}) = \frac{\partial}{\partial V_j}\left(V_i f_r^{(0)}\right) - \frac{1}{d}\delta_{ij}\frac{\partial}{\partial \mathbf{V}} \cdot \left(\mathbf{V} f_r^{(0)}\right).$$

$$\int d\mathbf{v}\left[f_r^{(1)}(\mathbf{v})-f_r^{(0)}(\mathbf{v})\right]=0,$$

$$\sum_r\int d\mathbf{v}m_r\mathbf{v}\;\left[f_r^{(1)}(\mathbf{v})-f_r^{(0)}(\mathbf{v})\right]=0,$$

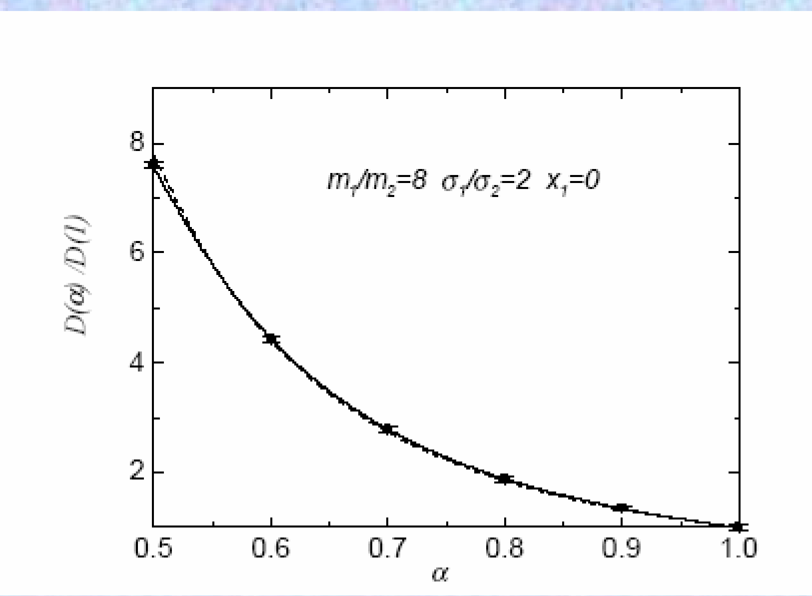
$$\sum_r\int d\mathbf{v}\frac{m_r}{2}v^2\;\left[f_r^{(1)}(\mathbf{v})-f_r^{(0)}(\mathbf{v})\right]=0.$$

Navier-Stokes transport coefficients

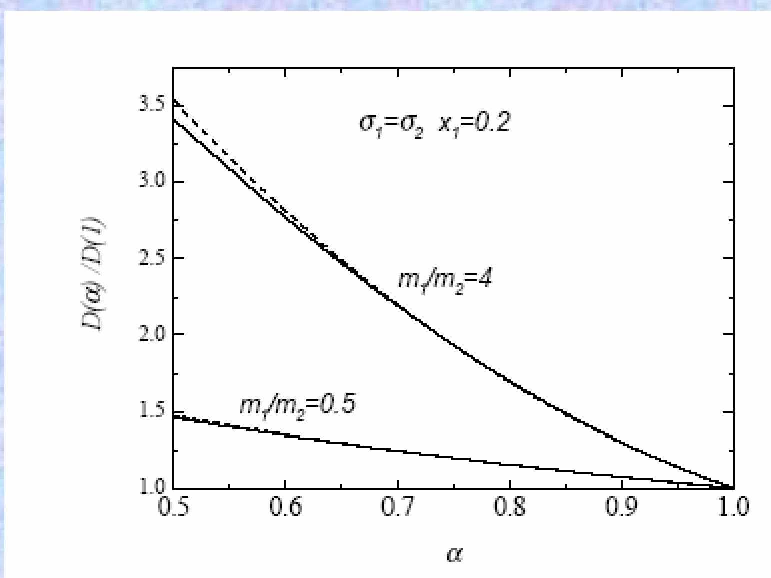
$$\mathbf{j}_1=-\frac{m_1m_2n}{\rho}D\nabla x_1-\frac{\rho}{p}D_p\nabla p-\frac{\rho}{T}D'\nabla T,\quad \mathbf{j}_2=-\mathbf{j}_1$$

