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Generalized transport coefficients in a gas with large shear rate

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We get a solution of the Bhatnagar–Gross–Krook (BGK) model kinetic equation by means of a perturbative expansion of a temperature gradient to study the transport properties in a gas with large shear rate. The irreversible fluxes are evaluated exactly to first order in the expansion for Maxwell molecules. The transport coefficients obtained are highly nonlinear functions of the shear rate. This dependence on shear rate is analysed and compared with previous results for several transport coefficients. Finally, we have found a solution for a simple model of constant collision frequency for which a large shear rate coexists with an arbitrary temperature gradient.

1. Introduction

The transport properties of a dilute gas are usually described by the well-known Boltzmann equation (BE) [1]. However, because of the complex structure of the Boltzmann's collision term, is very difficult to find an explicit solution. Studies have been restricted to very simple physical situations, and simplified collision models have been considered [2]. This problem stimulates the search for kinetic equations mathematically simpler than BE, but preserving the most important physical properties of the BE. One of the best known is the Bhatnagar–Gross–Krook (BGK) equation [3].

The BGK equation has been resolved by Zwanzig [4], in the study of a system in uniform shear flow (USF) for potentials of the form $r^{-\mu}$. The USF is characterized by constant density n, constant shear rate $a = \partial u_x/\partial y$ and uniform temperature $T(t)$. Because of the viscous heating, the temperature of the system monotonically increases in time. In the particular case of Maxwell molecules ($\mu = 4$), both the BE and the BGK equation yield the same results for the nonlinear shear viscosity [2, 4]. Recently a self-consistent solution of the BGK equation for a stationary heat flux has been found [5].

The coupling between the transport of momentum and energy has been treated previously by kinetic theory [6] and by fluctuating hydrodynamics [7]. In these papers, under adequate boundary conditions, steady state analyses are described.

The aim of the present paper is the study of the coupling between a velocity gradient and a temperature gradient in a non-stationary state. The physical situation corresponds to a gas subjected to a large shear rate and a weak temperature gradient. In this situation it seems natural to assume that the system is essentially described by the distribution function corresponding to USF. This state has therefore been taken as the reference one and we have perturbed around it, considering the temperature gradient as a small perturbation [8]. In our analysis, we compute the irreversible fluxes to first order in the temperature gradient. The proposed solution is similar to that obtained from conventional expansions of the Chapman–Enskog or Hilbert [1].

However, in our expansion the transport coefficients obtained are highly nonlinear functions of a, since they retain the full nonlinear dependence on this gradient. Due to the complexity of the problem, it has been possible only to consider Maxwell molecules. The reason of this choice is that the BGK equation has been solved exactly [4] in the USF for this interaction model.

According to the proposed expansion, a very simple case is that which corresponds to a constant collision frequency model. In this case, the USF coexists with a constant temperature gradient in all the hydrodynamics orders of temperature gradient [9]. So, every term of the expansion can be evaluated exactly. While the pressure tensor does not depend explicitly on temperature gradient, the heat flux is a linear function of the temperature gradient. In this way, we have obtained a generalized Fourier's law with a thermal conductivity tensor that depends on the shear rate.

The paper is organized as follows. In § 2, we present a short description of the BGK's solution in the USF for Maxwell molecules. In § 3, we introduce the proposed perturbative expansion. Furthermore, we obtain explicitly the Navier–Stokes generalized transport equations (first order in the expansion) to analyse their dependence on the shear rate. In § 4, we study the constant collision frequency model, for which the solution is exact. Finally, in the § 5, we give a brief summary of the results obtained.

2. Uniform shear flow

The uniform shear flow is a state characterized by uniform density and pressure and a velocity field given by

$$u_i(\mathbf{r};\, t) = a_{ij} r_j, \quad a_{ij} = a\delta_{ix}\delta_{jy} \tag{1}$$

where the shear rate a is constant. This state can be maintained in the system provided that the hydrostatic pressure p increases in time (viscous heating) according to the equation

$$\frac{\partial p}{\partial t} = -\tfrac{2}{3} a P_{xy} \tag{2}$$

where P_{ij} is the pressure tensor. This flow is simple enough to allow for a complete description when the BGK model kinetic equation is used. This equation can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f = -\zeta(f - f^{\mathrm{LE}}), \tag{3}$$

where $\zeta(\mathbf{r};\, t)$ is the collision frequency, $f(\mathbf{r},\, \mathbf{v};\, t)$ the one-particle distribution function and $f^{\mathrm{LE}}(\mathbf{r},\, \mathbf{v};\, t)$ the local equilibrium distribution function defined as,

$$f^{\mathrm{LE}} = n\left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} \exp\left[-\frac{m}{2k_{\mathrm{B}} T}(\mathbf{v} - \mathbf{u})^2\right]. \tag{4}$$

