Influence of gravity on nonlinear transport in the planar Couette flow

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The effect of gravity on a dilute gas subjected to the steady planar Couette flow with arbitrarily large velocity and temperature gradients is analyzed. The results are obtained from the Bhatnagar–Gross–Krook kinetic model by means of a perturbation expansion in powers of the external field. The reference state corresponds to the pure (nonlinear) Couette flow solution, which retains all the hydrodynamic orders in the shear rate and the thermal gradient. To first order in the gravity field, we explicitly obtain the hydrodynamic profiles and the five relevant nonlinear transport coefficients; the shear viscosity η, the two viscometric functions $\Psi_{1,2}$, and the two nonzero elements, κ_{xy} and κ_{yy}, of the thermal conductivity tensor. The results show that, in general, the influence of gravity on the rheological properties η and $\Psi_{1,2}$ tend to decrease as the shear rate increases, while this influence is especially important in the case of the thermal conductivity coefficient, κ_{yy}, which measures the heat flux parallel to the temperature gradient. *© 1999 American Institute of Physics.* [S1070-6631(99)00204-4]

I. INTRODUCTION

It is usual to ignore the effect of gravity on the properties of gases under ordinary conditions. This is justified by the fact that the action of gravity between two successive collisions of a molecule is negligible, i.e., $\lambda \ll h$, where λ is the mean free path and $h = v_0^2/g$ is the distance over which a particle feels the action of gravity, v_0 being the thermal velocity and g being the gravity acceleration. At most, one introduces g in the balance equations for momentum, but otherwise one assumes that the dependence of the momentum and heat fluxes on the hydrodynamic fields and their gradients is unchanged.[1] For instance, in the case of air under terrestrial conditions and at room temperature, $\lambda/h \sim 10^{-11}$. On the other hand, it is appealing to investigate if and how the transport equations are modified in situations where the role of gravity is not so tiny, say for example values of rarefaction and/or g for which $\lambda/h \sim 10^{-3}$. To the best of our knowledge, this problem has not been sufficiently studied. Recently, we have evaluated the corrections to the Navier–Stokes (NS) equations due to gravity in a dilute gas subjected to the planar Fourier flow.[2] This study was made from an exact perturbation solution of the Boltzmann equation for Maxwell molecules through order g^2. The zeroth order solution leads to an isotropic pressure tensor and to the fulfillment of the linear Fourier law, even for large thermal gradients.[3,4] We found that, because of gravity, the pressure tensor becomes anisotropic and the heat flux increases (decreases) with respect to its NS value when the gas is heated from above (below). This analysis has been extended[5] to higher orders in g by using the Bhatnagar–Gross–Krook (BGK) model.

The aim of this paper is to analyze the influence of gravity on a more complex state than the Fourier flow. Specifically, we will consider the steady planar Couette flow, which corresponds to a gas enclosed between two infinite, parallel plates in relative motion and, in general, kept at different temperatures. This state reduces to the Fourier flow in the special case where the gas is at rest. In the steady Couette flow, momentum as well as heat transport are present and the hydrodynamic balance equations become

$$\frac{\partial}{\partial y} P_{xy} = 0, \tag{1}$$

$$\frac{\partial}{\partial y} P_{yy} + \rho g = 0, \tag{2}$$

$$P_{xy} \frac{\partial u_x}{\partial y} + \frac{\partial}{\partial y} q_y = 0, \tag{3}$$

where P_{ij} is the pressure tensor, $\mathbf{q}$ is the heat flux, ρ is the mass density, $\mathbf{u}$ is the flow velocity, the *x*-axis is parallel to the direction of motion, and the *y*-axis is orthogonal to the plates. In the above equations we have assumed that there are gradients only along the *y*-axis and that gravity is antiparallel to that axis. According to Eq. (1), the viscous pressure is uniform across the system; otherwise, the state would not be a steady one. Equation (2) implies that the normal pressure at the bottom plate exceeds the one at the top plate in an amount equal to the weight of a fluid column of unit area. Finally, Eq. (3) expresses the fact that the rate of mechanical work introduced by the plates equal the rate of heat lost through the two surfaces. Note that gravity does not

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appear explicitly in Eqs. (1) and (3). The balance equations (1)–(3) do not constitute a closed set. Nevertheless, if the hydrodynamic gradients are weak, the fluxes are well described by the NS relations, namely the Newton and Fourier laws,

$$P_{xx}=P_{yy}=P_{zz}=p, \quad P_{xy}=-\eta_{\rm NS}\frac{\partial u_x}{\partial y}, \tag{4}$$

$$q_x=0, \quad q_y=-\kappa_{\rm NS}\frac{\partial T}{\partial y}, \tag{5}$$

where $p=nk_BT$ is the hydrostatic pressure (n being the number density, k_B being the Boltzmann constant, and T being the temperature), and $\eta_{\rm NS}$ and $\kappa_{\rm NS}$ are the NS shear viscosity and thermal conductivity, respectively. On the other hand, when the strength of the gradients is not small, the NS constitutive equations are not expected to apply and the transport must be described by nonlinear equations.

In the absence of gravity, the nonlinear regime has been studied in the past from different methods. For dense gases, the problem has been studied by molecular dynamics simulations[6] and by a modified moment method;[7] for dilute gases, studies have been carried out by molecular dynamics simulations,[8] by a perturbation solution of the Boltzmann equation,[9] by the Grad method,[8] and by exact solutions of the BGK model[10,11] and related models.[12–14] In all these works, the main motivation is to study the breakdown of the NS relations (4) and (5), what means that normal stress differences exist, the viscous pressure P_{xy} is not just proportional to the shear rate $\partial u_x/\partial y$, and the heat flux $\mathbf{q}$ is no longer proportional to the thermal gradient vector ∇T. In order to characterize these deviations from the NS relations, it is usual to introduce *generalized* transport coefficients, namely the viscometric functions Ψ_1 and Ψ_2, the generalized shear viscosity η, the generalized thermal conductivity κ_{yy}, and a coefficient κ_{xy} measuring cross effects. These nonlinear transport coefficients are defined by

$$P_{yy}-P_{xx}=\Psi_1\left(\frac{\partial u_x}{\partial y}\right)^2, \tag{6}$$

$$P_{zz}-P_{yy}=\Psi_2\left(\frac{\partial u_x}{\partial y}\right)^2, \tag{7}$$

$$P_{xy}=-\eta\frac{\partial u_x}{\partial y}, \tag{8}$$

$$q_x=-\kappa_{xy}\frac{\partial T}{\partial y}, \tag{9}$$

$$q_y=-\kappa_{yy}\frac{\partial T}{\partial y}. \tag{10}$$

In general, all these coefficients are functions of the hydrodynamic gradients $\partial u_x/\partial y$ and $\partial T/\partial y$. When the gradients become small, the NS coefficients are recovered, i.e., $\Psi_1\to 0$, $\Psi_2\to 0$, $\eta\to\eta_{\rm NS}$, $\kappa_{yy}\to\kappa_{\rm NS}$, and $\kappa_{xy}\to 0$.

As said above, we are interested in evaluating the effect of gravity on the coefficients defined by Eqs. (6)–(10). To this end, we will use the BGK model of the Boltzmann equation, for which an exact solution when $g=0$ is known.[10,11] Recent comparisons between the BGK results and those obtained from molecular dynamics[14] as well as Monte Carlo[15] simulations show a very good agreement. This gives us confidence in the use of the BGK approximation to describe the steady Couette flow in the presence of gravity. We solve the BGK kinetic equation by performing a perturbation expansion in powers of g, taking the pure (nonlinear) Couette flow solution as the zeroth order approximation. As a consequence, the successive approximations depend in a nonlinear way on the hydrodynamic gradients. Here, we will restrict ourselves to the first order corrections, what is justified by the fact that in practical applications the value of g is very small.

The paper is organized as follows. In Sec. II we describe the problem and give a brief summary of the main known results in the absence of gravity. The first order corrections are worked out in Sec. III, the mathematical details being given in the Appendices. By taking a given example, we illustrate the influence of gravity on the five relevant transport properties in Sec. IV. We close the paper with some concluding remarks in Sec. V.

