THE HILBERT EXPANSION OF THE BGK EQUATION

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The Hilbert method is applied to the steady-state Bhatnagar, Gross, Krook (BGK) equation and the Navier–Stokes steady-state hydrodynamic equations derived. The viscosity and thermal conductivity coefficients that appear in the equations are similar to the transport coefficients in the Chapman–Enskog expansion of the BGK equation.

The Chapman–Enskog solution [1] to the Boltzmann equation provides a useful way of obtaining transport coefficients for a low-density gas. These are the coefficients in an expansion of the average fluxes of energy, momentum, etc., in powers of appropriate uniformity parameters (temperature gradient, velocity gradient, etc.). When the relationships between fluxes and gradients are known, the hydrodynamic equations for the density n, velocity $\boldsymbol{u}$ and temperature T may be obtained. In this way, the Euler, Navier–Stokes, Burnett, ... transport equations can be derived by successive approximations.

Historically, the Chapman–Enskog method was preceded by an expansion due to Hilbert [2]. In this method, the conserved hydrodynamic variables n, $\boldsymbol{u}$ and T are expanded in powers of an auxiliary parameter instead of expanding the transport equations as in the Chapman–Enskog solution. This is the essential difference between the two perturbative expansions. Although the Hilbert method predated the Chapman–Enskog method, the latter has proven to be more popular in recent years. This may be due to the fact that the Chapman–Enskog expansion gives the transport coefficients in a more systematic way. The only author we are aware of who has used the Hilbert expansion for solving the Boltzmann equation is Delale [3]. In this work, we use Hilbert's method to obtain the Navier–Stokes steady-state hydrodynamic equations and expressions for the thermal conductivity and viscosity are shown to be precisely the same as those given by the Chapman–Enskog method.

Unfortunately, because of the complex structure of the Boltzmann collision term, it is difficult to find explicit expressions for the transport coefficients from either the Chapman–Enskog or the Hilbert expansion. This problem has stimulated the search for kinetic equations that are mathematically simpler than the Boltzmann equation. One of the best known is the model equation suggested by Bhatnagar, Gross and Krook [4] (BGK). The Chapman–Enskog expansion has also been applied to the BGK equation [5] and the pressure tensor and the heat flux vector have been evaluated to the super-Burnett hydrodynamic order (third order). The purpose of this Letter is to solve the BGK equation by means of the Hilbert expansion. Our study is centered on the Navier–Stokes order in the same way as in Delale's work [3].

Let us write the BGK equation in the form

$$(\partial/\partial t+v_j\nabla_j)f=-\epsilon^{-1}\zeta(f-f_{\mathrm{LE}})\,, \tag{1}$$

where ϵ is an auxiliary parameter which may be set equal to unity at the end of the calculations. In eq. (1), f is the one-particle distribution function, f_{LE} is the local-equilibrium distribution function and ζ is an average velocity-independent collision frequency that depends on space and time through its dependence on the density

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and temperature.

In the Hilbert expansion, we construct a normal solution for the distribution function f assuming that it does not explicitly depend on the coordinates $\boldsymbol{r}$ and time t. Thus f is given by

$$f=f^{(0)}+\epsilon f^{(1)}+\epsilon^2 f^{(2)}+\dots . \tag{2}$$

In a similar way, the density, velocity and temperature can be expanded according to

$$n=n^{(0)}+\epsilon n^{(1)}+\epsilon^2 n^{(2)}+\dots , \tag{3}$$

$$u_j=u_j^{(0)}+\epsilon u_j^{(1)}+\epsilon^2 u_j^{(2)}+\dots , \tag{4}$$

$$T=T^{(0)}+\epsilon T^{(1)}+\epsilon^2 T^{(2)}+\dots . \tag{5}$$

Obviously, every function that depends on n, $\boldsymbol{u}$ and T can be expanded in a similar way. In order to evaluate the terms that appear in expressions (3)–(5), we consider the corresponding expansion for the conservation equations and collect terms of the same power in ϵ. The transport equations are given by