Here, k_{B} is the Boltzmann constant, m is the mass of a particle, and $n(\mathbf{r};\, t)$, $\mathbf{u}(\mathbf{r};\, t)$ and $T(\mathbf{r};\, t)$ are the local density, velocity and temperature, respectively. In terms of the distribution function, they are given by

$$n(\mathbf{r};t)=\int d\mathbf{v}\, f(\mathbf{r},\mathbf{v};t), \tag{5}$$

$$n(\mathbf{r};t)\mathbf{u}(\mathbf{r};t)=\int d\mathbf{v}\,\mathbf{v}\, f(\mathbf{r},\mathbf{v};t), \tag{6}$$

$$\tfrac{3}{2}n(\mathbf{r};t)k_B T(\mathbf{r};t)=\frac{m}{2}\int d\mathbf{v}[\mathbf{v}-\mathbf{u}(\mathbf{r};t)]^2 f(\mathbf{r},\mathbf{v};t). \tag{7}$$

The details of the particle interactions are modelled through the density and temperature dependence of the collision frequency ζ. For repulsive potentials of the form $r^{-\mu}$, it is

$$\zeta\propto nT^{\alpha} \tag{8}$$

with $\alpha=\frac{1}{2}-2/\mu$.

The BGK equation is a model for the BE in which collisions are treated in a statistical way. Also, this model preserves the most important properties of the BE, such as the conservation laws and the irreversible tendency to equilibrium. By taking moments in velocity space, the BGK equation leads to familiar transport equations,

$$\frac{\partial n}{\partial t}+\mathbf{u}\,.\,\nabla n=-n\nabla\,.\,\mathbf{u}, \tag{9}$$

$$mn\left[\frac{\partial u_i}{\partial t}+\mathbf{u}\,.\,\nabla u_i\right]=-\nabla_j P_{ij}, \tag{10}$$

$$\frac{\partial p}{\partial t}+\mathbf{u}\,.\,\nabla p=-\tfrac{2}{3}\nabla\,.\,\mathbf{J}-\tfrac{1}{3}P_{ij}(\nabla_i u_j+\nabla_j u_i)-p\nabla\,.\,\mathbf{u}, \tag{11}$$

where we have introduced the pressure tensor and the heat flux given respectively by

$$P_{ij}=\int d\mathbf{v} m V_i V_j f, \tag{12}$$

$$J_i=\int d\mathbf{v}\,\frac{m}{2}V^2 V_i f, \tag{13}$$

with

$$\nabla_j\equiv\frac{\partial}{\partial r_j},\quad V_j=v_j-u_j\quad\text{and}\quad p=\frac{P_{ii}}{3}=nk_B T.$$

It is adequate to consider the pseudo-galilean transformation $V_i=v_i-a_{ij}r_j$ so that the distribution function can be written as $f(\mathbf{r},\mathbf{v};t)=g(\mathbf{r},\mathbf{V};t)$. In this description, the USF is then described by a homogeneous distribution function $g_0(\mathbf{V};t)$ that is even in $\mathbf{V}$. Then, equation (3) becomes [10]

$$\left(\frac{\partial}{\partial t}-aV_y\frac{\partial}{\partial V_x}\right)g_0=-\zeta_0\,(g_0-g_0^{\mathrm{LE}}), \tag{14}$$

where

$$g_0^{\mathrm{LE}}(\mathbf{V};t) = n_0\left(\frac{m}{2\pi k_B T_0(t)}\right)^{3/2} \exp\left[-\frac{mV^2}{2k_B T_0(t)}\right]. \tag{15}$$

For Maxwell molecules ($\mu = 4$), the collision frequency ζ_0 depends on the density n_0, and so, is a constant in the USF. From now on we shall restrict ourselves to Maxwell molecules. For this interaction, the USF has a clearer meaning as an arbitrary far from equilibrium state. In this case, after the initial transient has slowed down, the solution of equation (14) is

$$g_0(\mathbf{V};t) = \int_0^\infty ds \exp(-\zeta_0 s)\zeta_0 \exp\left[asV_y \frac{\partial}{\partial V_x}\right] g_0^{\mathrm{LE}}(\mathbf{V};t-s). \tag{16}$$

We shall call g_0 the uniform-shear-flow (USF) distribution function. In addition, from the equation (14) we obtain the relations [4]

$$P_{0xx}^* = (1 + 3\lambda_0^*)/(1 + \lambda_0^*), \tag{17}$$

$$P_{0yy}^* = P_{0zz}^* = 1/(1 + \lambda_0^*), \tag{18}$$

$$P_{0xy}^* = P_{0yx}^* = -\tfrac{3}{2}\lambda_0^*/a^*, \tag{19}$$

where $P_{0ij}^* = P_{0ij}/p_0$, is a dimensionless form of the pressure tensor, $a^* = a/\zeta_0$ is a dimensionless shear rate and $\lambda_0^*(a^*) = \frac{4}{3}\sinh^2[\frac{1}{6}ch^{-1}(1 + 9a^{*2})]$.

The relations (17)–(19) will be used below.