II. DESCRIPTION OF THE PROBLEM

Let us consider a dilute gas described by the BGK kinetic equation,[16]

$$\frac{\partial}{\partial t}f+\mathbf{v}\cdot\nabla f+\frac{\mathbf{F}}{m}\cdot\frac{\partial}{\partial \mathbf{v}}f=-\nu(f-f_L), \tag{11}$$

where $f(\mathbf{r},\mathbf{v};t)$ is the one-particle velocity distribution function, $\mathbf{F}$ is an external force, and f_L is the local equilibrium distribution function given by

$$f_L(\mathbf{r},\mathbf{v};t)=n\left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp\left[-m\frac{(\mathbf{v}-\mathbf{u})^2}{2k_BT}\right]. \tag{12}$$

Here, m is the mass of a particle. The local quantities $n(\mathbf{r},t)$, $\mathbf{u}(\mathbf{r},t)$, and $T(\mathbf{r},t)$ are defined in terms of the distribution function as

$$n=\int d\mathbf{v}f, \tag{13}$$

$$\mathbf{u}=\frac{1}{n}\int d\mathbf{v}\mathbf{v}f, \tag{14}$$

$$T=\frac{m}{3nk_B}\int d\mathbf{v}(\mathbf{v}-\mathbf{u})^2f. \tag{15}$$

Furthermore, Eq. (11) introduces a velocity-independent collision frequency, ν, which is proportional to the density and whose dependence on the temperature models the interaction potential. For instance, $\nu\propto n$ for Maxwell molecules, while $\nu\propto nT^{1/2}$ for hard spheres. Apart from the densities of conserved quantities, one can define the pressure tensor (related to the transport of momentum)

$$\mathsf{P}=m\int d\mathbf{v}(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})f \tag{16}$$

and the heat flux (related to the transport of energy)

$$\mathbf{q}=\frac{m}{2}\int d\mathbf{v}(\mathbf{v}-\mathbf{u})^2(\mathbf{v}-\mathbf{u})f. \tag{17}$$

The main motivation of this paper is to analyze the influence of gravity on the heat and momentum transport across a fluid. The physical situation is that of a dilute gas enclosed between two parallel plates moving at different velocities (planar Couette flow) and subjected to a constant gravitational field perpendicular to the plates. Let the x-axis be parallel to the direction of motion and the y-axis be orthogonal to the walls. We want to study a steady state with velocity and temperature gradients along the y direction coexisting with a field $\mathbf{F}=-mg\hat{\mathbf{y}}$, where g is the acceleration due to gravity. In addition, we are interested in a situation where the external field does not generate convective motion, so that the flow velocity profile is only due to the relative motion of the plates enclosing the gas (boundary conditions). This implies that the corresponding (local) Rayleigh number Ra, which is proportional to $g(-\partial T/\partial y)$, must be less than a certain critical value $\mathrm{Ra}_c\simeq 1700$.[17] In addition, we are interested in the properties of the gas in the bulk region rather than close to the walls. Thus, we will assume that the Knudsen number (which is defined as the ratio between the mean free path and the separation between the plates) is sufficiently small to identify such a region. In other words, we will look for a ''normal'' solution of Eq. (11) where all the space dependence of f is given trough its functional dependence on the pressure, the flow velocity and the temperature.

In order to simplify the analysis, it is convenient to introduce dimensionless quantities. To do so, we choose an *arbitrary* point y_0 belonging to the bulk domain as the origin and take the quantities at that point (henceforth denoted by a subscript 0) as reference units. Therefore, we define $T^*\equiv T/T_0$, $p^*\equiv p/p_0$, $\mathbf{u}^*\equiv\mathbf{u}/v_0$, $\mathbf{v}^*\equiv\mathbf{v}/v_0$, $f^*\equiv n_0^{-1}v_0^3 f$, and $g^*\equiv g/v_0\nu_0$. Here, $v_0=(k_BT_0/m)^{1/2}$ is a thermal velocity. One can define a mean free path (at $y=y_0$) as $\lambda_0=v_0/\nu_0$ and a characteristic length $h_0=v_0^2/g$. Thus, $g^*=\lambda_0/h_0$ represents the ratio between the mean free path and the distance over which a typical particle feels the action of the field. In the same way as in previous descriptions,[10,12,13,18] the spatial variable y must be conveniently rescaled in a nonlinear way that takes into account the local dependence of the collision frequency. Thus, we define

$$s=\frac{1}{v_0}\int_{y_0}^{y}dy'\,\nu(y'). \tag{18}$$

Under the above conditions, the BGK equation (11) becomes

$$\left(1+v_y^*\frac{\partial}{\partial s}-g^*\frac{T^*}{p^*}\frac{\partial}{\partial v_y^*}\right)f^*=f_L^*, \tag{19}$$

where, for the sake of concreteness, we have restricted ourselves to the case of Maxwell molecules.

A trivial solution of Eq. (19) corresponds to the *equilibrium* state characterized by $\mathbf{u}^*=0$, $T^*=1$ and $p^*=1-g^*s$. The latter equation leads to the well-known barometric formula $p(y)=p_0\exp[-mg(y-y_0)/k_BT]$. A much more interesting and nontrivial situation corresponds to the (pure) steady planar Couette flow, i.e., when $g^*=0$. In this case, Eq. (19) admits an *exact* solution characterized by a constant pressure $p^*=1$, and ''linear'' velocity and ''parabolic'' temperature profiles[10,11]

$$u_x^*(s)=u_x^*(0)+as, \tag{20}$$

$$T^*(s)=1+\epsilon s-\gamma_0(a)s^2, \tag{21}$$

where a and ϵ are independent constants fixed by the boundary conditions. These two quantities measure the departure of the system from equilibrium. The dimensionless parameter $\gamma_0(a)$ is a *nonlinear* function of the reduced shear rate a given implicitly through the equation[10]

$$a^2=\gamma_0\frac{2F_2(\gamma_0)+3F_1(\gamma_0)}{F_1(\gamma_0)}, \tag{22}$$

where

$$F_r(\gamma_0)=\left(\frac{d}{d\gamma_0}\gamma_0\right)^r F_0(\gamma_0) \tag{23}$$

and

$$F_0(\gamma_0)=\frac{2}{\gamma_0}\int_0^\infty dt\,t\exp(-t^2/2)K_0(2\gamma_0^{-1/4}t^{1/2}), \tag{24}$$

K_0 being the zeroth-order modified Bessel function. The relevant transport coefficients of the steady Couette flow are obtained from the pressure tensor and the heat flux. They are nonlinear functions of the reduced shear rate a given by[10,13,14]

$$P_{xx}^*=1+4\gamma_0(F_1+F_2), \tag{25}$$

$$P_{yy}^*=1-2\gamma_0(F_1+2F_2), \tag{26}$$

$$P_{zz}^*=1-2\gamma_0F_1, \tag{27}$$

$$P_{xz}^*=P_{yz}^*=0, \tag{28}$$

$$P_{xy}^*=-F_0\frac{\partial u_x^*}{\partial s}, \tag{29}$$

$$q_y^*=-\frac{a^2F_0}{2\gamma_0}\frac{\partial T^*}{\partial s}, \tag{30}$$

$$q_x^*=[5F_2+2F_3+2a^2(F_2+5F_3+8F_4+4F_5)]a\frac{\partial T^*}{\partial s}. \tag{31}$$

Here, we have introduced the dimensionless fluxes $\mathsf{P}^*\equiv\mathsf{P}/p_0$ and $\mathbf{q}^*\equiv\mathbf{q}/p_0v_0$. In Eqs. (25)–(31), $F_r\equiv F_r(\gamma_0)$. Notice that although the temperature gradient is only directed along the y-axis (so that there is a response in this direction through q_y^*), the shear flow induces a nonzero x component of the heat flux.[8,13,14] Furthermore, an explicit expression for the velocity distribution function f^* has also been derived.[11]

The presence of the term proportional to g^* in Eq. (19) complicates the problem significantly and we have not been able to get an explicit solution for arbitrary values of the gravity. However, given that the value of the gravity acceleration is small enough, for practical purposes it is sufficient to perform a perturbation analysis in the same way as in the case of the Fourier flow.[2] More specifically, we will carry out a perturbation expansion in powers of $\delta\equiv\epsilon g^*$,

$$f^*=f^{(0)}+f^{(1)}\delta+\cdots, \tag{32}$$

where the reference state $f^{(0)}$ represents the pure steady Couette flow corresponding to the actual values of pressure, flow velocity, temperature, and both velocity and thermal gradients at the point of interest $y=y_0$. The use of δ as the perturbation parameter rather than g^* is due to the fact that the product ϵg^* appears in a natural way, so that the final expressions are more compact. The parameter δ represents the combination $\lambda_0^2/l_0 h_0$, where $l_0^{-1}=\partial \ln T/\partial y|_{y=y_0}$ is the inverse of a characteristic hydrodynamic length. In consistency with the expansion (32), we expand the hydrodynamic fields and the dissipative fluxes as

$$p^*=p^{(0)}+p^{(1)}\delta+\cdots, \tag{33}$$

$$u_x^*=u_x^{(0)}+u_x^{(1)}\delta+\cdots, \tag{34}$$

$$T^*=T^{(0)}+T^{(1)}\delta+\cdots, \tag{35}$$

$$\mathsf{P}^*=\mathsf{P}^{(0)}+\mathsf{P}^{(1)}\delta+\cdots, \tag{36}$$

$$\mathbf{q}^*=\mathbf{q}^{(0)}+\mathbf{q}^{(1)}\delta+\cdots. \tag{37}$$

Here, $p^{(0)}=1$, $u_x^{(0)}=u_x^*(0)+as$, $T^{(0)}=1+\epsilon s-\gamma_0 s^2$, and the fluxes $\mathsf{P}^{(0)}$ and $\mathbf{q}^{(0)}$ are given by Eqs. (25)–(31). By definition, $p^{(k)}(0)=u_x^{(k)}(0)=T^{(k)}(0)=\partial u_x^{(k)}/\partial s|_{s=0}=\partial T^{(k)}/\partial s|_{s=0}=0$ for $k\geqslant 1$. According to the spirit of our expansion, the terms of order δ^k are *nonlinear* functions of the shear rate a and the thermal gradient ϵ. This is the main feature of the method. In this paper we will only consider the first order correction to the pure Couette flow.

