

Hydrodynamics for a granular binary mixture at low density

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Hydrodynamic equations for a binary mixture of inelastic hard spheres are derived from the Boltzmann kinetic theory. A normal solution is obtained via the Chapman–Enskog method for states near the local homogeneous cooling state. The mass, heat, and momentum fluxes are determined to first order in the spatial gradients of the hydrodynamic fields, and the associated transport coefficients are identified. In the same way as for binary mixtures with elastic collisions, these coefficients are determined from a set of coupled linear integral equations. Practical evaluation is possible using a Sonine polynomial approximation, and is illustrated here by explicit calculation of the relevant transport coefficients: the mutual diffusion, the pressure diffusion, the thermal diffusion, the shear viscosity, the Dufour coefficient, the thermal conductivity, and the pressure energy coefficient. All these coefficients are given in terms of the restitution coefficients and the ratios of mass, concentration, and particle sizes. Interesting and new effects arise from the fact that the reference states for the two components have different partial temperatures, leading to additional dependencies of the transport coefficients on the concentration. The results hold for arbitrary degree of inelasticity and are not limited to specific values of the parameters of the mixture. Applications of this theory will be discussed in subsequent papers. © 2002 American Institute of Physics. [DOI: 10.1063/1.1458007]

I. INTRODUCTION

The qualitative properties of a granular gas whose dynamics is dominated by pairwise collisions between the grains can be described by a Boltzmann equation, suitably modified to describe the more complex two particle collisions. In the simplest model the grains are taken to be smooth, hard spheres with inelastic collisions. In recent years, the derivation of hydrodynamic equations for a one-component granular gas from this idealized Boltzmann description has been worked out in detail to Navier–Stokes order,¹ with explicit expressions for the transport coefficients as a function of the degree of dissipation (the restitution coefficient). These recent results improve upon earlier studies^{2–4} by providing expressions that are accurate even for strong dissipation. As a consequence, there are now precise predictions from the Boltzmann equation suitable for detailed comparison with Monte Carlo simulation, molecular dynamics simulation, and the evolving new class of controlled experiments. This analysis for the one-component system also has been extended to dense gases described by the Enskog equation.⁵ Similar studies for multicomponent granular gases are more scarce. Existing work on multicomponent transport appears to be based on weak dissipation approximations.^{6–9} Our objective is to provide a description of hydrodynamics in binary granular mixtures with a comparable accuracy to that for the one-component system, valid over the broadest parameter range including strong dissipa-

tion. The reference homogeneous cooling state for a binary mixture has been discussed in detail recently¹⁰ to provide the proper basis for analysis of transport due to spatial inhomogeneities. Those results are used here to describe mass, heat, and momentum transport. The expressions for the distribution functions, fluxes, and transport coefficients are exact to Navier–Stokes order (within the context of the Boltzmann equation).

The hydrodynamic equations for a binary mixture at low density are derived from the coupled set of Boltzmann equations for the two species in the same manner as for a one-component system. The solutions for the distribution functions are expanded about a local homogeneous cooling state that is analogous to the local equilibrium state for a gas with elastic collisions. The expansion is in powers of the spatial gradients of the hydrodynamic fields (e.g., species densities, temperature, and flow velocity) and is an extension of the familiar Chapman–Enskog procedure for elastic collisions. The primary technical complication for inelastic collisions is an inherent time dependence of the reference state due to collisional cooling. In a one-component system this occurs through the time dependence of the temperature defined in terms of the mean square velocity for the homogeneous cooling distribution. For a two-component system the temperature is defined in terms of the algebraic average of the mean square velocities for the two distributions. In the case of elastic collisions the average temperature is the same as the kinetic temperatures for each species in the local equilibrium state. However, a surprising result of the study in Ref. 10 is that these temperatures are all different for inelastic collisions. This does not mean that there are additional hydrody-

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dynamic degrees of freedom since their cooling rates are the same and consequently, the partial kinetic temperatures still can be expressed in terms of the average temperature. But the relationships between these temperatures are functions of the densities for the two species and lead to a new dependence of the reference cooling states on these hydrodynamic variables. The consequences of this effect for the transport coefficients are significant, as shown in the following.

The hydrodynamic equations for a binary mixture are somewhat more complicated than for the one-component case: six coupled equations with eight transport coefficients. The irreversible (dissipative) parts of the mass, heat, and momentum fluxes are calculated to leading order in the spatial gradients of the hydrodynamic fields. For systems with elastic collisions the specific set of gradients contributing to each flux is restricted by fluid symmetry, time reversal invariance (Onsager relations), and the form of the entropy production.¹¹ For inelastic collisions only fluid symmetry applies so there is greater freedom in representing the fluxes and identifying associated transport coefficients. This is discussed further in Sec. III where the independent gradients are chosen to be those for the concentration of species one, the pressure, the temperature, and the components of the flow field, with eight independent scalar transport coefficients. Using the Chapman–Enskog expansion the solutions to the Boltzmann equations are obtained to leading order in these gradients, and the transport coefficients are expressed in terms of the solutions to a set of coupled linear integral equations.

The plan of the paper is as follows. In Sec. II, the coupled set of Boltzmann equations and the corresponding hydrodynamic equations are recalled. The Chapman–Enskog expansion adapted to the inelastic binary mixtures is described in Sec. III to construct the distribution function to linear order in the gradients. This solution is used to calculate the fluxes and identify associated transport coefficients. A Sonine polynomial approximation is applied to solve the linear integral equations defining selected transport coefficients in Sec. IV. We get explicit expressions for these transport coefficients in terms of the restitution coefficients and the masses, concentrations, and sizes of the constituents of the mixture. Finally, the results are summarized and discussed in Sec. V.