$$\mathrm{d}n/\mathrm{d}t=-n\nabla_i u_i , \tag{6}$$

$$mn\, \mathrm{d}u_i/\mathrm{d}t=-\nabla_j P_{ij} , \tag{7}$$

$$\tfrac{3}{2} n k_\mathrm{B}\, \mathrm{d}T/\mathrm{d}t=-\nabla_i J_i-\tfrac{1}{2}P_{ij}(\nabla_i u_j+\nabla_j u_i) , \tag{8}$$

where P_{ij} and J_i are the components of the pressure tensor and the heat flux vector, defined respectively by

$$P_{ij}=\int \mathrm{d}\boldsymbol{v}\, m(v_i-u_i)(v_j-u_j)\, f(\boldsymbol{r},\boldsymbol{v};t) , \tag{9}$$

$$J_i=\int \mathrm{d}\boldsymbol{v}\, \tfrac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2(v_i-u_i)\, f(\boldsymbol{r},\boldsymbol{v};t) . \tag{10}$$

When we substitute expansions (2) and (3)–(5) in the BGK equation (1) and separate the terms order-by-order, we obtain the equations

$$f^{(0)}=f^{(0)}_\mathrm{LE} ,$$

$$f^{(1)}=f^{(1)}_\mathrm{LE}-(1/\zeta^{(0)})(\partial/\partial t+v_j\nabla_j)f^{(0)} ,$$

$$f^{(2)}=f^{(2)}_\mathrm{LE}-(1/\zeta^{(0)})(\partial/\partial t+v_j\nabla_j)f^{(1)}-(\zeta^{(1)}/\zeta^{(0)})(f^{(1)}-f^{(1)}_\mathrm{LE}) ,$$

$$\vdots$$

$$f^{(k)}=f^{(k)}_\mathrm{LE}-(1/\zeta^{(0)})(\partial/\partial t+v_j\nabla_j)f^{(k-1)}-(1/\zeta^{(0)})\sum_{r=1}^{k-1}\zeta^{(r)}(f^{(k-r)}-f^{(k-r)}_\mathrm{LE}) . \tag{11}$$

Whereas the Boltzmann equation leads to sets of integral equations [3], the equations given by (11) are algebraic and may be solved sequentially. From now on, in order to avoid having to specify the initial conditions, let us restrict ourselves to the steady state. In this event the partial time derivates will disappear.

At zeroth order, the distribution function corresponds to the local equilibrium function defined by means of the variables $n^{(0)}$, $\boldsymbol{u}^{(0)}$ and $T^{(0)}$,

$$f^{(0)}=f^{(0)}_\mathrm{LE}=n^{(0)}(m/2\pi k_\mathrm{B}T^{(0)})^{3/2}\exp[-(m/2k_\mathrm{B}T^{(0)})\,|\boldsymbol{v}-\boldsymbol{u}^{(0)}|^2] , \tag{12}$$

from which we obtain the Euler steady-state transport equations,

$$\nabla_i(n^{(0)}u_i^{(0)})=0\,, \tag{13}$$

$$\rho^{(0)}u_j^{(0)}\,\nabla_j u_i^{(0)}=-\nabla_i p^{(0)}\,, \tag{14}$$

$$u_j^{(0)}\,\nabla_j T^{(0)}=-\tfrac{2}{3}T^{(0)}\,\nabla_r u_r^{(0)}\,, \tag{15}$$

where we have introduced the density $\rho^{(0)}=mn^{(0)}$ and the pressure $p^{(0)}=n^{(0)}k_\mathrm{B}T^{(0)}$.

The next approximation corresponds to the Navier–Stokes order. According to eq. (11), after some manipulation, the function $f^{(1)}$ may be written as

$$f^{(1)}=f^{(0)}_{\mathrm{LE}}\left[\frac{n^{(1)}}{n^{(0)}}+\left(\frac{m(V^{(0)})^2}{2k_\mathrm{B}T^{(0)}}-\frac{3}{2}\right)\frac{T^{(1)}}{T^{(0)}}+\frac{m}{k_\mathrm{B}T^{(0)}}V_j^{(0)}u_j^{(1)}-\frac{V_j^{(0)}}{\zeta^{(0)}}\left(\frac{m(V^{(0)})^2}{2k_\mathrm{B}T^{(0)}}-\frac{5}{2}\right)\frac{\nabla_j T^{(0)}}{T^{(0)}}\right.$$
$$\left.+\frac{m}{k_\mathrm{B}T^{(0)}}[V_j^{(0)}V_i^{(0)}-\tfrac{1}{3}(V^{(0)})^2\delta_{ij}]\nabla_j u_i^{(0)}\right], \tag{16}$$

where $V_i^{(0)}=v_i-u_i^{(0)}$.