III. STEADY COUETTE FLOW IN THE PRESENCE OF A WEAK GRAVITATIONAL FIELD

In this section we evaluate the hydrodynamic profiles as well as the momentum and heat fluxes to first order in δ. By substituting the expansions (32)–(35) into Eq. (19), one gets

$$\left(1+V_y\frac{\partial}{\partial s}\right)f^{(1)}-\frac{T^{(0)}}{\epsilon}\frac{\partial}{\partial V_y}f^{(0)}=f_L^{(1)}, \tag{38}$$

where $\mathbf{V}=\mathbf{v}^*-\mathbf{u}^{(0)}$, $f_L^{(1)}$ is given by

$$f_L^{(1)}=\left[p^{(1)}+\frac{V_x u_x^{(1)}}{T^{(0)}}+\left(\frac{V^2}{2T^{(0)}}-\frac{5}{2}\right)\frac{T^{(1)}}{T^{(0)}}\right]f_L^{(0)} \tag{39}$$

and

$$f_L^{(0)}=(2\pi)^{-3/2}T^{(0)-5/2}\exp\left(-\frac{V^2}{2T^{(0)}}\right). \tag{40}$$

It is convenient to recast Eq. (38) into the form

$$f^{(1)}=(1+V_y\partial_s)^{-1}\left(f_L^{(1)}+\frac{T^{(0)}}{\epsilon}\frac{\partial}{\partial V_y}f^{(0)}\right)$$
$$=\sum_{k=0}^{\infty}(-\partial_s)^k V_y^k\left(f_L^{(1)}+\frac{T^{(0)}}{\epsilon}\frac{\partial}{\partial V_y}f^{(0)}\right), \tag{41}$$

where $\partial_s\equiv\partial/\partial s$. This is only a *formal* solution since $f_L^{(1)}$ is a functional of $f^{(1)}$ through its dependence on the fields $p^{(1)}$, $u_x^{(1)}$, and $T^{(1)}$. In order to convert Eq. (41) into an explicit equation that can be solved, one needs to know the spatial dependence of the above hydrodynamic fields. As in the case of the Fourier flow,[2] we will follow a heuristic method, namely, we first guess simple profiles and then verify their consistency. Inspection of Eq. (41) suggests that the structure of the solution corresponding to the pure Couette flow, Eqs. (20) and (21) can be extended to the solution of first order. Therefore, we assume that $p^{(1)}$, $u_x^{(1)}$, and $T^{(1)}$ are polynomials in s of degree 1, 2, and 3, respectively, whose coefficients are nonlinear functions of the reduced shear rate a and of the reduced thermal gradient ϵ,

$$p^{(1)}=\alpha s, \tag{42}$$

$$u_x^{(1)}=\beta s^2, \tag{43}$$

$$T^{(1)}=\gamma_1 s^2+\gamma_2 s^3. \tag{44}$$

The unknown coefficients α, β, γ_1, and γ_2 are determined by requiring the self-consistency of the solution (41) characterized by the profiles (42)–(44). This means that $f^{(1)}$ and $f_L^{(1)}$ possess the same first five moments,

$$\int d\mathbf{v}\{1,\mathbf{V},V^2\}(f^{(1)}-f_L^{(1)})=\{0,\mathbf{0},0\}. \tag{45}$$

The fulfillment of these conditions leads (see Appendix A) to a system of four linear equations for the set $\{\alpha,\beta,\gamma_1,\gamma_2\}$ as functions of a and ϵ. The corresponding solution, Eqs. (A53)–(A59), provides the explicit form of the profiles. In particular, for small shear rates, these coefficients behave as $\alpha\approx-(1-\frac{6}{5}a^2)/\epsilon$, $\beta\approx(a/2\epsilon)(1+\frac{46}{5}a^2)$, $\gamma_1\approx\frac{1}{2}(1+\frac{494}{25}a^2)$, and $\gamma_2\approx-a^2/5\epsilon$. These results are consistent with those obtained from the BGK equation in the Fourier flow problem under gravitation (i.e., when $a=0$).[5]

Once the explicit form of the hydrodynamic fields is known, the goal now is to get the influence of gravity on the transport properties when only terms through first order in δ are considered in the momentum and heat fluxes. Since the reduced parameters a, ϵ, and $\delta=\epsilon g^*$ are defined at the (arbitrary) point $s=0$, we evaluate the fluxes at that point, without loss of generality. The idea is to express the fluxes at *any* point (represented by $s=0$ or, equivalently, $y=y_0$) in the bulk domain of the system in terms of the values of the hydrodynamic quantities and their gradients at that very point.

The xy element of the pressure tensor defines the nonlinear shear viscosity η, according to Eq. (8). In reduced units,

$$\eta^*(a,\epsilon,\delta)=-\left.\frac{P_{xy}^{(0)}+P_{xy}^{(1)}\delta}{a}\right|_{s=0}. \tag{46}$$

Normal stresses are measured through the viscometric functions Ψ_1 and Ψ_2 defined by Eqs. (6) and (7). To first order in the field, they are

$$\Psi_1^*(a,\epsilon,\delta)=\left.\frac{P_{yy}^{(0)}-P_{xx}^{(0)}+(P_{yy}^{(1)}-P_{xx}^{(1)})\delta}{a^2}\right|_{s=0}, \tag{47}$$

$$\Psi_2^*(a,\epsilon,\delta)=\left.\frac{P_{zz}^{(0)}-P_{yy}^{(0)}+(P_{zz}^{(1)}-P_{yy}^{(1)})\delta}{a^2}\right|_{s=0}. \tag{48}$$

As said before, in the Couette flow the heat flux vector defines a thermal conductivity tensor rather than a scalar due to the anisotropy of the problem. According to Eqs. (9) and (10), the nonzero elements are

$$\kappa_{xy}^*(a,\epsilon,\delta)=-\left.\frac{q_x^{(0)}+q_x^{(1)}\delta}{\epsilon}\right|_{s=0}, \tag{49}$$

$$\kappa_{yy}^*(a,\epsilon,\delta)=-\left.\frac{q_y^{(0)}+q_y^{(1)}\delta}{\epsilon}\right|_{s=0}. \tag{50}$$

Equations (46)–(50) define the five relevant transport coefficients in the problem. The explicit calculation of the nonzero elements of $\mathsf{P}^{(1)}$ and $\mathbf{q}^{(1)}$ is made in Appendix B. They are given by Eqs. (B4)–(B7), (B9) and (B10). The nonlinear dependence of these quantities on the shear rate is very apparent. From the above expressions, one can get the transport coefficients η, Ψ_1, Ψ_2, κ_{xy}, and κ_{yy}. For small shear rates, these coefficients behave as

$$\eta^*\approx 1+8\delta-\frac{18}{5}\left(1+\frac{958}{5}\delta\right)a^2, \tag{51}$$

$$\Psi_1^*\approx -\frac{14}{5}\left(1+\frac{1516}{35}\delta\right)+\frac{792}{25}\left(1+\frac{245\,083}{550}\delta\right)a^2, \tag{52}$$

$$\Psi_2^*\approx \frac{4}{5}\left(1+\frac{358}{5}\delta\right)-\frac{288}{25}\left(1+\frac{65\,179}{100}\delta\right)a^2, \tag{53}$$

$$\kappa_{xy}^*\approx -a[7+2(67\epsilon+81)\delta]+\frac{a^3}{5}\left[468+6\left(6090\epsilon+\frac{168\,969}{5}+31\epsilon^{-2}\right)\delta\right], \tag{54}$$

$$\kappa_{yy}^*\approx \frac{5}{2}\left(1+\frac{58}{5}\delta\right)-\frac{81}{5}\left[1+\left(\frac{35\,296}{135}+\frac{1549}{81}\epsilon^{-2}\right)\delta\right]a^2. \tag{55}$$

The dependence on the reduced thermal gradient ϵ appearing in Eqs. (51)–(55) is not restricted to small shear rates. The five transport coefficients are independent of ϵ in the absence of gravity ($g^*=0$), but this is not true when $g^*\neq 0$. According to the results of Appendix B, $\mathsf{P}^{(1)}$ is independent of ϵ, while $q_x^{(1)}=\cdots\epsilon^2+\cdots\epsilon+\cdots\epsilon^{-1}$, $q_y^{(1)}=\cdots\epsilon+\cdots\epsilon^{-1}$, where the ellipses denote nonlinear functions of the shear rate. This behavior of the heat flux gives rise to an interesting effect. Let us consider the layer where the temperature reaches its maximum value ($\epsilon=0$). In the absence of gravity, the heat flux across that layer vanishes, i.e., $\mathbf{q}^*|_{g^*=0,\epsilon=0}=0$. On the other hand, the coupling between shear flow and gravity induces a nonzero heat flux in spite of the fact that the (local) thermal gradient is zero, i.e., $\mathbf{q}^*|_{g^*\neq 0,\epsilon=0}\neq 0$. More specifically, $q_x^*|_{\epsilon=0}=-\frac{186}{5}g^*a^3[1+\mathcal{O}(a^2)]$ and $q_y^*|_{\epsilon=0}=\frac{1549}{5}g^*a^2[1+\mathcal{O}(a^2)]$.

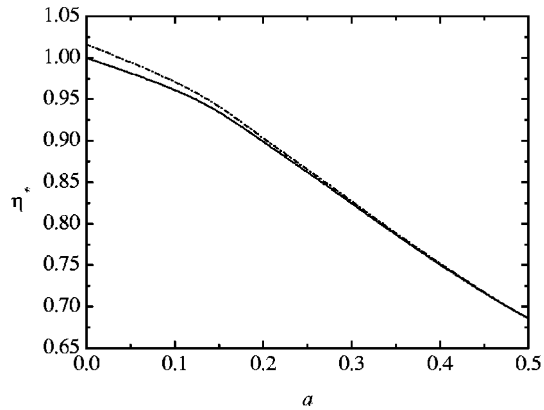

FIG. 1. Shear-rate dependence of the reduced shear viscosity η^* at $\epsilon=1$ and $g^*=0$ (solid line) and $g^*=0.002$ (dashed line).