II. BOLTZMANN EQUATION AND CONSERVATION LAWS

Consider a binary mixture of smooth hard spheres of masses m_1 and m_2 , and diameters σ_1 and σ_2 . The inelasticity of collisions among all pairs is characterized by three independent constant coefficients of normal restitution α_{11} , α_{22} , and $\alpha_{12} = \alpha_{21}$, where α_{ij} is the restitution coefficient for collisions between particles of species i and j . In the low-density regime, the distribution functions $f_i(\mathbf{r}, \mathbf{v}; t)$ ($i = 1, 2$) for the two species are determined from the set of nonlinear Boltzmann equations¹²

$$(\partial_t + \mathbf{v}_1 \cdot \nabla) f_i(\mathbf{r}, \mathbf{v}_1, t) = \sum_j J_{ij}[\mathbf{v}_1 | f_i(t), f_j(t)]. \quad (1)$$

The Boltzmann collision operator $J_{ij}[\mathbf{v}_1 | f_i, f_j]$ describing the scattering of pairs of particles is

$$J_{ij}[\mathbf{v}_1 | f_i, f_j] = \sigma_{ij}^2 \int d\mathbf{v}_2 \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \times [\alpha_{ij}^{-2} f_i(\mathbf{r}, \mathbf{v}'_1, t) f_j(\mathbf{r}, \mathbf{v}'_2, t) - f_i(\mathbf{r}, \mathbf{v}_1, t) f_j(\mathbf{r}, \mathbf{v}_2, t)], \quad (2)$$

where $\sigma_{ij} = (\sigma_i + \sigma_j)/2$, $\hat{\boldsymbol{\sigma}}$ is a unit vector along their line of centers, Θ is the Heaviside step function, and $\mathbf{g}_{12} = \mathbf{v}_1 - \mathbf{v}_2$. The primes on the velocities denote the initial values $\{\mathbf{v}'_1, \mathbf{v}'_2\}$ that lead to $\{\mathbf{v}_1, \mathbf{v}_2\}$ following a binary (restituting) collision:

$$\begin{aligned} \mathbf{v}'_1 &= \mathbf{v}_1 - \mu_{ji}(1 + \alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \hat{\boldsymbol{\sigma}}, \\ \mathbf{v}'_2 &= \mathbf{v}_2 + \mu_{ij}(1 + \alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \hat{\boldsymbol{\sigma}}, \end{aligned} \quad (3)$$

where $\mu_{ij} = m_i / (m_i + m_j)$. The relevant hydrodynamic fields are the number densities n_i , the flow velocity \mathbf{u} , and the temperature T . They are defined in terms of moments of the distributions f_i as

$$n_i = \int d\mathbf{v}_1 f_i(\mathbf{v}_1), \quad \rho \mathbf{u} = \sum_i \int d\mathbf{v}_1 m_i \mathbf{v}_1 f_i(\mathbf{v}_1), \quad (4)$$

$$nT = p = \sum_i \int d\mathbf{v}_1 \frac{m_i}{3} V_1^2 f_i(\mathbf{v}_1), \quad (5)$$

where $\mathbf{V}_1 = \mathbf{v}_1 - \mathbf{u}$ is the peculiar velocity, $n = n_1 + n_2$ is the total number density, $\rho = m_1 n_1 + m_2 n_2$ is the total mass density, and p is the pressure.

The collision operators conserve the particle number of each species and the total momentum but the total energy is not conserved:

$$\int d\mathbf{v}_1 J_{ij}[\mathbf{v}_1 | f_i, f_j] = 0, \quad (6)$$

$$\sum_{i,j} \int d\mathbf{v}_1 m_i \mathbf{v}_1 J_{ij}[\mathbf{v}_1 | f_i, f_j] = 0, \quad (7)$$

$$\sum_{i,j} \int d\mathbf{v}_1 \frac{1}{2} m_i v_1^2 J_{ij}[\mathbf{v}_1 | f_i, f_j] = -\frac{3}{2} nT\zeta, \quad (8)$$

where ζ is identified as the “cooling rate” due to inelastic collisions among all species. From Eqs. (4) to (8), the macroscopic balance equations for the mixture can be obtained. They are given by

$$D_t n_i + n_i \nabla \cdot \mathbf{u} + \frac{\nabla \cdot \mathbf{j}_i}{m_i} = 0, \quad (9)$$

$$D_t \mathbf{u} + \rho^{-1} \nabla P = 0, \quad (10)$$

$$D_t T - \frac{T}{n} \sum_i \frac{\nabla \cdot \mathbf{j}_i}{m_i} + \frac{2}{3n} (\nabla \cdot \mathbf{q} + P: \nabla \mathbf{u}) = -\zeta T. \quad (11)$$

In Eqs. (9)–(11), $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the material derivative,

$$\mathbf{j}_i = m_i \int d\mathbf{v}_1 \mathbf{V}_1 f_i(\mathbf{v}_1) \quad (12)$$

is the mass flux for species i relative to the local flow,

$$P = \sum_i \int d\mathbf{v}_1 m_i \mathbf{V}_1 \mathbf{V}_1 f_i(\mathbf{v}_1) \tag{13}$$

is the total pressure tensor, and

$$\mathbf{q} = \sum_i \int d\mathbf{v}_1 \frac{1}{2} m_i V_1^2 \mathbf{V}_1 f_i(\mathbf{v}_1) \tag{14}$$

is the total heat flux.