3. Description of the perturbative expansion. Navier–Stokes order

We assume that the system is slightly perturbed from the USF by the introduction of a weak temperature gradient. In general, the existence of a temperature gradient induces the presence of a density gradient, and consequently the temperature and the density are not uniform.

Under these conditions, we propose the following perturbative expansion for the hydrodynamics fields

$$n(\mathbf{r};t) = n_0(\mathbf{r}) + n_2(\mathbf{r};t) + \dots, \tag{20}$$

$$p(\mathbf{r};t) = p_0(\mathbf{r};t) + p_2(\mathbf{r};t) + \dots, \tag{21}$$

$$u_i(\mathbf{r};t) = a_{ij}r_j + u_{1i}(\mathbf{r};t) + \dots, \tag{22}$$

where the subindex 0, 1, 2, ..., means that the corresponding quantity is of order 0, 1, 2, ..., in ∇n_0 used as perturbative parameter. The USF is equivalent to taking n_0 constant. We define, consistently with equation (2), the pressure as

$$p_0(\mathbf{r};t) = n_0(\mathbf{r})k_B T_0(\mathbf{r}) \exp[\lambda_0^*(\mathbf{r})\zeta_0(\mathbf{r})t], \tag{23}$$

where $n_0(\mathbf{r})k_B T_0(\mathbf{r}) \equiv$ const., and so $\nabla n_0/n_0 = -\nabla T_0/T_0$. This relation shows that either ∇n_0 or ∇T_0 can be chosen as a perturbation parameter.

In the same way we have defined the hydrodynamics fields, so that the distribution function $g_0(\mathbf{r}, \mathbf{V}; t)$ can be written in the following way

$$g(\mathbf{r}, \mathbf{V}; t) = g_0(\mathbf{r}, \mathbf{V}; t) + g_1(\mathbf{r}, \mathbf{V}; t) + g_2(\mathbf{r}, \mathbf{V}; t) + \dots, \tag{24}$$

where the zeroth order function g_0 corresponds to the USF distribution function given in (16), but introducing the local dependence on $n_0(\mathbf{r})$, $p_0(\mathbf{r}; t)$ and $\zeta_0(\mathbf{r})$. Thus, the reference function g_0 plays an analogous role to that of the local equilibrium function in the conventional perturbative expansions (Chapman–Enskog, Hilbert). But, g_0 is now a highly nonlinear function of the gradient a since it retains all the hydrodynamics orders from it. Thus, each of the successive approximations g_k will be composed by terms which will retain the full nonlinear dependence on a and will be of order k in ∇n_0 (or ∇T_0). The derived transport equations will be a generalization of the Navier–Stokes, Burnett . . . equations since, in spite of their characteristic of retaining all the orders in the velocity gradient, they will be of first, second . . . order in the gradients of the remaining thermodynamic forces. This is the main characteristic of the proposed expansion.

The solution given in (24) shows some similarities with Hilbert's expansion [11]. In fact, when we consider the expansion of the fields in the equations (20)–(22) we are pre-fixing the space-temporal dependence of the parameters which define the reference function g_0. Thus, the formal expressions of the irreversible fluxes will be given in powers of the gradients of the variables which characterize the expansion of the fields and they are not in powers of the full density, velocity and temperature gradients as in the Chapman–Enskog's solution.

The functions $g_k(\mathbf{r}, \mathbf{V}; t)$ are even (odd) functions of $\mathbf{V}$ if k is an even (odd) integer. This symmetry justifies the parity of the terms in the proposed expansion given in (20)–(22). So, according with this symmetry, the momentum and heat fluxes can be written respectively by

$$P_{ij}(\mathbf{r}; t) = P_{0ij}(\mathbf{r}; t) + P_{2ij}(\mathbf{r}; t) + \dots, \tag{25}$$

$$J_i(\mathbf{r}; t) = J_{1i}(\mathbf{r}; t) + J_{3i}(\mathbf{r}; t) + \dots, \tag{26}$$

Let us use the hydrodynamic balance equations (9)–(11) in order to get information about the terms appearing in equations (20)–(22). For that, we must consider in equations (9)–(11) the expansions given in (20)–(22) and (25)–(26) and collect the terms of equal power in ∇n_0 together. Thus, a self-consistent solution of the BGK equation is ensured since $g(\mathbf{r}, \mathbf{V}; t)$ reproduces the five hydrodynamic moments n, $n\mathbf{u}$ and p, for each approximation.

According to the transport equations (9)–(11), it is obvious that at the zeroth order the USF equations are verified, thus

$$\frac{\partial n_0}{\partial t} = 0, \tag{27}$$

$$\nabla_j P_{0ij} = 0, \tag{28}$$

$$\frac{\partial p_0}{\partial t} = -\tfrac{2}{3} a P_{0xy}, \tag{29}$$

which correspond to a generalization of the Euler equations [11].