$$\mathbf{q}=-T^2D''\nabla x_1-L\nabla p-\lambda\nabla T.$$

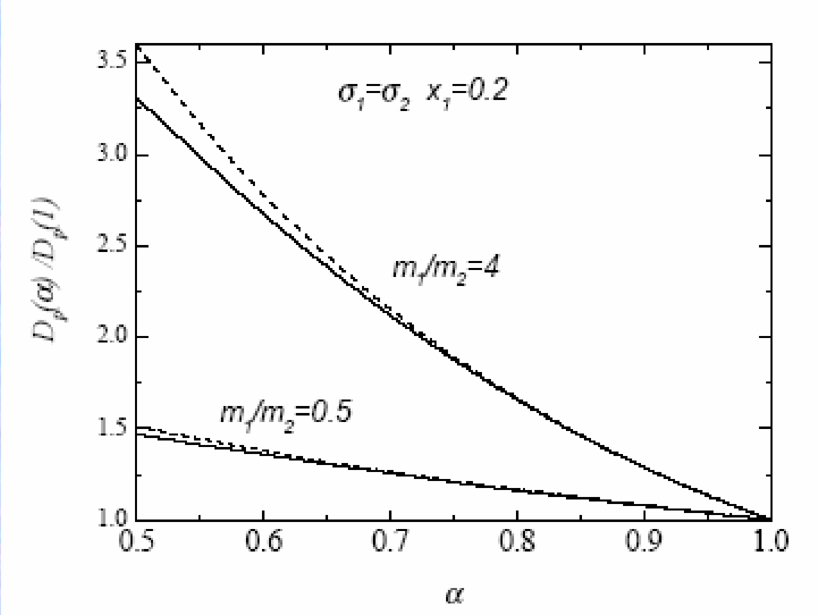
$$P_{ij}=p\delta_{ij}-\eta\left(\nabla_j u_i+\nabla_i u_j-\frac{2}{d}\delta_{ij}\nabla\cdot\mathbf{u}\right)$$



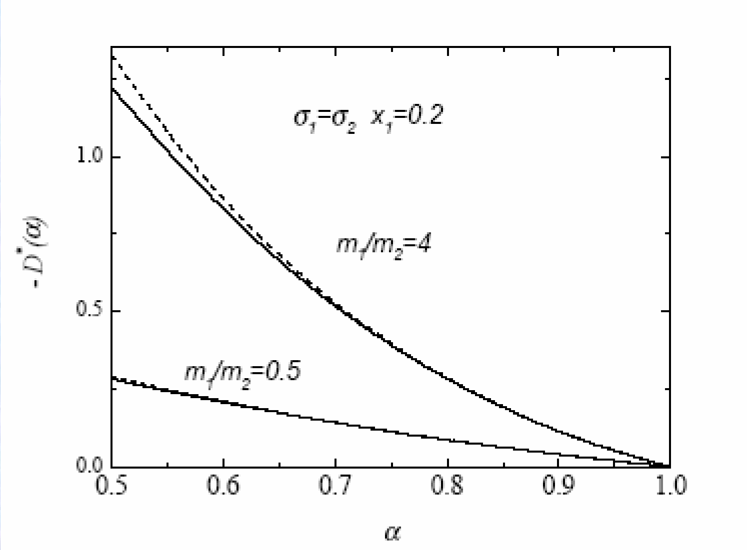
$m_1/m_2=8$ $\sigma_1/\sigma_2=2$ $x_1=0$
$D(\alpha)/D(1)$
α
8
6
4
2
0.5
0.6
0.7
0.8
0.9
1.0



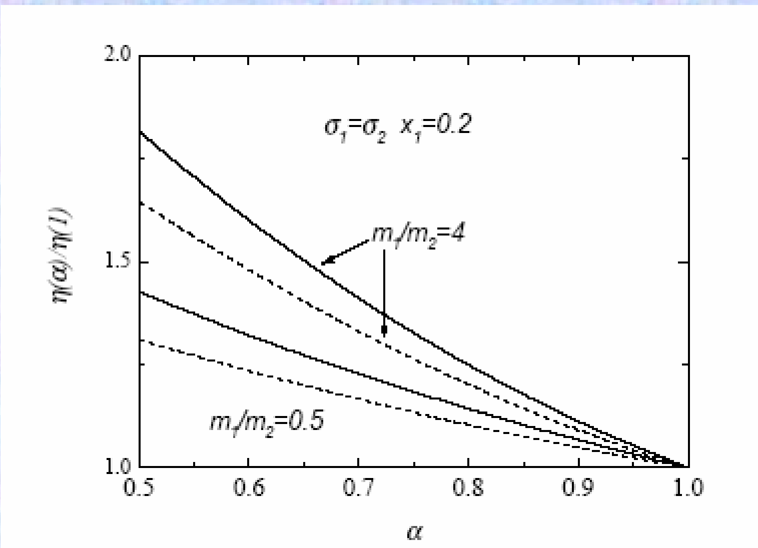
$\sigma_1=\sigma_2$ $x_1=0.2$
$m_1/m_2=4$
$m_1/m_2=0.5$
$D(\alpha)/D(1)$
α
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.6
0.7
0.8
0.9
1.0