The balance equations become a closed set of hydrodynamic equations for the fields n_i , \mathbf{u} , and T once the fluxes (11)–(14) and the cooling rate ζ are obtained in terms of the hydrodynamic fields and their gradients. The resulting equations constitute the hydrodynamics for the mixture. Since these fluxes are explicit linear functionals of f_i , a representation in terms of the fields results when a solution to the Boltzmann equation can be obtained as a function of the fields and their gradients. Such a solution is called a “normal” solution, and a practical means to generate it for small spatial gradients is provided by the Chapman–Enskog method.¹³

III. CHAPMAN–ENSKOG SOLUTION OF THE BOLTZMANN EQUATIONS

The analysis of transport phenomena in fluid mixtures is considerably more complicated than in the case of a one-component system. Not only is the number of transport coefficients larger but these coefficients also are functions of more parameters such as the concentrations, mass ratios, size ratios, and the three coefficients of restitution. It follows from fluid symmetry that the pressure tensor has the same form to first order in the gradients as for a one-component system. As noted in Sec. I, there is more flexibility in the representation of the heat and mass fluxes. Even in the case of elastic collisions, several different (but equivalent) choices of hydrodynamic fields are used so some care is required in comparing transport coefficients in the different representations. The choice here is to use the concentration of species 1, defined in terms of the densities by $x_1 = n_1 / (n_1 + n_2)$, together with the pressure, the temperature, and the three components of the local flow velocity.¹⁴ The fluxes then have the forms

$$\mathbf{j}_1 = - \left(\frac{m_1 m_2 n}{\rho} \right) D \nabla x_1 - \frac{\rho}{p} D_p \nabla p - \frac{\rho}{T} D' \nabla T, \tag{15}$$

$$\mathbf{j}_2 = - \mathbf{j}_1,$$

$$\mathbf{q} = -T^2 D'' \nabla x_1 - L \nabla p - \lambda \nabla T, \tag{16}$$

$$P_{\alpha\beta} = p \delta_{\alpha\beta} - \eta (\nabla_\beta u_\alpha + \nabla_\alpha u_\beta - \frac{2}{3} \delta_{\alpha\beta} \nabla \cdot \mathbf{u}) - \kappa \delta_{\alpha\beta} \nabla \cdot \mathbf{u}. \tag{17}$$

The transport coefficients in these equations are

$$\begin{pmatrix} D \\ D' \\ D_p \\ D'' \\ \lambda \\ L \\ \eta \\ \kappa \end{pmatrix} = \begin{pmatrix} \text{diffusion coefficient} \\ \text{thermal diffusion coefficient} \\ \text{pressure diffusion coefficient} \\ \text{Dufour coefficient} \\ \text{thermal conductivity} \\ \text{pressure energy coefficient} \\ \text{shear viscosity} \\ \text{bulk viscosity} \end{pmatrix}. \tag{18}$$

For systems with elastic collisions, the thermal conductivity in a mixture is generally measured at $\mathbf{j}_1 = \text{constant}$, based on Onsager’s relations between coefficients in \mathbf{j}_1 and \mathbf{q} . This is no longer an experimentally useful choice here. If in addition it is required that $p = \text{constant}$, then ∇x_1 can be eliminated to give \mathbf{q} in terms of \mathbf{j}_1 and ∇T . The coefficient of ∇T in this representation is then the thermal conductivity.¹¹ The bulk viscosity vanishes at low density, as shown in the following, just as in the case of elastic collisions. The objective here is to apply the Chapman–Enskog method for a solution to the Boltzmann equation to first order in the gradients, confirm the forms (15)–(17), and determine a means to calculate the transport coefficients as functions of the parameters of the system.

The Chapman–Enskog method assumes the existence of a “normal” solution in which all space and time dependence of the distribution function occurs through a functional dependence on the hydrodynamic fields

$$f_i(\mathbf{r}, \mathbf{v}, t) = f_i[\mathbf{v}_1 | x_1(\mathbf{r}, t), p(\mathbf{r}, t), T(\mathbf{r}, t), \mathbf{u}(\mathbf{r}, t)]. \tag{19}$$

For small spatial variations, this functional dependence can be made local in space and time through an expansion in gradients of the fields. To generate the expansion, f_i is written as a series expansion in a formal parameter ϵ measuring the uniformity of the system,

$$f_i = f_i^{(0)} + \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \dots, \tag{20}$$

where each factor of ϵ means an implicit gradient of a hydrodynamic field. The local reference states $f_i^{(0)}$ are chosen such that they have the same moments as in Eqs. (4) and (5), or equivalently, the remainder of the expansion must obey the orthogonality conditions

$$\int d\mathbf{v}_1 [f_i(\mathbf{v}_1) - f_i^{(0)}(\mathbf{v}_1)] = 0, \tag{21}$$

$$\sum_i \int d\mathbf{v}_1 m_i \mathbf{v}_1 [f_i(\mathbf{v}_1) - f_i^{(0)}(\mathbf{v}_1)] = \mathbf{0},$$

$$\sum_i \int d\mathbf{v}_1 \frac{m_i}{2} v_1^2 [f_i(\mathbf{v}_1) - f_i^{(0)}(\mathbf{v}_1)] = 0. \tag{22}$$

The time derivatives of the fields are also expanded as $\partial_t = \partial_t^{(0)} + \epsilon \partial_t^{(1)} + \dots$. The coefficients of the time derivative expansion are identified from the balance equations (9)–(11) with a representation of the fluxes and the cooling rate in the macroscopic balance equations as a similar series through their definitions as functionals of the distributions f_i . This is the usual Chapman–Enskog method for solving kinetic equations.¹³ The main difference in the case of inelastic collisions is that the reference state has a time dependence as-

sociated with the cooling that is not proportional to a spatial gradient. Consequently, terms from the time derivative $\partial_t^{(0)}$ are not zero.