Equations (51)–(53) and (55) show that, in the limit of zero shear rate, the gravitational field induces a decrease (increase) of the viscosity, the normal stress differences, and the thermal conductivity on those points where the thermal gradient is parallel (antiparallel) to the field. A subtler effect appears on the heat flux vector $\mathbf{q}$. In the absence of gravitation, the vector $\mathbf{q}$ rotates anticlockwise with respect to the direction of $-\nabla T$ a small angle $\theta\simeq\frac{14}{5}a$ for small shear rates. In the presence of gravitation, this angle becomes $\theta\simeq\frac{14}{5}a(1+\frac{404}{35}\epsilon g^*)$, i.e., it decreases (increases) if the thermal gradient (here assumed to be small) is parallel (antiparallel) to the field. All these effects become much more involved when the shear rate and/or the thermal conductivity are not small, as shown in the example given below.

IV. AN ILLUSTRATIVE EXAMPLE

Thus far, all the results are valid for arbitrary values of the reduced shear rate a and the reduced thermal gradient ϵ and to first order in the reduced gravitational field g^*. While, without loss of generality, a and g^* can be chosen as positive, the sign of ϵ denotes two distinct situations: $\epsilon<0$ ($\epsilon>0$) corresponds to points that are ''heated from below (above),'' i.e., points where the thermal gradient is parallel (antiparallel) to the field. For illustrative purposes, we fix $\epsilon=1$ and consider $g^*=0$ and $g^*=0.002$. Then we can get the five transport coefficients as nonlinear functions of the shear rate only. Figures 1–5 display η^*, Ψ_1^*, Ψ_2^*, κ_{xy}^*, and κ_{yy}^*, respectively, in the range $0\leqslant a\leqslant 0.5$. In general, we see that the presence of the field does not change the trends observed for the transport coefficients in the absence of gravitation. In particular, the shear viscosity η^* and the thermal conductivity κ_{yy}^* are smaller than their corresponding values at zero shear rate and they monotonically decrease as the shear rate increases. We also observe that, given a value of the shear rate, the gravitation increases the values of η^*, $-\Psi_1^*$, Ψ_2^*, and $-\kappa_{xy}^*$ if $\epsilon>0$. On the other hand, the rheological properties (shear viscosity and viscometric functions) with and without gravitational field tend to overlap as the shear rate increases so that the effect of the gravitation is practically negligible for large shear rates. This effect is especially sig-

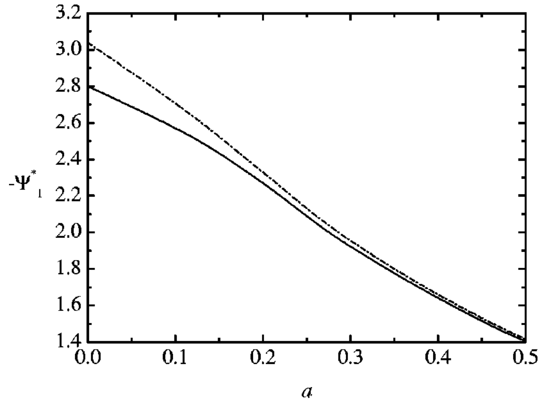

FIG. 2. Shear-rate dependence of the reduced first viscometric function $-\Psi_1^*$ at $\epsilon=1$ and $g^*=0$ (solid line) and $g^*=0.002$ (dashed line).

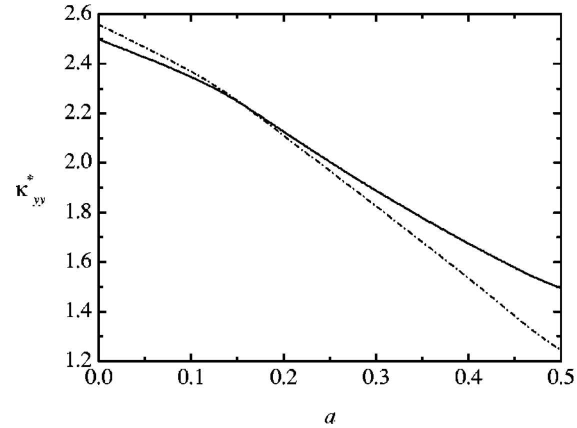

FIG. 4. Shear-rate dependence of the reduced diagonal thermal conductivity κ_{yy}^* at $\epsilon=1$ and $g^*=0$ (solid line) and $g^*=0.002$ (dashed line).

nificant in the case of the viscometric functions, where the influence of the field is quite remarkable for small shear rates.

The most interesting influence of the gravitational field on the transport properties appears in the case of the thermal conductivity coefficient κ_{yy}^*. In contrast to what happened in Figs. 1–3, the effect of the field is now more notorious as the shear rate becomes larger. In addition, while $\kappa_{yy}^*|_{\delta>0} > \kappa_{yy}^*|_{\delta=0}$ at small shear rates, the opposite happens for large shear rates. In particular, at $a=0\,\kappa_{yy}^*$ has increased by about a 2% with respect to its zero-field value, while it has decreased by a 17% at $a=0.5$. This crossover effect is an unexpected consequence of the highly nonlinear dependence of the transport coefficient κ_{yy}^* on the shear rate. Concerning the heat transport along the flow direction, we observe that the magnitude of the associated off-diagonal element κ_{xy}^* increases with a, its value being larger with than without gravity. From Figs. 4 and 5 it follows that when going from $g^*=0$ to $g^*=0.002$ at $a=0.5$ and $\epsilon=1$, the heat flux vector decreases its magnitude by a 7.5% and rotates from an angle $\theta\simeq 45°$ with the $-\hat{\mathbf{y}}$ direction to $\theta\simeq 50°$.

Given that our results are restricted to the first order in the field, it must be pointed out that the dashed curves in Figs. 1–5 do not intend to represent strictly the complete behavior at $\delta=0.002$, since higher order terms could also be important, especially in the cases of the viscometric functions and of the diagonal thermal conductivity coefficient. The main objective of Figs. 1–5 was to highlight the trends to be expected in the presence of nonzero gravity.

V. DISCUSSION

The aim of this paper has been to analyze the influence of gravitation on the transport properties of a dilute gas far from equilibrium. Specifically, we have considered the steady planar Couette flow described by the BGK model kinetic equation. In the absence of gravity, the exact solution of the problem is known for arbitrary values of the local shear rate and thermal gradient. Taking this solution as a reference state, we have performed a perturbation expansion in powers of the external field, the successive coefficients being nonlinear functions of both gradients. Here we have restricted ourselves to the first-order correction, although the method could be extended to higher orders. On the other hand, not only the algebraic complexity increases dramatically as one considers higher-order approximations, but for practical purposes the first-order approximation should be sufficient.

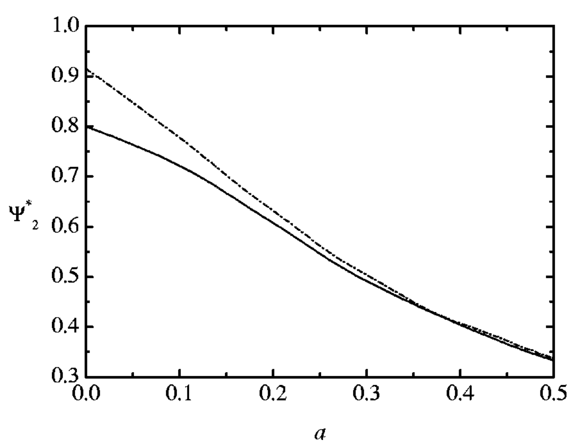

FIG. 3. Shear-rate dependence of the reduced second viscometric function Ψ_2^* at $\epsilon=1$ and $g^*=0$ (solid line) and $g^*=0.002$ (dashed line).

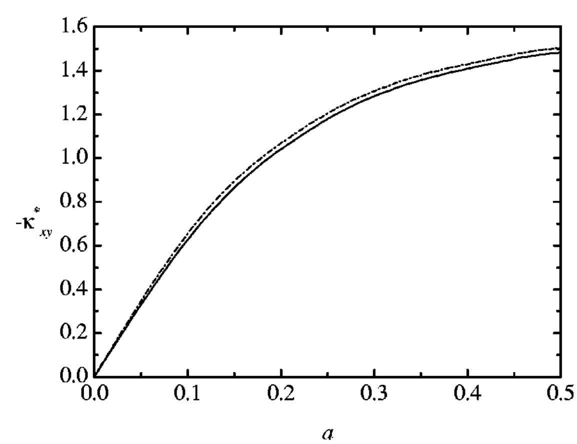

FIG. 5. Shear-rate dependence of the reduced off-diagonal thermal conductivity $-\kappa_{xy}^*$ at $\epsilon=1$ and $g^*=0$ (solid line) and $g^*=0.002$ (dashed line).