In this paper, we restrict our study to first order of the expansion (Navier–Stokes). The study in higher orders is very complex although *a priori* it seems possible to obtain the fluxes from the proposed expansion. However, our purpose will be to find the constitutive relations between fluxes and gradients in the case of coupling between a velocity gradient (considering all their orders) and other gradients considered small enough that only their linear contributions need to be retained.

At the Navier–Stokes order, the transport equations are given by

$$\frac{\partial n_0}{\partial x} = 0, \tag{30}$$

$$mn_0\left[\frac{\partial u_{1i}}{\partial t} + u_{1j}a_{ij}\right] = -\nabla_j P_{0ij}, \tag{31}$$

$$\frac{\partial p_0}{\partial x} = 0. \tag{32}$$

Equations (30) and (32) imply that ∇n_0 (and ∇T_0) are orthogonal to the unperturbed velocity field $a_{ij}r_j$. The general solution of equation (31) is

$$u_{1i}(\mathbf{r}; t) = \exp(-\lambda_0^* \zeta_0 t)\frac{p_0}{mn_0}\left[\left(\delta_{jl} - \frac{a_{jl}^*}{\lambda_0^*}\right)\nabla_k\left(\frac{P_{0lk}^*}{\lambda_0^*\zeta_0}\right) + \frac{P_{0lk}^*}{\lambda_0^{*2}\zeta_0^2}\frac{a_{jl}^*}{\lambda_0^*}\nabla_k(\lambda_0^*\zeta_0)\right]$$
$$\times(\delta_{ij} - a_{ij}^*\zeta_0 t) - \frac{1}{mn_0}\left(\delta_{ij} - \frac{a_{ij}^*}{\lambda_0^*}\right)\nabla_k\left(\frac{P_{0jk}}{\lambda_0^*\zeta_0}\right) - \frac{P_{0jk}}{mn_0\lambda_0^{*2}\zeta_0^2}\frac{a_{ij}^*}{\lambda_0^*}\nabla_k(\lambda_0^*\zeta_0). \tag{33}$$

In this equation, we have taken $\mathbf{u}_1(\mathbf{r}; 0) = 0$, and so $\mathbf{u}_1 \equiv 0$ when $a^* \equiv 0$. In any case, at finite a^*, the first term contribution in (33) (initial term) is negligible for long times and the two remainder terms are the leading ones of $\mathbf{u}_1(\mathbf{r}; t)$ behaviour. Therefore, the contribution of the initial term when considering long times won't be considered henceforth.

In order to obtain the functions $g_k(\mathbf{r}, \mathbf{V}; t)$, the BGK equation can be written in general as

$$\left[\frac{\partial}{\partial t} - a_{ij}V_j\frac{\partial}{\partial V_i} + (V_i + a_{ij}r_j)\frac{\partial}{\partial r_i}\right]g = -\zeta[g - g^{\mathrm{LE}}]. \tag{34}$$

By inserting the expressions (20)–(22) and (24) into the BGK equation (34) and taking into account only first order terms, we get for the function $g_1(\mathbf{r}, \mathbf{V}; t)$ the equation,

$$\left[\frac{\partial}{\partial t} - aV_y\frac{\partial}{\partial V_x} + \zeta_0\right]g_1 = -\mathbf{V}\cdot\nabla g_0 + \zeta_0 g_1^{\mathrm{LE}}, \tag{35}$$

where we have considered the following relations

$$g^{\mathrm{LE}}(\mathbf{r}, \mathbf{V}; t) = g_0^{\mathrm{LE}}(\mathbf{r}, \mathbf{V}; t) + g_1^{\mathrm{LE}}(\mathbf{r}, \mathbf{V}; t) + \ldots$$
$$= n_0(\mathbf{r})\left(\frac{mn_0(\mathbf{r})}{2\pi p_0(\mathbf{r}; t)}\right)^{3/2}\exp\left[-\frac{mn_0(\mathbf{r})V^2}{2p_0(\mathbf{r}; t)}\right]\left[1 + \frac{mn_0(\mathbf{r})}{p_0(\mathbf{r}; t)}\mathbf{V}\cdot\mathbf{u}_1(\mathbf{r}; t) + \ldots\right], \tag{36}$$

$$\zeta(\mathbf{r}; t) = \zeta_0(\mathbf{r}) + \zeta_2(\mathbf{r}; t) + \ldots. \tag{37}$$

The solution of the equation (35) is given by

$$g_1(\mathbf{r}, \mathbf{V}; t) = \int_0^\infty ds \exp(-\zeta_0 s)\zeta_0 \exp\left(asV_y \frac{\partial}{\partial V_x}\right)$$
$$\times [g_1^{\mathrm{LE}}(\mathbf{r}, \mathbf{V}; t-s) - s\mathbf{V} \cdot \nabla g_0^{\mathrm{LE}}(\mathbf{r}, \mathbf{V}; t-s)] + (\mathbf{V} \cdot \nabla\zeta_0) \int_0^\infty ds$$
$$\times \exp(-\zeta_0 s) \exp\left(asV_y \frac{\partial}{\partial V_x}\right)\left[\zeta_0 \frac{s^2}{2} - s\right] g_0^{\mathrm{LE}}(\mathbf{r}, \mathbf{V}; t-s), \qquad (38)$$

where again we have neglected the initial term.