To zeroth order in ϵ , the kinetic equations (1) become

$$\partial_t^{(0)} f_i^{(0)} = \sum_j J_{ij}[f_i^{(0)}, f_j^{(0)}]. \quad (23)$$

The mass and energy balance equations to this order give

$$\partial_t^{(0)} x_i = 0, \quad T^{-1} \partial_t^{(0)} T = p^{-1} \partial_t^{(0)} p = -\zeta^{(0)}, \quad (24)$$

where the cooling rate $\zeta^{(0)}$ is determined by Eq. (8) to zeroth order

$$\zeta^{(0)} = -\frac{2}{3p} \sum_{i,j} \int d\mathbf{v}_1 \frac{1}{2} m_i v_1^2 J_{ij}[\mathbf{v}_1 | f_i^{(0)}, f_j^{(0)}]. \quad (25)$$

This homogeneous state has been studied recently¹⁰ and is discussed in more detail in Sec. IV. The time derivative in Eq. (23) can be represented more usefully as

$$\partial_t^{(0)} f_i^{(0)} = -\zeta^{(0)} (T \partial_T + p \partial_p) f_i^{(0)} = \frac{1}{2} \zeta^{(0)} \nabla_{v_1} \cdot (\mathbf{V}_1 f_i^{(0)}), \quad (26)$$

where $\nabla_{v_1} = \partial / \partial \mathbf{v}_1$. The second equality follows from dimensional analysis which requires that the temperature dependence of $f_i^{(0)}$ must be of the form

$$f_i^{(0)} = x_i \frac{p}{T} v_0^{-3} \Phi_i(V_1 / v_0), \quad (27)$$

where $v_0^2(t) = 2T(t)(m_1 + m_2)/(m_1 m_2)$ is a thermal velocity defined in terms of the temperature $T(t)$ of the mixture. The dependence on the magnitude of \mathbf{V}_1 follows from the requirement that to zeroth order in gradients f_i must be isotropic with respect to the peculiar velocity. The Boltzmann equations at this order can be written finally as

$$\frac{1}{2} \zeta^{(0)} \nabla_{v_1} \cdot (\mathbf{V}_1 f_i^{(0)}) = \sum_j J_{ij}[f_i^{(0)}, f_j^{(0)}]. \quad (28)$$

Since the distribution functions are isotropic, it follows from Eqs. (12) and (14) that the zeroth-order mass and heat fluxes vanish while, for the same reason, the momentum flux is diagonal with a coefficient that is just the sum of the partial pressures, i.e.,

$$\mathbf{j}_i^{(0)} = \mathbf{0}, \quad \mathbf{q}^{(0)} = \mathbf{0}, \quad P_{\alpha\beta}^{(0)} = p \delta_{\alpha\beta}. \quad (29)$$

To first order in the gradients, the equation for $f_i^{(1)}$ is

$$(\partial_t^{(0)} + \mathcal{L}_i) f_i^{(1)} + \mathcal{M}_i f_j^{(1)} = -(\partial_t^{(1)} + \mathbf{v}_1 \cdot \nabla) f_i^{(0)}, \quad (30)$$

where it is understood that $i \neq j$ and the linear operators \mathcal{L}_i and \mathcal{M}_i are

$$\begin{aligned} \mathcal{L}_i f_i^{(1)} = & -(J_{ii}[f_i^{(0)}, f_i^{(1)}] + J_{ii}[f_i^{(1)}, f_i^{(0)}] \\ & + J_{ij}[f_i^{(1)}, f_j^{(0)}]), \end{aligned} \quad (31)$$

$$\mathcal{M}_i f_i^{(1)} = -J_{ij}[f_i^{(0)}, f_j^{(1)}]. \quad (32)$$

The action of the time derivatives $\partial_t^{(1)}$ on the hydrodynamic fields is

$$D_i^{(1)} x_1 = 0, \quad (33)$$

$$D_i^{(1)} p = -\frac{5p}{3} \nabla \cdot \mathbf{u}, \quad (34)$$

$$D_i^{(1)} T = -\frac{2T}{3} \nabla \cdot \mathbf{u}, \quad (35)$$

$$D_i^{(1)} \mathbf{u} = -\rho^{-1} \nabla p, \quad (36)$$

where $D_i^{(1)} = \partial_t^{(1)} + \mathbf{u} \cdot \nabla$ and use has been made of the results $\mathbf{j}_i^{(0)} = \mathbf{q}^{(0)} = \zeta^{(1)} = 0$. The last equality follows from the fact that the cooling rate is a scalar, and corrections to first order in the gradients can arise only from the divergence of the vector field. However, as demonstrated in the following, there is no contribution to the distribution function proportional to this divergence. We note that this is special to the low density Boltzmann equation and such terms do occur at higher densities. Evaluating the right-hand side of Eq. (30) gives

$$\begin{aligned} & -(\partial_t^{(1)} + \mathbf{v}_1 \cdot \nabla) f_i^{(0)} \\ & = -\left(\frac{\partial}{\partial x_1} f_i^{(0)} \right)_{p,T} \mathbf{V}_1 \cdot \nabla x_1 \\ & \quad - \left[f_1^{(0)} \mathbf{V}_1 + \frac{nT}{\rho} \left(\frac{\partial}{\partial \mathbf{V}_1} f_i^{(0)} \right) \right] \cdot \nabla \ln p \\ & \quad + \left[f_i^{(0)} + \frac{1}{2} \frac{\partial}{\partial \mathbf{V}_1} \cdot (\mathbf{V}_1 f_i^{(0)}) \right] \mathbf{V}_1 \cdot \nabla \\ & \quad \times \ln T + \left(V_{1\alpha} \frac{\partial}{\partial V_{1\beta}} f_i^{(0)} - \frac{1}{3} \delta_{\alpha\beta} \mathbf{V}_1 \cdot \frac{\partial}{\partial \mathbf{V}_1} f_i^{(0)} \right) \nabla_{\alpha} u_{\beta}. \end{aligned} \quad (37)$$