In the pure Couette flow $(g=0)$, the solution is characterized by a uniform pressure, a linear velocity profile, and a quadratic temperature profile, the latter two with respect to a conveniently scaled space variable. The results reported here show that, to first order in g, the pressure, the velocity, and the temperature become linear, quadratic, and cubic functions, respectively. The most important quantities of the problem are the five independent transport coefficients, measuring the heat and momentum fluxes across the system. Apart from generalizations of the shear viscosity, η, and the thermal conductivity, κ_{yy}, coefficients, one can define the viscometric functions Ψ_1 and Ψ_2, measuring normal stresses, and a new coefficient κ_{xy} that measures a component of the heat flux parallel to the flow and orthogonal to the temperature gradient. All of these coefficients have been explicitly obtained in terms of the shear rate and the thermal gradient. In the case of the rheological properties $(\eta, \Psi_{1,2})$, we have found that the influence of gravity tends to decrease as the shear rate increases, i.e., in the non-Newtonian regime. The quantity where the effect of gravity is the largest one is the thermal conductivity κ_{yy}. In addition, $\kappa_{yy}|_{g\neq 0}-\kappa_{yy}|_{g=0}$ changes sign as one goes from small to large shear rates.

Although the results reported in this paper have been obtained from the BGK model, we expect that most of the main trends will also be present in a more detailed description in the framework of the Boltzmann equation. In fact, this is what happens in the pure Couette flow[9] and in the Fourier flow under gravitation.[2] A more quantitative agreement could be expected if one conveniently defines reduced units in order to get the correct Prandtl number. This strategy has proven to be useful in the pure Couette flow[14] and in the Poiseuille flow.[19] More specifically, we propose to define $a=(\eta_{\rm NS}/p)\partial u_x/\partial y$, $\epsilon=(\sqrt{k_BT/m}\,\kappa_{\rm NS}/pc_p)\partial\ln T/\partial y$, and $g^*=(\eta_{\rm NS}/p\sqrt{k_BT/m})g$, where $c_p=\frac{5}{2}k_B/m$ and $\eta_{\rm NS}$ and $\kappa_{\rm NS}$ are, respectively, the correct Navier–Stokes shear viscosity and thermal conductivity coefficients. Thus, we conjecture that a good approximation to the nonlinear Boltzmann transport coefficients (to first order in g) could be obtained by making $\eta\rightarrow\eta_{\rm NS}\eta^*(a,\epsilon,g^*)$, $\Psi_i\rightarrow(\eta_{\rm NS}^2/p)\Psi_i^*(a,\epsilon,g^*)$, and $\kappa_{ij}\rightarrow\frac{2}{5}\kappa_{\rm NS}\kappa_{ij}^*(a,\epsilon,g^*)$, where the dimensionless transport coefficients are those explicitly derived here from the BGK model.

ACKNOWLEDGMENTS

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APPENDIX A: CONSISTENCY CONDITIONS FOR THE HYDRODYNAMIC FIELDS

In this Appendix we prove the consistency between the hydrodynamic profiles (42)–(44) and the solution given by Eq. (41). To this end, it is convenient to rewrite the formal solution (41) as

$$f^{(1)}-f_L^{(1)}=\sum_{k=1}^{\infty}(-\partial_s)^kV_y^kf_L^{(1)}+\epsilon^{-1}\sum_{k=0}^{\infty}(-\partial_s)^kT^{(0)}V_y^k\frac{\partial}{\partial V_i}f^{(0)}\equiv\Lambda^{\rm I}+\epsilon^{-1}\Lambda^{\rm II}. \tag{A1}$$

Let us introduce the integrals

$$\begin{aligned}\Phi^{\rm I}_{n_1n_2n_3}&=\int d\mathbf{V}V_x^{n_1}V_y^{n_2}V_z^{n_3}\Lambda^{\rm I}\\&=\sum_{k=1}^{\infty}(-\partial_s)^k\int d\mathbf{V}V_x^{n_1}V_y^{n_2+k}V_z^{n_3}f_L^{(1)}\\&\quad+\sum_{l=1}^{n_1}\sum_{k=l}^{\infty}\binom{k}{l}\frac{n_1!}{(n_1-l)!}(-a)^l(-\partial_s)^{k-l}\\&\quad\times\int d\mathbf{V}V_x^{n_1-l}V_y^{n_2+k}V_z^{n_3}f_L^{(1)},\end{aligned} \tag{A2}$$

where in the last step we have made use of the identities

$$A(-\partial_s)^kB=\sum_{l=0}^{k}\binom{k}{l}(-\partial_s)^{k-l}(\partial_s^lA)B, \tag{A3}$$

$$\partial_s^lV_x^{n_1}=\frac{n_1!}{(n_1-l)!}(-a)^lV_x^{n_1-l}. \tag{A4}$$

By taking into account that

$$\int d\mathbf{V}V_x^{n_1}V_y^{n_2}V_z^{n_3}f_L^{(0)}=K_{n_1}K_{n_2}K_{n_3}T^{(0)(n_1+n_2+n_3-2)/2}, \tag{A5}$$

where $K_n=(n-1)!!$ if $n=$even (with the convention $(-1)!!=1$), being $K_n=0$ otherwise, Eq. (39) yields

$$\begin{aligned}\int d\mathbf{V}V_x^{n_1}V_y^{n_2}V_z^{n_3}f_L^{(1)}&=\left[p^{(1)}+\frac{n_1+n_2+n_3-2}{2}\frac{T^{(1)}}{T^{(0)}}\right]\\&\quad\times K_{n_1}K_{n_2}K_{n_3}T^{(0)(n_1+n_2+n_3-2)/2}\\&\quad+u_x^{(1)}K_{n_1+1}K_{n_2}K_{n_3}T^{(0)(n_1+n_2+n_3-3)/2}.\end{aligned} \tag{A6}$$

The integrals of this type contributing to Eq. (A2) are of the form $n_1+n_2+n_3\geqslant 2$ with $n_1=$even and of the form $n_1+n_2+n_3\geqslant 3$ with $n_1=$odd. Thus, Eqs. (42)–(44) imply that the right-hand side of Eq. (A6) is a polynomial in s of degree $n_1+n_2+n_3-1$. Consequently, $\Phi^{\rm I}_{n_1n_2n_3}$ is also a polynomial of degree $n_1+n_2+n_3-1$. In particular,

$$\Phi^{\rm I}_{000}=\Phi^{\rm I}_{001}=0, \tag{A7}$$

$$\Phi_{100}^{\mathrm{I}}=\sum_{k=1}^{\infty}(2k-1)!!\,\partial_s^{2k}u_x^{(1)}T^{(0)k-1}+2a\sum_{k=1}^{\infty}k(2k-1)!!\,\partial_s^{2k-1}\left[p^{(1)}+(k-1)\frac{T^{(1)}}{T^{(0)}}\right]T^{(0)k-1}$$

$$=\beta\sum_{k=1}^{\infty}(2k)!(2k-1)!!(-\gamma_0)^{k-1}+2a\sum_{k=1}^{\infty}k(2k-1)!(2k-1)!![\alpha(-\gamma_0)^{k-1}+(k-1)\gamma_2(-\gamma_0)^{k-2}], \tag{A8}$$

$$\Phi_{010}^{\mathrm{I}}=-\sum_{k=0}^{\infty}(2k+1)!!\,\partial_s^{2k+1}\left[p^{(1)}+k\frac{T^{(1)}}{T^{(0)}}\right]T^{(0)k}$$

$$=-\sum_{k=0}^{\infty}(2k+1)!(2k+1)!![\alpha(-\gamma_0)^{k}+k\gamma_2(-\gamma_0)^{k-1}], \tag{A9}$$

$$\Phi_{200}^{\mathrm{I}}=\sum_{k=1}^{\infty}(2k-1)!!\,\partial_s^{2k}\left[p^{(1)}+k\frac{T^{(1)}}{T^{(0)}}\right]T^{(0)k}+4a\sum_{k=1}^{\infty}k(2k-1)!!\,\partial_s^{2k-1}u_x^{(1)}T^{(0)k-1}$$

$$+2a^2\sum_{k=1}^{\infty}k(2k-1)(2k-1)!!\,\partial_s^{2k-2}\left[p^{(1)}+(k-1)\frac{T^{(1)}}{T^{(0)}}\right]T^{(0)k-1}$$

$$=\sum_{k=1}^{\infty}(2k)!(2k-1)!!\{(2k+1)[\gamma_2k(-\gamma_0)^{k-1}+\alpha(-\gamma_0)^{k}]s+k[(\gamma_1+\epsilon\alpha)(-\gamma_0)^{k-1}+\epsilon\gamma_2(k-1)(-\gamma_0)^{k-2}]\}$$

$$+4a\beta\sum_{k=1}^{\infty}k(2k-1)!(2k-1)!![2k(-\gamma_0)^{k-1}s+(k-1)\epsilon(-\gamma_0)^{k-2}]$$

$$+2a^2\sum_{k=1}^{\infty}k(2k-1)(2k-2)!(2k-1)!!\{(2k-1)[\gamma_2(k-1)(-\gamma_0)^{k-2}+\alpha(-\gamma_0)^{k-1}]s$$

$$+(k-1)[(\gamma_1+\epsilon\alpha)(-\gamma_0)^{k-2}+\epsilon\gamma_2(k-2)(-\gamma_0)^{k-3}]\}, \tag{A10}$$

$$\Phi_{020}^{\mathrm{I}}=\sum_{k=1}^{\infty}(2k+1)!!\,\partial_s^{2k}\left[p^{(1)}+k\frac{T^{(1)}}{T^{(0)}}\right]T^{(0)k}$$

$$=\sum_{k=1}^{\infty}(2k)!(2k+1)!!\{(2k+1)[\gamma_2k(-\gamma_0)^{k-1}+\alpha(-\gamma_0)^{k}]s+k[(\gamma_1+\epsilon\alpha)(-\gamma_0)^{k-1}+\epsilon\gamma_2(k-1)(-\gamma_0)^{k-2}]\}, \tag{A11}$$

$$\Phi_{002}^{\mathrm{I}}=\sum_{k=1}^{\infty}(2k-1)!!\,\partial_s^{2k}\left[p^{(1)}+k\frac{T^{(1)}}{T^{(0)}}\right]T^{(0)k}$$

$$=\sum_{k=1}^{\infty}(2k)!(2k-1)!!\{(2k+1)[\gamma_2k(-\gamma_0)^{k-1}+\alpha(-\gamma_0)^{k}]s+k[(\gamma_1+\epsilon\alpha)(-\gamma_0)^{k-1}+\epsilon\gamma_2(k-1)(-\gamma_0)^{k-2}]\}. \tag{A12}$$