At this order, the only nontrivial flux is the heat flux J_{1i}. Taking into account the perturbation $\mathbf{u}_1$ at stationary velocity, J_{1i} is given by

$$J_{1i} = \tilde{J}_{1i} - \tfrac{3}{2}p_0 u_{1i} - u_{1j} P_{0ij}, \qquad (39)$$

where we have introduced the quantity

$$\tilde{J}_{1i} = \frac{m}{2} \int d\mathbf{V}\, V^2 V_i g_1(\mathbf{r}, \mathbf{V}; t). \qquad (40)$$

The evaluation of $\tilde{J}_{1i}$ is not straighforward and is shown in the Appendix. So, according to the obtained results for $\tilde{J}_{1i}$, the heat flux J_{1i} can be expressed finally in the compact form,

$$J_{1i} = -\lambda_{ij} \nabla_j T_0 - \theta_{ij} \nabla_j p_0 \cdot a, \qquad (41)$$

where the coefficients λ_{ij}, θ_{ij} are also defined in the Appendix. They are functions of the dimensionless shear rate a^*. Obviously, in the limit $a^* \to 0$, Fourier's linear law is obtained again with an expression for the Navier–Stokes thermal conductivity given by

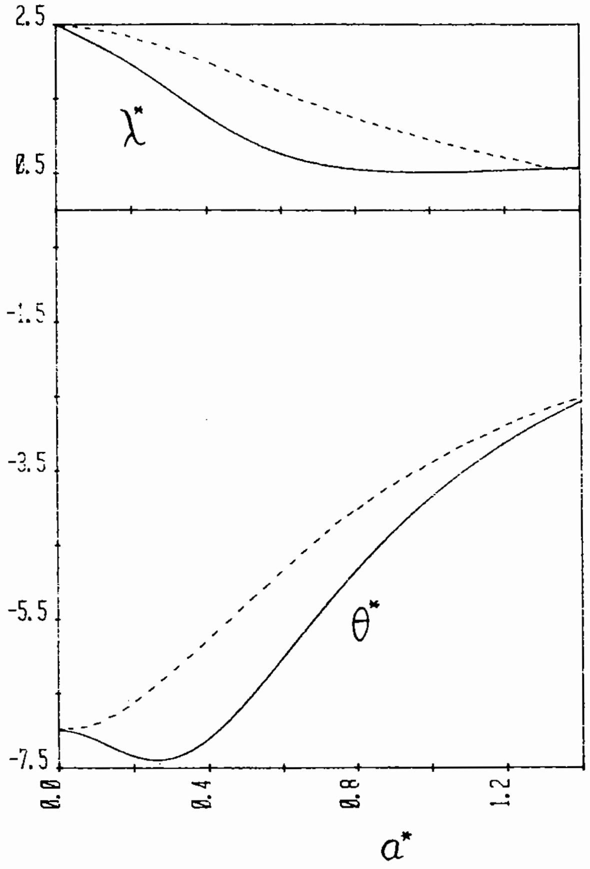
$$\lambda_0 = \frac{5}{2} \frac{n_0 k_B^2 T_0}{m\zeta_0}. \qquad (42)$$

From now on, for simpler calculations, we shall restrict ourselves to the particular case of parallel gradients. We define the dimensionless scalar functions,

$$\lambda^*(a^*) = \frac{mn_0 T_0 \zeta_0}{p_0^2} \lambda_{yy}(a^*), \qquad (43)$$

$$\theta^*(a^*) = \frac{mn_0 \zeta_0}{p_0} \theta_{xy}(a^*). \qquad (44)$$

λ^* can then be interpreted as a generalized thermal conductivity coefficient, while θ^* is a generalization of a Burnett coefficient [12]. Marchetti and Dufty [8] studied the dependence on a^* for coefficients analogous to λ^* and θ^*. They used Boltzmann equation's nonequilibrium time correlation functions for a low-density gas of Maxwell molecules. In the figure, we have plotted the functions $\lambda^*(a^*)$ and $\theta^*(a^*)$, obtained from the results of [8] (dot-dash lines) and the one from our solution of the BGK equation (solid lines). It can be observed that the results obtained by both models are qualitatively in agreement, particularly for large dimensionless shear rate a^*.



Dimensionless shear rate dependence of $\lambda^*(a^*)$ and $\theta^*(a^*)$. The solid lines correspond to the BGK solutions and the dot-dash lines correspond to the Boltzmann solutions. The latter curves have been taken from [8].