The equation for $f_i^{(1)}$ is now

$$\begin{aligned} (\partial_t^{(0)} + \mathcal{L}_i) f_i^{(1)} + \mathcal{M}_i f_j^{(1)} = & \mathbf{A}_i \cdot \nabla x_1 + \mathbf{B}_i \cdot \nabla p + \mathbf{C}_i \cdot \nabla T \\ & + D_{i,\alpha\beta} \nabla_{\alpha} u_{\beta}. \end{aligned} \quad (38)$$

The coefficients of the field gradients on the right-hand side are functions of \mathbf{V}_1 and the hydrodynamic fields. They are given by

$$\mathbf{A}_i(\mathbf{V}_1) = -\left(\frac{\partial}{\partial x_1} f_i^{(0)} \right)_{p,T} \mathbf{V}_1, \quad (39)$$

$$\mathbf{B}_i(\mathbf{V}_1) = -\frac{1}{p} \left[f_i^{(0)} \mathbf{V}_1 + \frac{nT}{\rho} \left(\frac{\partial}{\partial \mathbf{V}_1} f_i^{(0)} \right) \right], \quad (40)$$

$$\mathbf{C}_i(\mathbf{V}_1) = \frac{1}{T} \left[f_i^{(0)} + \frac{1}{2} \frac{\partial}{\partial \mathbf{V}_1} \cdot (\mathbf{V}_1 f_i^{(0)}) \right] \mathbf{V}_1, \quad (41)$$

$$D_{i,\alpha\beta}(\mathbf{V}_1) = V_{1\alpha} \frac{\partial}{\partial V_{1\beta}} f_i^{(0)} - \frac{1}{3} \delta_{\alpha\beta} \mathbf{V}_1 \cdot \frac{\partial}{\partial \mathbf{V}_1} f_i^{(0)}. \quad (42)$$

Note that the trace of $D_{i,\alpha\beta}$ vanishes, confirming that the distribution function does not have contribution from the divergence of the flow field. The solutions to Eq. (38) is of the form

$$f_i^{(1)} = \mathcal{A}_i \cdot \nabla x_1 + \mathcal{B}_i \cdot \nabla p + \mathcal{C}_i \cdot \nabla T + \mathcal{D}_{i,\alpha\beta} \nabla_{\alpha} u_{\beta}. \quad (43)$$

The coefficients \mathcal{A}_i , \mathcal{B}_i , \mathcal{C}_i , and $\mathcal{D}_{i,\alpha\beta}$ are functions of the peculiar velocity \mathbf{V}_1 and the hydrodynamic fields. The cooling rate depends on space through its dependence on x_1 , p , and T . The time derivative $\partial_t^{(0)}$ acting on these quantities can be evaluated by the replacement $\partial_t^{(0)} \rightarrow -\zeta^{(0)}(T\partial_T + p\partial_p)$. In addition, there are contributions from $\partial_t^{(0)}$ acting on the temperature and pressure gradients given by

$$\begin{aligned} \partial_t^{(0)}\nabla T &= -\nabla(T\zeta^{(0)}) \\ &= -\zeta^{(0)}\nabla T - T\nabla\zeta^{(0)} \\ &= -\frac{\zeta^{(0)}}{2T}\nabla T - T\left[\left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T}\nabla x_1 + \frac{\zeta^{(0)}}{p}\nabla p\right], \end{aligned} \quad (44)$$

$$\begin{aligned} \partial_t^{(0)}\nabla p &= -\nabla(p\zeta^{(0)}) \\ &= -\zeta^{(0)}\nabla p - p\nabla\zeta^{(0)} \\ &= -2\zeta^{(0)}\nabla p - p\left[\left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T}\nabla x_1 - \frac{\zeta^{(0)}}{2T}\nabla T\right]. \end{aligned} \quad (45)$$

The integral equations for \mathcal{A}_i , \mathcal{B}_i , \mathcal{C}_i , and $\mathcal{D}_{i,\alpha\beta}$ are identified as coefficients of the independent gradients in Eq. (43):

$$\begin{aligned} [-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_i]\mathcal{A}_i + \mathcal{M}_i\mathcal{A}_j \\ = \mathbf{A}_i + \left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T}(p\mathcal{B}_i + T\mathcal{C}_i), \end{aligned} \quad (46)$$

$$\begin{aligned} [-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_i - 2\zeta^{(0)}]\mathcal{B}_i + \mathcal{M}_i\mathcal{B}_j \\ = \mathbf{B}_i + \frac{T\zeta^{(0)}}{p}\mathcal{C}_i, \end{aligned} \quad (47)$$

$$\begin{aligned} \left[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_i - \frac{1}{2}\zeta^{(0)}\right]\mathcal{C}_i + \mathcal{M}_i\mathcal{C}_j \\ = \mathbf{C}_i - \frac{p\zeta^{(0)}}{2T}\mathcal{B}_i, \end{aligned} \quad (48)$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \mathcal{L}_i]\mathcal{D}_{i,\alpha\beta} + \mathcal{M}_i\mathcal{D}_{j,\alpha\beta} = \mathcal{D}_{i,\alpha\beta}. \quad (49)$$