The series expansions appearing in Eqs. (A8)–(A12) are asymptotic. They can be expressed in terms of the functions $F_r(\gamma_0)$ defined by Eqs. (23) and (24) by taking into account their asymptotic expansions[10]

$$F_r\equiv F_r(\gamma_0)=\sum_{k=0}^{\infty}(k+1)^r(2k+1)!(2k+1)!!(-\gamma_0)^k. \tag{A13}$$

It must be noticed that the functions $\{F_r, r\geqslant 3\}$ can be easily expressed in terms of F_0, F_1, and F_2 as

$$F_3=\frac{1}{8\gamma_0}(1-F_0)-F_2-\frac{1}{4}F_1, \tag{A14}$$

$$F_r=\frac{1}{8\gamma_0}\sum_{m=0}^{r-3}\binom{r-3}{m}(-1)^{m+r}F_m-F_{r-1}-\frac{1}{4}F_{r-2},\quad r\geqslant 4. \tag{A15}$$

Making use of Eq. (A13), one can rewrite Eqs. (A8)–(A12) as

$$\Phi_{100}^{\mathrm{I}}=2(\beta+a\alpha)F_1-2a\frac{\gamma_2}{\gamma_0}(F_2-F_1), \tag{A16}$$

$$\Phi^{\rm I}_{010}=-\alpha F_0+\frac{\gamma_2}{\gamma_0}(F_1-F_0), \tag{A17}$$

$$\begin{aligned}\Phi^{\rm I}_{200}=&2[\gamma_2(2F_3+F_2)-\alpha\gamma_0(2F_2+F_1)]s+2(\gamma_1+\epsilon\alpha)F_2\\&+4\epsilon\gamma_2(4F_6+12F_5+13F_4+6F_3+F_2)\\&+8a\beta[F_2s+\epsilon(4F_5+8F_4+5F_3+F_2)]\\&+2a^2\left\{\left[\alpha(2F_2-F_1)-\frac{\gamma_2}{\gamma_0}(2F_3-3F_2+F_1)\right]s\right.\\&\left.-\frac{\gamma_1+\epsilon\alpha}{\gamma_0}(F_2-F_1)+\frac{\gamma_2}{\gamma_0^2}\epsilon(F_3-3F_2+2F_1)\right\},\end{aligned} \tag{A18}$$

$$\begin{aligned}\Phi^{\rm I}_{020}=&2[\gamma_2(4F_4+4F_3+F_2)-\alpha\gamma_0(4F_3+4F_2+F_1)]s\\&+2(\gamma_1+\epsilon\alpha)(2F_3+F_2)+4\epsilon\gamma_2(8F_7+36F_6+62F_5\\&+51F_4+20F_3+3F_2),\end{aligned} \tag{A19}$$

$$\begin{aligned}\Phi^{\rm I}_{002}=&2[\gamma_2(2F_3+F_2)-\alpha\gamma_0(2F_2+F_1)]s+2(\gamma_1+\epsilon\alpha)F_2\\&+4\epsilon\gamma_2(4F_6+12F_5+13F_4+6F_3+F_2).\end{aligned} \tag{A20}$$

Now let us consider the term $\Lambda^{\rm II}$ defined in the last line of Eq. (A1). By taking into account that $T^{(0)}$ is a quadratic function of s and considering an identity similar to (A3), we can carry out the decomposition

$$\Lambda^{\rm II}=T^{(0)}\Lambda^{\rm II,0}-(\partial_sT^{(0)})\Lambda^{\rm II,1}+(\partial_s^2T^{(0)})\Lambda^{\rm II,2}, \tag{A21}$$

where

$$\Lambda^{{\rm II},m}=\sum_{k=0}^{\infty}\binom{k+m}{m}(-\partial_s)^kV_y^{k+m}\frac{\partial}{\partial V_y}f^{(0)}. \tag{A22}$$

To proceed, we need the expression of the distribution function in the absence of gravitation, $f^{(0)}$. While its explicit form is known,[11] here it is more convenient to make use of the formal solution[10]

$$f^{(0)}=\sum_{l=0}^{\infty}(-\partial_s)^lV_y^lf_L^{(0)}. \tag{A23}$$

Substitution into Eq. (A22) yields

$$\begin{aligned}\Lambda^{{\rm II},m}=&\sum_{k=0}^{\infty}\binom{k+m+1}{m+1}\left(\frac{\partial}{\partial V_y}V_y-m-\frac{m+1}{m+2}k\right)\\&\times V_y^{k+m-1}(-\partial_s)^kf_L^{(0)},\end{aligned} \tag{A24}$$

where we have made use of the identities

$$\sum_{l=0}^{k}\binom{k-l+m}{m}=\binom{k+m+1}{m+1}, \tag{A25}$$

$$\sum_{l=0}^{k}\binom{k-l+m}{m}(k-l+m)=\binom{k+m+1}{m+1}\left(m+\frac{m+1}{m+2}k\right). \tag{A26}$$

In the same way as in Eq. (A2), we define the integrals

$$\begin{aligned}\Phi^{{\rm II},m}_{n_1n_2n_3}=&\int d\mathbf{V}V_x^{n_1}V_y^{n_2}V_z^{n_3}\Lambda^{{\rm II},m}\\=&-\sum_{l=0}^{n_1}\sum_{k=l}^{\infty}\binom{k}{l}\binom{k+m+1}{m+1}\frac{n_1!}{(n_1-l)!}\\&\times\left(n_2+m+\frac{m+1}{m+2}k\right)(-a)^l(-\partial_s)^{k-l}\\&\times\int d\mathbf{V}V_x^{n_1-l}V_y^{n_2+k+m-1}V_z^{n_3}f_L^{(0)},\end{aligned} \tag{A27}$$

where in the last step we have used again Eqs. (A3) and (A4). The evaluation of $\Phi^{{\rm II},m}_{n_1n_2n_3}$ is now straightforward by using Eq. (A5) and it is easy to see that $\Phi^{{\rm II},m}_{n_1n_2n_3}$ is in general a polynomial of degree $n_1+n_2+n_3+m-3$. The most important integrals are

$$\Phi^{\rm II,0}_{000}=-\partial_sT^{(0)-1},\quad \Phi^{\rm II,1}_{000}=T^{(0)-1},\quad \Phi^{\rm II,2}_{000}=0, \tag{A28}$$

$$\Phi^{\rm II,0}_{100}=aT^{(0)-1},\quad \Phi^{\rm II,1}_{100}=0, \tag{A29}$$

$$\Phi^{\rm II,2}_{100}=\frac{a}{24}\sum_{k=0}^{\infty}(6k+11)(2k+4)!(2k+1)!!(-\gamma_0)^k, \tag{A30}$$

$$\Phi^{\rm II,0}_{010}=-T^{(0)-1},\quad \Phi^{\rm II,1}_{010}=0, \tag{A31}$$

$$\Phi^{\rm II,2}_{010}=-\frac{1}{8}\sum_{k=0}^{\infty}(2k+4)!(2k+1)!!(-\gamma_0)^k, \tag{A32}$$

$$\Phi^{\rm II,0}_{001}=\Phi^{\rm II,1}_{001}=\Phi^{\rm II,2}_{001}=0, \tag{A33}$$

$$\Phi^{\rm II,0}_{200}=0, \tag{A34}$$

$$\begin{aligned}\Phi^{\rm II,1}_{200}=&-\frac{1}{3}\sum_{k=0}^{\infty}(4k+3)(k+1)(2k)!(2k+1)!!(-\gamma_0)^k\\&-\frac{a^2}{6}\sum_{k=0}^{\infty}(4k+7)(2k+4)!(2k+1)!!(-\gamma_0)^k,\end{aligned} \tag{A35}$$

$$\begin{aligned}\Phi^{\rm II,2}_{200}=&\frac{1}{24}(\partial_sT^{(0)})\sum_{k=0}^{\infty}(6k+11)(k+1)(2k+4)!\\&\times(2k+1)!!(-\gamma_0)^k+\frac{a^2}{24}(\partial_sT^{(0)})\sum_{k=0}^{\infty}(6k+17)\\&\times(k+1)(2k+6)!(2k+3)!!(-\gamma_0)^k,\end{aligned} \tag{A36}$$

$$\Phi^{\rm II,0}_{020}=0, \tag{A37}$$

$$\Phi^{\rm II,1}_{020}=-\frac{1}{6}\sum_{k=0}^{\infty}(4k+9)(2k+2)!(2k+1)!!(-\gamma_0)^k, \tag{A38}$$

$$\begin{aligned}\Phi^{\rm II,2}_{020}=&\frac{1}{24}(\partial_sT^{(0)})\sum_{k=0}^{\infty}(6k+19)(k+1)(2k+4)!\\&\times(2k+3)!!(-\gamma_0)^k,\end{aligned} \tag{A39}$$

$$\Phi^{\rm II,0}_{002}=0, \tag{A40}$$

$$\Phi_{002}^{\mathrm{II},1}=-\frac{1}{3}\sum_{k=0}^{\infty}(4k+3)(k+1)(2k)!(2k+1)!!(-\gamma_0)^k, \tag{A41}$$

$$\Phi_{002}^{\mathrm{II},2}=\frac{1}{24}(\partial_s T^{(0)})\sum_{k=0}^{\infty}(6k+11)(k+1)(2k+4)! \times(2k+1)!!(-\gamma_0)^k. \tag{A42}$$