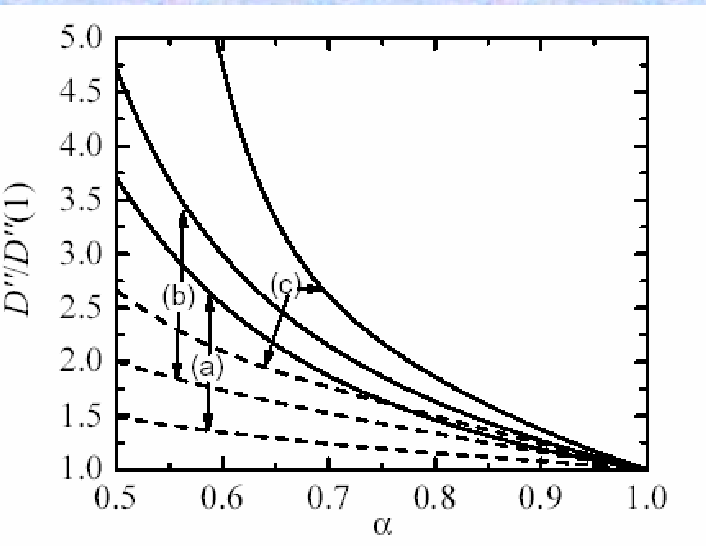


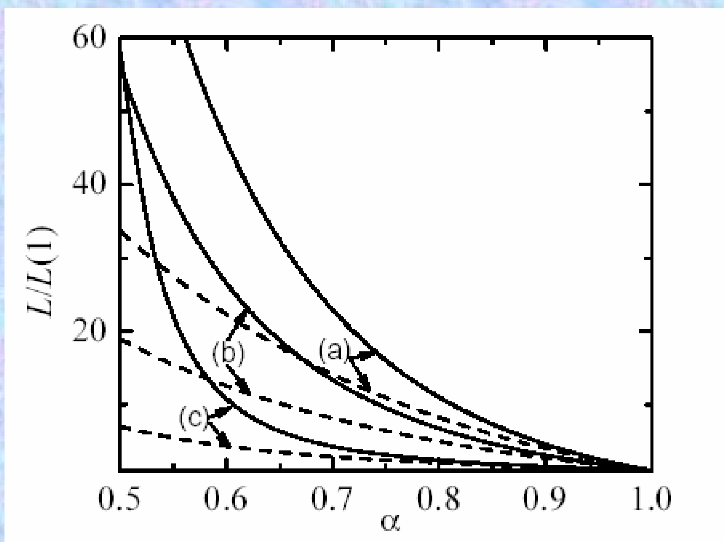
$\sigma_1=\sigma_2$ $x_1=0.2$
$m_1/m_2=4$
$m_1/m_2=0.5$
$D_j(\alpha)/D_j(1)$
α
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.6
0.7
0.8
0.9
1.0

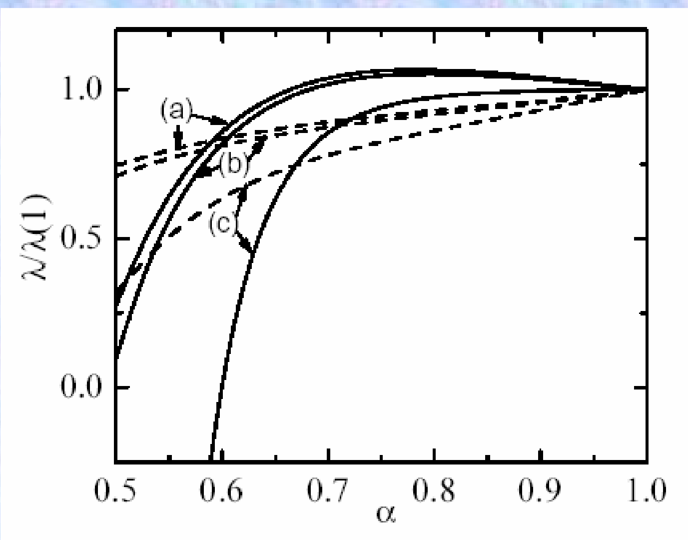


$\sigma_1=\sigma_2$ $x_1=0.2$
$m_1/m_2=4$
$m_1/m_2=0.5$
$-D^*(\alpha)$
α
1.0
0.5
0.0
0.5
0.6
0.7
0.8
0.9
1.0