4. Generalized thermal conductivity for a constant collision frequency model

The equations describing the USF for Maxwell molecules, equations (16)–(19), are characterized by a constant collision frequency ζ_0. However when there is a temperature (or density) gradient, ζ_0 is not uniform, as we saw in the previous section, and this gives rise to a perturbation in the local velocity field. However, if we consider a very simplified collision model with constant ζ_0, the coexistence of a constant velocity gradient and a constant orthogonal temperature gradient becomes possible.

We consider the hydrodynamics fields,

$$n(\mathbf{r};\, t) = n_0(\mathbf{r}), \tag{45}$$

$$p(\mathbf{r};\, t) = p_0(t), \tag{46}$$

$$u_i(\mathbf{r};\, t) = a_{ij} r_j. \tag{47}$$

This macroscopic state corresponds to ignoring higher than zeroth order terms in the expansion given by the equations (20)–(22). In these conditions, it is straightforward to verify that the USF coexists with a constant temperature gradient, ∇T_0 retaining all the hydrodynamics orders from it. In fact, in order to be $g_k(\mathbf{r}, \mathbf{V}; t)$ functions consistent with the proposed state (45)–(47) it must be necessarily fulfilled

$$\int d\mathbf{V}\psi^{\alpha}(\mathbf{V})g_k = 0, \quad (k \geqslant 1), \tag{48}$$

where $\psi^{\alpha}(\mathbf{V}) = \{1, m\mathbf{V}, (m/2)V^2\}$ are the so-called collisional invariants. If ∇T_0 is constant, the g_k functions are given by

$$\begin{aligned} g_k(\mathbf{r}, \mathbf{V}; t) &= \zeta_0 \int_0^{\infty} ds \exp(-\zeta_0 s) \frac{(-)^k}{k!} s^k \exp\left(asV_y \frac{\partial}{\partial V_x}\right)(\mathbf{V} \,.\, \nabla)^k g_0^{\mathrm{LE}}(\mathbf{r}, \mathbf{V}; t-s) \\ &= \zeta_0 \int_0^{\infty} ds \exp(-\zeta_0 s) \frac{(-)^k}{k!} s^k (\mathbf{V} \,.\, \nabla T_0)^k \frac{p_0[t-s]}{k_{\mathrm{B}}} \left(\frac{m}{2\pi k_{\mathrm{B}}}\right)^{3/2} \\ &\quad \times \left(\frac{d}{dT_0}\right)^k T_0^{-5/2}[t-s] \exp\left(asV_y \frac{\partial}{\partial V_x}\right) \exp\left[-\frac{mV^2}{2k_{\mathrm{B}} T_0[t-s]}\right]. \end{aligned} \tag{49}$$

It is easy to show that the first moments of the functions g_k are zeroth [9]. Specifically, conditions (48) are verified and in addition

$$P_{kij} = 0. \tag{50}$$

The pressure tensor is not affected explicitly by the temperature gradient ∇T_0. The first nonzeroth moment corresponds to the heat flux J_i. It is exactly a linear function of the temperature gradient. Therefore,

$$\mathbf{J} = -\mathbf{\Lambda} \,.\, \nabla T_0, \tag{51}$$

where $\mathbf{\Lambda}$ is a generalized thermal conductivity tensor defined by

$$\mathbf{\Lambda}(a^*) = \frac{n_0 k_{\mathrm{B}}^2 T_0}{2m\zeta_0} \begin{pmatrix} \frac{5}{(1+2\lambda_0^*)^2} + 60\frac{a^{*2}}{(1+2\lambda_0^*)^4} + 360\frac{a^{*4}}{(1+2\lambda_0^*)^6} & -14\frac{a^*}{(1+2\lambda_0^*)^3} - 72\frac{a^{*3}}{(1+2\lambda_0^*)^5} & 0 \\ -14\frac{a^*}{(1+2\lambda_0^*)^3} - 72\frac{a^{*3}}{(1+2\lambda_0^*)^5} & \frac{5}{(1+2\lambda_0^*)^2} + 18\frac{a^{*2}}{(1+2\lambda_0^*)^4} & 0 \\ 0 & 0 & \frac{5}{(1+2\lambda_0^*)^2} + 6\frac{a^{*2}}{(1+2\lambda_0^*)^4} \end{pmatrix}. \tag{52}$$

If we consider ζ_0 constant, this result coincides with that obtained in (41). However, while (41) is valid only to first order in ∇T_0, the expression (52) is exact for arbitrarily large ∇T_0. According to (52), we have obtained a generalization of Fourier's linear law. Although the imposed temperature gradient may be very large, the system always follows a linear law for the heat conduction but with a thermal conductivity tensor which depends nonlinearly on a^*.