If we define

$$\Phi_{n_1n_2n_3}^{\mathrm{II}}=\int d\mathbf{V}V_x^{n_1}V_y^{n_2}V_z^{n_3}\Lambda^{\mathrm{II}} =T^{(0)}\Phi_{n_1n_2n_3}^{\mathrm{II},0}-(\partial_s T^{(0)})\Phi_{n_1n_2n_3}^{\mathrm{II},1} +(\partial_s^2 T^{(0)})\Phi_{n_1n_2n_3}^{\mathrm{II},2}, \tag{A43}$$

we easily get

$$\Phi_{000}^{\mathrm{II}}=\Phi_{001}^{\mathrm{II}}=0, \tag{A44}$$

$$\Phi_{100}^{\mathrm{II}}=a\left[1-\frac{\gamma_0}{3}(12F_4+28F_3+21F_2+5F_1)\right], \tag{A45}$$

$$\Phi_{010}^{\mathrm{II}}=-1+\gamma_0(2F_3+3F_2+F_1), \tag{A46}$$

$$\Phi_{200}^{\mathrm{II}}=(\epsilon-2\gamma_0 s)\left[1-\frac{\gamma_0}{3}(12F_5+44F_4+57F_3+31F_2 +6F_1)+\frac{a^2}{3}(16F_4+36F_3+26F_2+6F_1) -\frac{2}{3}a^2\gamma_0(48F_8+352F_7+1024F_6+1500F_5 +1157F_4+443F_3+66F_2)\right], \tag{A47}$$

$$\Phi_{020}^{\mathrm{II}}=(\epsilon-2\gamma_0 s)\left[\frac{4}{3}F_2+\frac{5}{3}F_1-\frac{\gamma_0}{3} \times(24F_6+100F_5+134F_4+71F_3+13F_2)\right], \tag{A48}$$

$$\Phi_{002}^{\mathrm{II}}=(\epsilon-2\gamma_0 s)\left[1-\frac{\gamma_0}{3} \times(12F_5+44F_4+57F_3+31F_2+6F_1)\right]. \tag{A49}$$

Once the integrals $\Phi_{n_1n_2n_3}^{\mathrm{II}}$ and $\Phi_{n_1n_2n_3}^{\mathrm{II}}$ associated with the collisional invariants have been expressed in terms of ϵ, a, $\gamma_0(a)$, α, β, γ_1, and γ_2, the consistency conditions, Eq. (45), imply

$$\Phi_{100}^{\mathrm{I}}+\epsilon^{-1}\Phi_{100}^{\mathrm{II}}=0, \tag{A50}$$

$$\Phi_{010}^{\mathrm{I}}+\epsilon^{-1}\Phi_{010}^{\mathrm{II}}=0, \tag{A51}$$

$$\Phi_{200}^{\mathrm{I}}+\Phi_{020}^{\mathrm{I}}+\Phi_{002}^{\mathrm{I}}+\epsilon^{-1}(\Phi_{200}^{\mathrm{II}}+\Phi_{020}^{\mathrm{II}}+\Phi_{002}^{\mathrm{II}})=0. \tag{A52}$$

Since the left-hand side of Eq. (A52) is a linear function of s, Eqs. (A50)–(A52) constitute a closed set of four linear equations for the unknowns α, β, γ_1, and γ_2. The solution for γ_2 is

$$\gamma_2=-\frac{\gamma_0}{3\epsilon}\frac{R}{S}, \tag{A53}$$

where we have introduced the coefficients

$$R=a^2\{2F_0F_2[\gamma_0(12F_4+28F_3+21F_2+5F_1)-3] -3F_1(2F_2+F_1)[\gamma_0(2F_3+3F_2+F_1)-1]\} -3F_1\gamma_0\{F_0\lambda-[1-\gamma_0(2F_3+3F_2+F_1)] \times(4F_3+8F_2+3F_1)\}, \tag{A54}$$

$$S=a^2\{F_0[F_1(F_2-2F_3)+4F_2^2)]-F_1^2(2F_2+F_1)\} +F_1\gamma_0\{F_0[4(F_4+3F_3)+11F_2+3F_1)] -F_1(4F_3+8F_2+3F_1)\}, \tag{A55}$$

with

$$\lambda=2+\frac{4}{3}F_2+\frac{5}{3}F_1-\frac{\gamma_0}{3}[24F_6+124F_5+222F_4 +185F_3+75F_2+12F_1]+\frac{2}{3}a^2(8F_4+18F_3+13F_2 +3F_1)-\frac{2}{3}a^2\gamma_0(48F_8+352F_7+1024F_6+1500F_5 +1157F_4+443F_3+66F_2). \tag{A56}$$

The remaining parameters are

$$\alpha=\frac{1}{\epsilon F_0}\left[\gamma_0(2F_3+3F_2+F_1)+\epsilon\frac{\gamma_2}{\gamma_0}(F_1-F_0)-1\right], \tag{A57}$$

$$\beta=-a\left[\alpha-\frac{\gamma_2}{\gamma_0}\frac{F_2-F_1}{F_1}-\frac{\gamma_0}{6\epsilon}\frac{12F_4+28F_3+21F_2+5F_1}{F_1} +\frac{1}{2\epsilon F_1}\right], \tag{A58}$$

$$\gamma_1=-\alpha\epsilon+\frac{1}{2a^2(F_2-F_1)/\gamma_0-4F_3-6F_2} \times\left[\lambda+4\epsilon\gamma_2(8F_7+44F_6+86F_5+77F_4+32F_3+5F_2) +8a\beta\epsilon(4F_5+8F_4+5F_3+F_2) +2a^2\epsilon\frac{\gamma_2}{\gamma_0^2}(F_3-3F_2+2F_1)\right]. \tag{A59}$$

Equations (A53)–(A59) show the highly nonlinear dependence of the profiles on the shear rate. On the other hand, the dependence on the reduced thermal gradient ϵ is quite simple; α, β, and γ_2 are inversely proportional to ϵ, while γ_1 does not depend on ϵ.

APPENDIX B: CALCULATION OF THE FLUXES

This appendix is concerned with the evaluation of the momentum and heat fluxes at the (arbitrary) point of interest $s=0$.

Let us start with the xy element of the pressure tensor. In the first order approximation, it is defined as

$$P_{xy}^{(1)}|_{s=0}=\int d\mathbf{V}V_xV_yf^{(1)}=\Phi_{110}^{\mathrm{I}}|_{s=0}+\epsilon^{-1}\Phi_{110}^{\mathrm{II}}|_{s=0}. \tag{B1}$$

From Eqs. (A2) and (A6) we have

$$\begin{aligned}\Phi_{110}^{\mathrm{I}}|_{s=0}=&-\beta\epsilon\sum_{k=0}^{\infty}k(2k+1)!(2k+1)!!(-\gamma_0)^{k-1}\\&-a\sum_{k=0}^{\infty}k(2k+1)!(2k+1)!![(\gamma_1+\alpha\epsilon)\\&\times(-\gamma_0)^{k-1}+(k-1)\epsilon\gamma_2(-\gamma_0)^{k-2}]\\=&\frac{\epsilon(\beta+a\alpha)+a\gamma_1}{\gamma_0}(F_1-F_0)\\&-a\epsilon\frac{\gamma_2}{\gamma_0^2}(F_2-3F_1+2F_0).\end{aligned} \tag{B2}$$

Next, by taking into account Eq. (A27), we get

$$\begin{aligned}\Phi_{110}^{\mathrm{II}}|_{s=0}=&-\frac{1}{3}a\epsilon\sum_{k=0}^{\infty}(2k+4)!(2k+1)!!(-\gamma_0)^k\\&+\frac{1}{4}a\epsilon\gamma_0\sum_{k=0}^{\infty}(k+1)(2k+6)!(2k+3)!!(-\gamma_0)^k\\=&a\epsilon\Big[2\gamma_0(8F_7+44F_6+90F_5+85F_4+37F_3\\&+6F_2)-\frac{4}{3}(2F_3+3F_2+F_1)\Big].\end{aligned} \tag{B3}$$

Inserting Eqs. (B2) and (B3) into Eq. (B1), one finally gets

$$\begin{aligned}P_{xy}^{(1)}|_{s=0}=&\frac{\beta\epsilon+a(\alpha\epsilon+\gamma_1)}{\gamma_0}(F_1-F_0)\\&-a\epsilon\frac{\gamma_2}{\gamma_0^2}(F_2-3F_1+2F_0)\\&+a\Big[2\gamma_0(8F_7+44F_6+90F_5+85F_4+37F_3\\&+6F_2)-\frac{4}{3}(2F_3+3F_2+F_1)\Big].\end{aligned} \tag{B4}$$