The solutions to these linear integral equations are made unique by the orthogonality conditions (21) and (22), i.e.,

$$\int d\mathbf{v}_1 \begin{pmatrix} \mathcal{A}_i \\ \mathcal{B}_i \\ \mathcal{C}_i \\ \mathcal{D}_{i,\alpha\beta} \end{pmatrix} = 0, \quad \sum_i \int d\mathbf{v}_1 m_i \mathbf{V}_1 \begin{pmatrix} \mathcal{A}_i \\ \mathcal{B}_i \\ \mathcal{C}_i \\ \mathcal{D}_{i,\alpha\beta} \end{pmatrix} = \mathbf{0}, \quad (50)$$

$$\sum_i \int d\mathbf{v}_1 \frac{m_i}{2} V_1^2 \begin{pmatrix} \mathcal{A}_i \\ \mathcal{B}_i \\ \mathcal{C}_i \\ \mathcal{D}_{i,\alpha\beta} \end{pmatrix} = 0. \quad (51)$$

With the functions $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{i,\alpha\beta})$ determined in this way, the solutions to the Boltzmann equations are determined by (43) exactly to first order in the spatial gradients.

Use of Eq. (43) in the definitions (12)–(14) gives the expected forms (15)–(17) for the fluxes. The transport coefficients associated with \mathbf{j}_1 are identified as

$$D = -\frac{\rho}{3m_2n} \int d\mathbf{v}_1 \mathbf{V}_1 \cdot \mathcal{A}_1, \quad (52)$$

$$D_p = -\frac{m_1 p}{3\rho} \int d\mathbf{v}_1 \mathbf{V}_1 \cdot \mathcal{B}_1, \quad (53)$$

$$D' = -\frac{m_1 T}{3\rho} \int d\mathbf{v}_1 \mathbf{V}_1 \cdot \mathcal{C}_1. \quad (54)$$

The transport coefficients for the heat flux are

$$D'' = -\frac{1}{3T^2} \sum_i \int d\mathbf{v}_1 \frac{1}{2} m_i V_1^2 \mathbf{V}_1 \cdot \mathcal{A}_i, \quad (55)$$

$$L = -\frac{1}{3} \sum_i \int d\mathbf{v}_1 \frac{1}{2} m_i V_1^2 \mathbf{V}_1 \cdot \mathcal{B}_i, \quad (56)$$

$$\lambda = -\frac{1}{3} \sum_i \int d\mathbf{v}_1 \frac{1}{2} m_i V_1^2 \mathbf{V}_1 \cdot \mathcal{C}_i. \quad (57)$$

Finally, the shear and bulk viscosities are

$$\eta = -\frac{1}{10} \sum_i \int d\mathbf{v}_1 m_i V_{1\alpha} V_{1\beta} \mathcal{D}_{i,\alpha\beta}, \quad (58)$$

$$\kappa = -\frac{1}{9} \sum_i \int d\mathbf{v}_1 m_i V_1^2 \mathcal{D}_{i,\alpha\alpha} = 0. \quad (59)$$

The bulk viscosity vanishes since the trace of $\mathcal{D}_{i,\alpha\beta}$ vanishes, as follows from Eq. (49).

To summarize the results to this point, the solutions to the Boltzmann equations to first order in the spatial gradients are

$$f_i = f_i^{(0)} + \mathcal{A}_i \cdot \nabla x_1 + \mathcal{B}_i \cdot \nabla p + \mathcal{C}_i \cdot \nabla T + \mathcal{D}_{i,\alpha\beta} \nabla_\alpha \nabla_\beta. \quad (60)$$

The solution to zeroth order, $f_i^{(0)}$, is obtained from Eq. (49) while the functions \mathcal{A}_i , \mathcal{B}_i , \mathcal{C}_i , and $\mathcal{D}_{i,\alpha\beta}$ characterizing the solution to first order in the gradients are determined from the integral equations (46) to (49). Calculating the mass, heat, and momentum fluxes from this solution one can identify the transport coefficients in terms of the integrals (52)–(59). These fluxes, together with the macroscopic balance equations (9)–(11), provide the closed set of Navier–Stokes order hydrodynamic equations for a granular binary mixture. All of these results are still exact and valid for arbitrary values of the restitution coefficients.

IV. SONINE POLYNOMIAL APPROXIMATION

Accurate approximations to the solutions to the integral equations for $f_i^{(0)}$ and $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{i,\alpha\beta})$ may be obtained using low order truncation of expansions in a series of Sonine polynomials. The determination of $f_i^{(0)}$ to leading order in the Sonine expansion has been analyzed elsewhere¹⁰ and only the main result is quoted here. The polynomials are defined with respect to a Gaussian weight factor. The parameters of this Gaussian are chosen such that the leading term in the expansion yields the exact moments of the entire distribution with respect to 1, v_1 , and v_1^2 . The latter defines the kinetic temperatures for each species

$$\frac{3}{2} n_i T_i = \int d\mathbf{v}_1 \frac{m_i}{2} V_1^2 f_i^{(0)}. \tag{61}$$