5. Summary

In this paper we propose a perturbative expansion, formally analogous to Hilbert's expansion, for the study of the transport properties in a gas subject to large shear rate and to other types of gradients (temperature, pressure . . .). We use the BGK equation and we consider the simple case of Maxwell molecules. The zeroth order expansion corresponds to the USF distribution function that contains all the hydrodynamics orders on the shear rate *a*. The irreversible fluxes are evaluated exactly to first order. The transport equations obtained are a generalization of the usual Navier–Stokes hydrodynamic equations, but the transport coefficients depend now on the dimensionless shear rate a^* in a highly nonlinear manner. The dependence on shear rate for several transport coefficients is examined and compared with previous results obtained by the correlation function formalism of the Boltzmann equation.

There is a particular case for which the result is exact. This corresponds to the constant collision frequency model. For this model, the pressure tensor does not depend on the temperature gradient ∇T_0 and the heat flux is linear with ∇T_0. Then, we obtain a generalized Fourier's law with a generalized thermal conductivity tensor that is function of the reduced gradient a^*.

Although in general, the model has been only studied to first order we think that it is possible to resolve the proposed transport equations in higher orders. The transport equations obtained in this way are generalizations of the Burnett, super-Burnett . . . hydrodynamic equations, and the transport coefficients retain all the hydrodynamics orders in the linear velocity gradient.

Appendix

In order to evaluate the quantity $\tilde{J}_{1i}$ that appears in equation (40), we need the results

$$\int d\mathbf{V}\, V^2 V_i \exp\left[bV_y \frac{\partial}{\partial V_x}\right] V_j \exp(-cV^2) = \frac{\pi^{3/2}}{4} c^{-7/2} A_{ij}(b), \tag{A 1}$$

$$\int d\mathbf{V}\, V^2 V_i V_j \exp\left[bV_y \frac{\partial}{\partial V_x}\right] \exp(-cV^2) = \frac{\pi^{3/2}}{4} c^{-7/2} B_{ij}(b), \tag{A 2}$$

where we have introduced the tensors **A** and **B**, whose components are given by

$$A_{ij}(b) = (5 + 3b^2)\delta_{ij} - 2b^2\delta_{iz}\delta_{jz} - b(7 + 3b^2)\delta_{ix}\delta_{jy} - 2b\delta_{iy}\delta_{jx}, \tag{A 3}$$

$$B_{ij}(b) = (5 + 10b^2 + 3b^4)\delta_{ix}\delta_{jx} + (5 + 3b^2)\delta_{iy}\delta_{jy} + (5 + b^2)\delta_{iz}\delta_{jz} - b(7 + 3b^2)(\delta_{ix}\delta_{jy} + \delta_{jx}\delta_{iy}). \tag{A 4}$$

In this way, $\tilde{J}_{1i}$ can be expressed formally by

$$\tilde{J}_{1i} = \tilde{J}_{1i}^{(1)} + \tilde{J}_{1i}^{(2)} + \tilde{J}_{1i}^{(3)} \tag{A 5}$$

where

$$\tilde{J}_{1i}^{(1)} = \frac{1}{2}\int_0^\infty ds \exp(-\zeta_0 s)\zeta_0 p_0[t-s] u_{1j}[t-s] A_{ij}(as) \tag{A 6}$$

$$\tilde{J}_{1i}^{(2)} = -\frac{1}{2m}\int_0^\infty ds \exp(-\zeta_0 s)\zeta_0 s B_{ij}(as)\nabla_j \frac{p_0^2[t-s]}{n_0} \tag{A 7}$$

$$\tilde{J}_{1i}^{(3)} = \frac{1}{2mn_0}(\nabla_j \zeta_0)\int_0^\infty ds \exp(-\zeta_0 s)(\tfrac{1}{2}\zeta_0 s^2 - s) p_0^2[t-s] B_{ij}(as). \tag{A 8}$$

To compute these expressions, it is necessary to take into account the temporal dependence of the hydrostatic pressure (equation (23)) and the velocity $\mathbf{u}_1$ (equation

(33)). We obtain straightforwardly

$$\tilde{J}_{1i}^{(1)} = \frac{p_0[t]}{2}\left[u_{1j}\hat{A}_{ij} - \frac{p_0[t]}{2mn_0\zeta_0}\frac{P^*_{0kl}}{\lambda^*_0\zeta_0}\left(\delta_{jk} - \frac{a^*_{jk}}{\lambda^*_0}\right)\nabla_l(\lambda^*_0\zeta_0)\left(\frac{\partial \hat{A}_{ij}}{\partial \lambda^*_0}\right)\right], \qquad (A\,9)$$

$$\tilde{J}_{1i}^{(2)} = \frac{1}{4m\zeta_0}\nabla_j\left(\frac{p_0^2[t]}{n_0}\right)\left(\frac{\partial \hat{B}_{ij}}{\partial \lambda^*_0}\right) + \frac{p_0^2[t]}{4mn_0\zeta_0^2}\left(\frac{\partial^2 \hat{B}_{ij}}{\partial \lambda^{*2}_0}\right)\nabla_j(\lambda^*_0\zeta_0), \qquad (A\,10)$$

$$\tilde{J}_{1i}^{(3)} = \frac{p_0^2[t]}{4mn_0\zeta_0^2}\left[\left(\frac{\partial \hat{B}_{ij}}{\partial \lambda^*_0}\right) + \frac{1}{4}\left(\frac{\partial^2 \hat{B}_{ij}}{\partial \lambda^{*2}_0}\right)\right](\nabla_j\zeta_0). \qquad (A\,11)$$