(a) $m_1/m_2 = 0.5$, (b) $m_1/m_2 = 2$, and (c) $m_1/m_2 = 4$

$\sigma_1 = \sigma_2,\ x_1 = 0.2$

Onsager's constitutive equations for elastic mixtures:

$$\mathbf{j}_s = -\sum_{r=1}^{N} L_{sr} \left(\frac{\nabla \mu_r}{T}\right)_T - L_{sq}\frac{\nabla T}{T^2},$$

$$\mathbf{J}_q \equiv \mathbf{q} - \frac{d+2}{2} T \sum_{s=1}^{N} \frac{\mathbf{j}_s}{m_s}$$

$$= -L_{qq}\nabla T - \sum_{s=1}^{N} L_{qs} \left(\frac{\nabla \mu_s}{T}\right)_T .$$

$$\left(\frac{\nabla \mu_s}{T}\right)_T = \frac{1}{m_s}\nabla \ln(x_s p)$$

Onsager's reciprocal relations ⟶ $L_{sr} = L_{rs}, \quad L_{sq} = L_{qs}$

Time reversal invariance

$$\mathbf{j}_1 = -\frac{m_1 m_2 \rho_1 \rho_2}{\rho^2} D \frac{(\nabla \mu_1)_T - (\nabla \mu_2)_T}{T} - C_p \nabla \ln p - \frac{\rho}{T} D' \nabla T,$$

$$\mathbf{J}_q = -\left(\frac{\rho_1 \rho_2 T^2}{n\rho} D'' - \frac{d+2}{2} T \frac{m_2 - m_1}{\rho^2} \rho_1 \rho_2 D \right) \frac{(\nabla \mu_1)_T - (\nabla \mu_2)_T}{T}$$
$$- C_p' \nabla \ln p - \left(\lambda - \frac{d+2}{2} \frac{m_2 - m_1}{m_1 m_2} \rho D' \right) \nabla T,$$

Onsager´s phenomenological coefficients:

$$L_{11} = -L_{12} = \frac{m_1 m_2 \rho_1 \rho_2}{\rho^2} D, \quad L_{1q} = \rho T D',$$

$$L_{q1} = -L_{q2} = \frac{T^2 \rho_1 \rho_2}{n\rho} D'' - \frac{d+2}{2} \frac{T \rho_1 \rho_2}{\rho^2} (m_2 - m_1) D, \quad L_{qq} = \lambda - \frac{d+2}{2} \rho \frac{m_2 - m_1}{m_1 m_2} D'$$

Contributions **not** present in the elastic case: $C_p = C_p' = 0$

First Onsager´s relation $L_{12} = L_{21}$ holds since D is symmetric under the change $1 \leftrightarrow 2$