For elastic collisions the temperatures T_i are the same as the global temperature T defined in Eq. (5). The condition that $f_i^{(0)}$ is normal implies that T_i is a function of n_i and T , or equivalently

$$\frac{T_i}{T} \equiv \gamma_i(x_1), \tag{62}$$

where γ_i depends on the hydrodynamic state through the concentration. The determination of this functional dependence is worked out in Ref. 10. The Sonine polynomials used here are defined with respect to Maxwellians characterized by the temperatures T_i rather than those characterized by T for elastic collisions. With this choice the leading deviation from the Maxwellians is a polynomial of degree 4,

$$f_i^{(0)} = n_i v_0^{-3} \Phi_i(V_1^*), \tag{63}$$

$$\Phi_i(V_1^*) \rightarrow \left(\frac{\theta_i}{\pi} \right)^{3/2} e^{-\theta_i V_1^{*2}} \left[1 + \frac{c_i}{4} \left(\theta_i^2 V_1^{*4} - 5 \theta_i V_1^{*2} + \frac{15}{4} \right) \right], \tag{64}$$

where $V_1^* = V_1/v_0$, and $\theta_i = (\mu_{ji} \gamma_i)^{-1}$. If polynomials defined in terms of T are used the leading correction is a polynomial of degree 2 in v_1^* , proportional to $T - T_i$. The choice of polynomials defined in terms of T_i effectively resums an infinite set of terms in this second type of expansion. Since γ_i is a function of the concentration, a significant new contribution to the parameters of the integral equations for the transport coefficients occurs through the additional concentration dependence associated with temperature difference of the two species. The coefficients c_i in Eq. (64) are determined by substitution of Eq. (63) into Eq. (26) and retaining all terms linear in c_i for the leading Sonine polynomial approximation. The reader is referred to Ref. 10 for further details.

In Secs. IV A, IV B, and IV C, respectively, we will compute the mass flux, the pressure tensor, and the heat flux in the leading Sonine polynomial approximations. Let us consider each one separately.

A. Mass flux

Here, the lowest order Sonine polynomial approximations for \mathcal{A}_i , \mathcal{B}_i , and \mathcal{C}_i are obtained and applied to the calculation of the transport coefficients in the mass fluxes, D , D_p , and D' . The leading Sonine approximations (lowest degree polynomial) of the quantities \mathcal{A}_i , \mathcal{B}_i , and \mathcal{C}_i are

$$\begin{pmatrix} \mathcal{A}_i \\ \mathcal{B}_i \\ \mathcal{C}_i \end{pmatrix} \rightarrow f_{i,M} \mathbf{V}_1 \begin{pmatrix} a_{i,1} \\ b_{i,1} \\ c_{i,1} \end{pmatrix}. \tag{65}$$

The coefficients $\{a_{i,1}, b_{i,1}, c_{i,1}\}$ are related to the transport coefficients in this approximation through (52)–(54),

$$a_{1,1} = -\frac{1}{\delta\gamma} a_{2,1} = -\frac{m_1 m_2 n}{\rho n_1 T_1} D, \tag{66}$$

$$b_{1,1} = -\frac{1}{\delta\gamma} b_{2,1} = -\frac{\rho}{p n_1 T_1} D_p, \tag{67}$$

$$c_{1,1} = -\frac{1}{\delta\gamma} c_{2,1} = -\frac{\rho}{T n_1 T_1} D'. \tag{68}$$

Here, $\gamma = T_1/T_2$ and $\delta = n_1/n_2$. The coefficients $\{a_{i,1}; b_{i,1}; c_{i,1}\}$ are determined by substitution of Eq. (65) into the integral equations (46)–(48). The details are carried out in Appendix A with the results

$$a_{1,1} = -\left(\nu - \frac{1}{2} \zeta^{(0)} \right)^{-1} \left[\left(\frac{\partial}{\partial x_1} \ln n_1 T_1 \right)_{p,T} - \left(\frac{\partial \zeta^{(0)}}{\partial x_1} \right)_{p,T} (p b_{1,1} + T c_{1,1}) \right], \tag{69}$$

$$b_{1,1} = -\frac{1}{p} \left(1 - \frac{m_1 n T}{\rho T_1} \right) \left(\nu - \frac{3}{2} \zeta^{(0)} + \frac{\zeta^{(0)2}}{2\nu} \right)^{-1}, \tag{70}$$

$$c_{1,1} = -\frac{p \zeta^{(0)}}{2 T \nu} b_{1,1}, \tag{71}$$

where the collision frequency ν is given by

$$\begin{aligned} \nu = & -\frac{m_1}{3 n_1 T_1} \int d\mathbf{v}_1 \mathbf{V}_1 \cdot (J_{12}[\mathbf{v}_1 | f_{1,M} \mathbf{V}_1, f_2^{(0)}], \\ & - \delta \gamma J_{12}[\mathbf{v}_1 | f_1^{(0)}, f_{2,M} \mathbf{V}_2]). \end{aligned} \tag{72}$$

This integral is evaluated in Appendix B with the result

$$\begin{aligned} \frac{\nu}{\nu_e} = & \frac{(1 + \alpha_{12})}{2} \sqrt{\mu_{21} \gamma_1 + \mu_{12} \gamma_2} \\ & \times \left[1 - \frac{1}{16\rho} \frac{\rho_1 (m_2 \gamma_1)^2 c_1 + \rho_2 (m_1 \gamma_2)^2 c_2}{(m_2 \gamma_1 + m_1 \gamma_2)^2} \right], \end{aligned} \tag{73}$$

where ν_e is the corresponding result for the elastic case

$$\nu_e = \frac{8}{3} \sqrt{2\pi} \sigma_{12}^2 \sqrt{\frac{T}{m_1 m_2}} \frac{\rho}{\sqrt{m_1 + m_2}}, \tag{74}$$

and $\rho_i = m_i n_i$ is the mass density of species i . These results, together with $a_{2,1} = -\delta\gamma a_{1,1}$, $b_{2,1} = -\delta\gamma b_{1,1}$, and $c_{2,1} = -\delta\gamma c_{1,1}$ completely determine the distribution functions to first order in the Sonine polynomial expansion.