The diagonal elements of the pressure tensor have already been evaluated in Appendix A,

$$\begin{aligned}P_{yy}^{(1)}|_{s=0}=&\Phi_{020}^{\mathrm{I}}|_{s=0}+\epsilon^{-1}\Phi_{020}^{\mathrm{II}}|_{s=0}\\=&2(\alpha\epsilon+\gamma_1)(2F_3+F_2)+4\gamma_2\epsilon(8F_7+36F_6\\&+62F_5+51F_4+20F_3+3F_2)\\&+\tfrac{1}{3}(4F_2+5F_1)-\tfrac{1}{3}\gamma_0(24F_6+100F_5+134F_4\\&+71F_3+13F_2),\end{aligned} \tag{B5}$$

$$\begin{aligned}P_{zz}^{(1)}|_{s=0}=&\Phi_{002}^{\mathrm{I}}|_{s=0}+\epsilon^{-1}\Phi_{002}^{\mathrm{II}}|_{s=0}\\=&2(\alpha\epsilon+\gamma_1)F_2+4\gamma_2\epsilon(4F_6+12F_5+13F_4\\&+6F_3+F_2)+1-\tfrac{1}{3}\gamma_0(12F_5+44F_4+57F_3\\&+31F_2+6F_1),\end{aligned} \tag{B6}$$

$$P_{xx}^{(1)}|_{s=0}=-P_{yy}^{(1)}|_{s=0}-P_{zz}^{(1)}|_{s=0}. \tag{B7}$$

The heat flux vector is defined at this order as

$$\mathbf{q}^{(1)}=\frac{1}{2}\int d\mathbf{V}V^2\mathbf{V}f^{(1)}. \tag{B8}$$

In the same way as in the pure Couette flow problem, the two nonzero components of the heat flux are $q_x^{(1)}$ and $q_y^{(1)}$. They require the evaluation of $\Phi_{n_1n_2n_3}^{\mathrm{I,II}}$ with $n_1+n_2+n_3=3$, which can be obtained from Eqs. (A2) and (A27). After rather tedious algebra, one obtains the expressions

$$\begin{aligned}q_y^{(1)}|_{s=0}=&-\frac{\alpha}{2}(5+18a^2)-18\gamma_2(7+100a^2)-\beta a\left[2+\frac{\epsilon^2}{2\gamma_0^2}(F_2-3F_1+2F_0)\right]+\frac{\alpha\epsilon^2+2\gamma_1\epsilon}{4\gamma_0}[2F_3+F_2-3F_1\\&+2a^2(4F_4-3F_2-F_1)]-\frac{\gamma_2\epsilon^2}{4\gamma_0^2}[2F_4-3F_3-5F_2+6F_1+2a^2(4F_5-8F_4-3F_3+5F_2+2F_1)]\\&-\frac{3+2F_1}{2\epsilon}+\frac{\gamma_0}{\epsilon}(2F_3+5F_2+3F_1+17)-\frac{a^2}{\epsilon}\left(F_3+\frac{3}{2}F_2+\frac{1}{2}F_1-270\right)-\epsilon\left(F_6+6F_5+\frac{47}{4}F_4+\frac{25}{3}F_3+\frac{23}{12}F_2\right)\\&-a^2\epsilon\left(4F_8+26F_7+67F_6+\frac{175}{2}F_5+61F_4+\frac{43}{2}F_3+3F_2\right),\end{aligned} \tag{B9}$$

$$
\begin{aligned}
q_x^{(1)}|_{s=0}=&\frac{a\alpha}{\gamma_0}\Bigg\{\epsilon^2\left[a^2(F_2+4F_3+3F_4-4F_5-4F_6)+\frac{1}{2}(5F_2-3F_3-2F_4)\right]+\gamma_0[a^2(8F_5+16F_4+10F_3\\
&+2F_2)+2F_3+5F_2]\Bigg\}+\frac{\beta}{2\gamma_0^2}\{\epsilon^2[3a^2(F_3-3F_2+2F_1)+\gamma_0(5F_2-3F_3-2F_4)]+\gamma_0[6a^2(F_1-F_2)\\
&+2\gamma_0(2F_3+5F_2)]\}+\frac{a\epsilon\gamma_1}{\gamma_0}[2a^2(F_2+4F_3+3F_4-4F_5-4F_6)+5F_2-3F_3-2F_4]\\
&+\frac{a\gamma_2}{2\gamma_0^2}\{\epsilon^2[2a^2(2F_2+7F_3+2F_4-11F_5-4F_6+4F_7)+10F_2-11F_3-F_4+2F_5]\\
&+\gamma_0[2a^2(6F_4+8F_3+2F_2-8F_5-8F_6)+10F_2-6F_3-4F_4]\}+\frac{a}{2\epsilon}(2F_2+3F_1+2F_0)\\
&-\frac{a\gamma_0}{6\epsilon}(24F_6+148F_5+246F_4+155F_3+33F_2)+\frac{a^3}{\epsilon}(4F_4+8F_3+5F_2+F_1)-\frac{a^3\gamma_0}{3\epsilon}(48F_8+352F_7+1024F_6\\
&+1500F_5+1157F_4+443F_3+66F_2)-\epsilon\left[\frac{a}{12}(33F_2+122F_3+91F_4-98F_5-124F_6-24F_7)+\frac{a^3}{6}(66F_2\right.\\
&\left.+377F_3+714F_4+343F_5-476F_6-672F_7-304F_8-48F_9)\right]+\frac{\epsilon^2}{3}[a(16F_6+100F_5+164F_4+101F_3+21F_2)\\
&+a^3(64F_8+464F_7+1336F_6+1940F_5+1486F_4+566F_3+84F_2)]. \qquad \text{(B10)}
\end{aligned}
$$

[1] M. Mareschal, M. Malek Mansour, A. Puhl, and E. Kestemont, ''Molecular dynamics vs hydrodynamics in a two-dimensional Rayleigh–Bénard system,'' Phys. Rev. Lett. **61**, 2550 (1988); A. Puhl, M. Malek Mansour, and M. Mareschal, ''Quantitative comparison of molecular dynamics with hydrodynamics in Rayleigh–Bénard convection,'' Phys. Rev. A **40**, 1999 (1989); D. Risso and P. Cordero, ''Two-dimensional gas of disks: Thermal conductivity,'' J. Stat. Phys. **82**, 1453 (1996).

[2] M. Tij, V. Garzó, and A. Santos, ''Nonlinear heat transport in a dilute gas in the presence of gravitation,'' Phys. Rev. E **56**, 6729 (1997).

[3] E. S. Asmolov, N. K. Makashev, and V. I. Nosik, ''Heat transfer between parallel plates in a gas of Maxwellian molecules,'' Sov. Phys. Dokl. **24**, 892 (1979).

[4] A. Santos and V. Garzó, ''Exact nonlinear transport from the Boltzmann equation,'' in *Rarefied Gas Dynamics 19*, edited by J. Harvey and G. Lord (Oxford University Press, Oxford, 1995), Vol. 1, pp. 13–22.

[5] M. Tij, V. Garzó, and A. Santos, ''On the influence of gravity on the thermal conductivity,'' in Rarefied Gas Dynamics 21 (Cépaduès, Toulouse, 1999) (to be published).

[6] C. Trozzi and G. Ciccotti, ''Stationary nonequilibrium states by molecular dynamics. II. Newton's law,'' Phys. Rev. A **29**, 916 (1984).

[7] D. K. Bhattacharya and B. C. Eu, ''Nonlinear transport processes and fluid dynamics: Effects of thermoviscous coupling and nonlinear transport coefficients on plane Couette flow of Lennard-Jones fluids,'' Phys. Rev. A **35**, 821 (1987).

[8] D. Risso and P. Cordero, ''Dilute gas Couette flow: Theory and molecular dynamics simulation,'' Phys. Rev. E **56**, 489 (1997).

[9] M. Tij and A. Santos, ''Combined heat and momentum transport in a dilute gas,'' Phys. Fluids **7**, 2858 (1995).

[10] J. J. Brey, A. Santos, and J. W. Dufty, ''Heat and momentum transport far from equilibrium,'' Phys. Rev. A **36**, 2842 (1987).

[11] C. S. Kim, J. W. Dufty, A. Santos, and J. J. Brey, ''Analysis of nonlinear transport in Couette flow,'' Phys. Rev. A **40**, 7165 (1989).

[12] V. Garzó and M. López de Haro, ''Kinetic model for heat and momentum transport,'' Phys. Fluids **6**, 3787 (1994).

[13] V. Garzó and M. López de Haro, ''Nonlinear transport for a dilute gas in steady Couette flow,'' Phys. Fluids **9**, 776 (1997).

[14] J. M. Montanero and V. Garzó, ''Nonlinear Couette flow in a dilute gas: Comparison between theory and molecular dynamics simulation,'' Phys. Rev. E **58**, 1836 (1998).

[15] J. M. Montanero, V. Garzó, and A. Santos (unpublished).

[16] P. L. Bhatnagar, E. P. Gross, and M. Krook, ''A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems,'' Phys. Rev. **94**, 511 (1954).

[17] D. J. Tritton, *Physical Fluid Dynamics* (Oxford University Press, Oxford, 1988).

[18] J. W. Dufty, ''Kinetic Theory of Fluids Far From Equilibrium: Exact Results,'' in *Lectures on Thermodynamics and Statistical Mechanics*, edited by M. López de Haro and C. Varea (World Scientific, Singapore, 1990), pp. 166–181.

[19] M. Malek Mansour, F. Baras, and A. L. Garcia, ''On the validity of hydrodynamics in plane Poiseuille flows,'' Physica A **240**, 255 (1997).