Second Onsager´s relation $L_{1q} = L_{q1}$ along with $C_p = C_p' = 0$

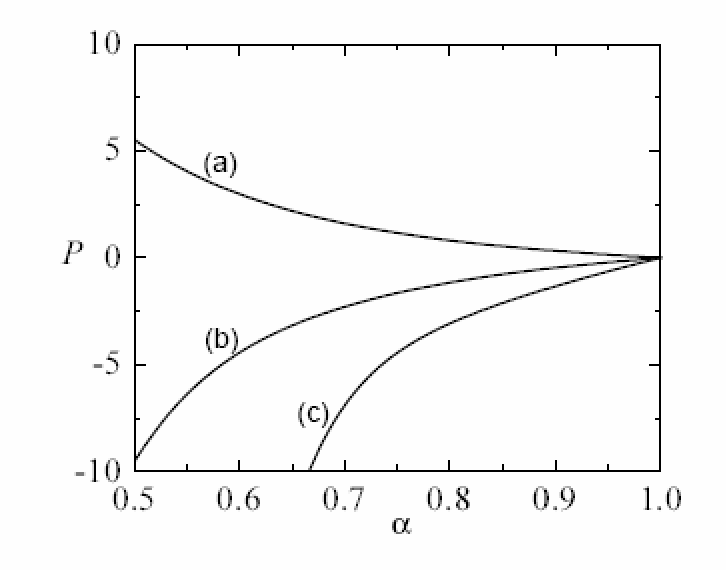
implies that

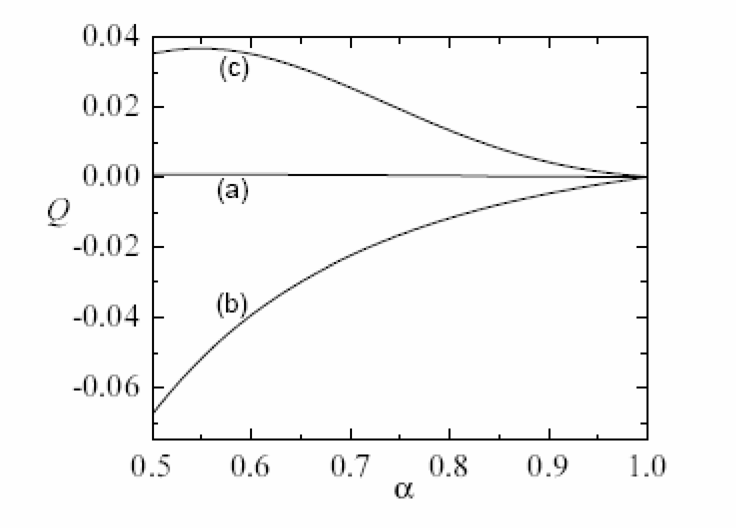
$$P(\alpha_{rs}) \equiv D''^{*} - \frac{d+2}{2}\frac{1-\mu^2}{\mu}D^{*} - \frac{1+\mu}{\mu}\frac{x_2+\mu x_1}{x_1 x_2}D'^{*} = 0,$$

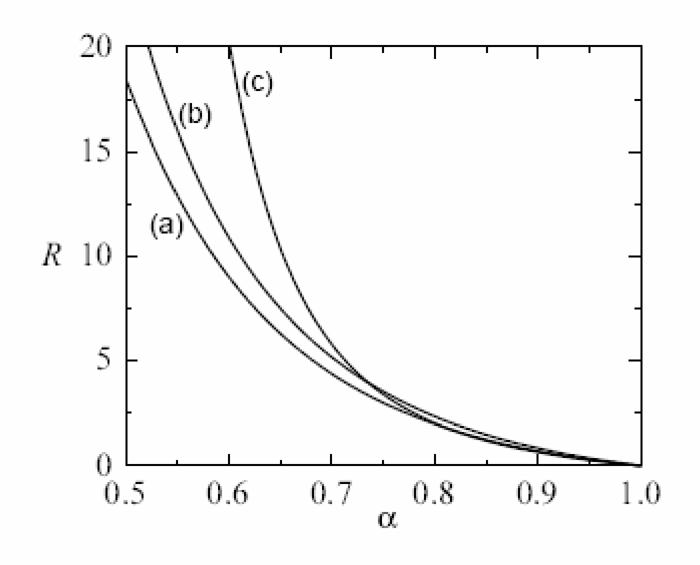
$$Q(\alpha_{rs}) \equiv D_p^{*} - x_1 x_2 \frac{1-\mu}{x_2+\mu x_1}D^{*} = 0,$$

$$R(\alpha_{rs}) \equiv L^{*} - \frac{d+2}{2}\frac{1-\mu^2}{\mu}Q - x_1 x_2 \frac{1-\mu}{x_2+\mu x_1}D''^{*} = 0.$$

In the elastic case, $P=Q=R=0$







$$\sigma_1 = \sigma_2, \; x_1 = 0.2$$

(a) $m_1/m_2 = 0.5$, (b) $m_1/m_2 = 2$, and (c) $m_1/m_2 = 4$

CONCLUSIONS

1. Hydrodynamic equations for a binary mixture of inelastic Maxwell molecules have been derived from the Boltzmann kinetic theory
2. Seven relevant transport coefficients given **exactly** in terms of the coefficients of restitution and parameters of the mixture.
3. Comparison with inelastic hard spheres shows reasonably good qualitative agreement, especially for the mass flux.
4. As expected, Onsager's relations do not apply for inelastic collisions.

Thanks for your attention