In order to obtain the corresponding hydrodynamic equations we need to compute the pressure tensor and the heat flux in that order. Taking into account only first-order terms, we obtain

$$P_{ij}^{(1)}=\int \mathrm{d}\boldsymbol{v}\, mV_i^{(0)}V_j^{(0)}f^{(1)}=p^{(1)}\delta_{ij}-\eta(\nabla_j u_i^{(0)}+\nabla_i u_j^{(0)}-\tfrac{2}{3}\delta_{ij}\nabla_r u_r^{(0)})\,, \tag{17}$$

$$J_i^{(1)}=\int \mathrm{d}\boldsymbol{v}\,\tfrac{1}{2}m(V^{(0)})^2V_i^{(0)}f^{(1)}-\tfrac{5}{2}p^{(0)}u_i^{(1)}=-\lambda\nabla_i T^{(0)}\,, \tag{18}$$

where $p^{(1)}=n^{(1)}k_\mathrm{B}T^{(0)}+n^{(0)}k_\mathrm{B}T^{(1)}$, and we have introduced the dynamic viscosity $\eta=n^{(0)}k_\mathrm{B}T^{(0)}/\zeta^{(0)}$ and the thermal conductivity $\lambda=\tfrac{5}{2}n^{(0)}k_\mathrm{B}^2T^{(0)}/m\zeta^{(0)}$. These expressions are similar to those obtained using the Chapman–Enskog expansion [5]. If we introduce expressions (17) and (18) into the transport equations (6)–(8), we finally obtain the Navier–Stokes equations which are given by

$$\nabla_j(n^{(0)}u_j^{(1)}+n^{(1)}u_j^{(0)})=0, \tag{19}$$

$$\nabla_j[\rho^{(0)}u_j^{(0)}u_i^{(1)}+\rho^{(0)}u_i^{(0)}u_j^{(1)}+\rho^{(1)}u_i^{(0)}u_j^{(0)}+p^{(1)}\delta_{ij}-\eta(\nabla_i u_j^{(0)}+\nabla_j u_i^{(0)}-\tfrac{2}{3}\delta_{ij}\,\nabla_r u_r^{(0)})]=0\,, \tag{20}$$

$$\tfrac{3}{2}n^{(0)}k_\mathrm{B}[u_i^{(0)}\,\nabla_i T^{(1)}+u_i^{(1)}\,\nabla_i T^{(0)}]=\nabla_i(\lambda\,\nabla_i T^{(0)})-p^{(0)}\,\nabla_i u_i^{(1)}-n^{(0)}k_\mathrm{B}T^{(1)}\,\nabla_r u_r^{(0)}$$
$$+\eta(\nabla_i u_j^{(0)}\,\nabla_j u_i^{(0)}+\nabla_j u_i^{(0)}\,\nabla_j u_i^{(0)})-\tfrac{2}{3}\eta(\nabla_r u_r^{(0)})^2\,. \tag{21}$$

These equations are the same as those obtained from the Boltzmann equation [3]. The steady-state equations (19)–(21) are linear in $n^{(1)}$, $\boldsymbol{u}^{(1)}$ and $T^{(1)}$ and can be solved from the solutions to the Euler equations.

In summary, we have applied the perturbative Hilbert expansion to the BGK equation. The Navier–Stokes hydrodynamic equations are the same as those obtained from the Boltzmann equation. Additionally the expressions for the dynamic viscosity and thermal conductivity coefficients are similar to those obtained from the Chapman–Enskog expansion. The proposed method can be similarly used to obtain higher-order hydrodynamic equations (Burnett, super-Burnett, ...) and the transport coefficients obtained will be directly comparable with those from the Chapman–Enskog theory. Work along these lines is in progress.

References

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