In these expressions, we define

$$\begin{aligned}\hat{A}_{ij}(\lambda^*_0, a^*) &= \int_0^\infty d\alpha \exp\left[-\alpha(1+2\lambda^*_0)\right]A_{ij}(\alpha a^*)\\ &= \frac{1}{(1+2\lambda^*_0)}\left\{\left[5 + 6\frac{a^{*2}}{(1+2\lambda^*_0)^2}\right]\delta_{ij} - \frac{4a^{*2}}{(1+2\lambda^*_0)^2}\delta_{iz}\delta_{jz}\right.\\ &\quad \left. - \left[\frac{7a^*}{(1+2\lambda^*_0)} + 18\frac{a^{*3}}{(1+2\lambda^*_0)^3}\right]\delta_{ix}\delta_{jy} - \frac{2a^*}{(1+2\lambda^*_0)}\delta_{iy}\delta_{jx}\right\}, \end{aligned} \qquad (A\,12)$$

$$\begin{aligned}\hat{B}_{ij}(\lambda^*_0, a^*) &= \int_0^\infty d\alpha \exp\left[-\alpha(1+2\lambda^*_0)\right]B_{ij}(\alpha a^*)\\ &= \frac{1}{(1+2\lambda^*_0)}\left\{\left[5 + 20\frac{a^{*2}}{(1+2\lambda^*_0)^2} + 72\frac{a^{*4}}{(1+2\lambda^*_0)^4}\right]\delta_{ix}\delta_{jx}\right.\\ &\quad + \left[5 + \frac{6a^{*2}}{(1+2\lambda^*_0)^2}\right]\delta_{iy}\delta_{jy} + \left[5 + \frac{2a^{*2}}{(1+2\lambda^*_0)^2}\right]\delta_{iz}\delta_{jz}\\ &\quad \left. - \left[\frac{7a^*}{(1+2\lambda^*_0)} + \frac{18a^{*3}}{(1+2\lambda^*_0)^3}\right](\delta_{ix}\delta_{jy} + \delta_{ix}\delta_{jy})\right\}, \end{aligned} \qquad (A\,13)$$

where their derivatives with respect to λ^*_0 can be easily obtained.

If we write J_{1i} in the compact form given in (41), we can obtain the transport coefficients

$$\begin{aligned}\lambda_{ij} &= \frac{p_0^2[t]}{mn_0T_0\zeta_0}\left\{\frac{1}{16}\frac{\partial^2 \hat{B}_{ij}}{\partial \lambda^{*2}_0} - \frac{1}{\lambda^*_0}\frac{\partial}{\partial \zeta_0}(\lambda^*_0\zeta_0)\left\{\left(\frac{\hat{A}_{ik}}{2} - \frac{3}{2}\delta_{ik} - P^*_{0ik}\right)\right.\right.\\ &\quad \times \left[\left(\delta_{kl} - \frac{a^*_{kl}}{\lambda^*_0}\right)\left(\frac{\partial P^*_{0lj}}{\partial \lambda^*_0} - \lambda^*_0\left[\frac{\partial}{\partial \zeta_0}(\lambda^*_0\zeta_0)\right]^{-1}\frac{\partial P^*_{0lj}}{\partial \lambda^*_0} - \frac{P^*_{0lj}}{\lambda^*_0}\right) + \frac{P^*_{0lj}}{\lambda^*_0}\frac{a^*_{kl}}{\lambda^*_0}\right]\\ &\quad \left.\left. + \left(\delta_{ik} - \frac{a^*_{kl}}{\lambda^*_0}\right)\frac{P^*_{0lj}}{4}\left(\frac{\partial \hat{A}_{ik}}{\partial \lambda^*_0}\right) - \frac{\lambda^*_0}{4}\left(\frac{\partial^2 \hat{A}_{ij}}{\partial \lambda^{*2}_0}\right)\right\}\right\}, \end{aligned} \qquad (A\,14)$$

$$\theta_{ij} = \frac{p_0[t]}{mn_0\zeta_0^2a^*}\left\{\left(\frac{\hat{A}_{ik}}{2} - \frac{3}{2}\delta_{ik} - P^*_{0ik}\right)\frac{P^*_{0lj}}{4}\left(\delta_{kl} - \frac{a^*_{kl}}{\lambda^*_0}\right) - \frac{1}{2}\left(\frac{\partial \hat{B}_{ij}}{\partial \lambda^*_0}\right)\right\}. \qquad (A\,15)$$

All the quantities that appear in these expressions are known. The highly nonlinear dependence of these coefficients on the dimensionless shear rate is apparent.

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