The transport coefficients are identified from (69) to (71) as

$$\begin{aligned} D = & \frac{\rho}{m_1 m_2 n} \left(\nu - \frac{1}{2} \zeta^{(0)} \right)^{-1} \left[\left(\frac{\partial}{\partial x_1} n_1 T_1 \right)_{p,T} \right. \\ & \left. + \rho \left(\frac{\partial \zeta^{(0)}}{\partial x_1} \right)_{p,T} (D_p + D') \right], \end{aligned} \tag{75}$$

$$D_p = \frac{n_1 T_1}{\rho} \left(1 - \frac{m_1 n T}{\rho T_1} \right) \left(\nu - \frac{3}{2} \zeta^{(0)} + \frac{\zeta^{(0)2}}{2\nu} \right)^{-1}, \tag{76}$$

$$D' = -\frac{\zeta^{(0)}}{2\nu} D_p. \tag{77}$$

Since $\mathbf{j}_1 = -\mathbf{j}_2$ and $\nabla_{x_1} = -\nabla_{x_2}$, it is expected that D should be symmetric with respect to interchange of particles

1 and 2 while D_p and D' should be antisymmetric. This can be verified by noting that $n_1 T_1 + n_2 T_2 = nT$. The expression for ν agrees with the known result for elastic collisions.¹³ For the case of mechanically equivalent particles ($m_1 = m_2, \alpha_{11} = \alpha_{22} = \alpha_{12} \equiv \alpha, \sigma_{11} = \sigma_{22}$) in the dilute concentration limit ($\rho_2 \rightarrow 0$), the expression for the diffusion coefficient D coincides with the one recently derived in the self-diffusion problem.¹⁵

B. Pressure tensor

The leading Sonine approximation for the function $\mathcal{D}_{i,\alpha\beta}$ is

$$\mathcal{D}_{i,\alpha\beta} \rightarrow f_{1,M} d_{i,1} R_{i,\alpha\beta}(\mathbf{V}_1), \tag{78}$$

where

$$R_{i,\alpha\beta}(\mathbf{V}_1) = m_i (V_{1\alpha} V_{1\beta} - \frac{1}{3} V_1^2 \delta_{\alpha\beta}), \tag{79}$$

and

$$d_{i,1} = \frac{1}{10} \frac{1}{n_i T_i^2} \int d\mathbf{v}_1 R_{i,\alpha\beta} \mathcal{D}_{i,\alpha\beta}. \tag{80}$$

The coefficients $d_{i,1}$ are related to the shear viscosity η in this approximation through Eq. (58) as

$$\eta = -T^2 (n_1 \gamma_1^2 d_{1,1} + n_2 \gamma_2^2 d_{2,1}). \tag{81}$$

The integral equations for the coefficients $d_{i,1}$ are decoupled from the remaining transport coefficients. The two coefficients $d_{i,1}$ are obtained by multiplying Eq. (49) with $R_{i,\alpha\beta}$ and integrating over the velocity to get the coupled set of equations

$$\begin{pmatrix} \tau_{11} - \frac{1}{2} \zeta^{(0)} & \tau_{12} \\ \tau_{21} & \tau_{22} - \frac{1}{2} \zeta^{(0)} \end{pmatrix} \begin{pmatrix} d_{1,1} \\ d_{2,1} \end{pmatrix} = - \begin{pmatrix} T_1^{-1} \\ T_2^{-1} \end{pmatrix}. \tag{82}$$

The frequencies τ_{ij} are given in terms of the collision operator by

$$\tau_{ii} = \frac{1}{10} \frac{1}{n_i T_i^2} \int d\mathbf{v}_1 R_{i,\alpha\beta} \mathcal{L}_i (f_{i,M} R_{i,\alpha\beta}), \tag{83}$$

$$\tau_{ij} = \frac{1}{10} \frac{1}{n_i T_i^2} \int d\mathbf{v}_1 R_{i,\alpha\beta} \mathcal{M}_i (f_{j,M} R_{j,\alpha\beta}), \quad i \neq j. \tag{84}$$

The evaluation of these collision integrals is given in Appendix C. The solution with the matrix elements known is elementary so that η can be calculated directly from Eq. (81).

C. Heat flux

The computation of the heat flux requires going to the second Sonine approximation. In this case, the quantities $\mathcal{A}_i, \mathcal{B}_i,$ and \mathcal{C}_i are taken to be

$$\mathcal{A}_1(\mathbf{V}_1) \rightarrow f_{1,M} \left[- \frac{m_1 m_2 n}{\rho n_1 T_1} D \mathbf{V}_1 + a_{1,2} \mathbf{S}_1(\mathbf{V}_1) \right], \tag{85}$$

$$\mathcal{A}_2(\mathbf{V}_1) \rightarrow f_{2,M} \left[\frac{m_1 m_2 n}{\rho n_2 T_2} D \mathbf{V}_1 + a_{2,2} \mathbf{S}_2(\mathbf{V}_1) \right],$$

$$\mathcal{B}_1(\mathbf{V}_1) \rightarrow f_{1,M} \left[- \frac{\rho}{\rho n_1 T_1} D_p \mathbf{V}_1 + b_{1,2} \mathbf{S}_1(\mathbf{V}_1) \right], \tag{86}$$

$$\mathcal{B}_2(\mathbf{V}_1) \rightarrow f_{2,M} \left[\frac{\rho}{\rho n_2 T_2} D_p \mathbf{V}_1 + b_{2,2} \mathbf{S}_2(\mathbf{V}_1) \right],$$

$$\mathcal{C}_1(\mathbf{V}_1) \rightarrow f_{1,M} \left[- \frac{\rho}{T n_1 T_1} D' \mathbf{V}_1 + c_{1,2} \mathbf{S}_1(\mathbf{V}_1) \right], \tag{87}$$

$$\mathcal{C}_2(\mathbf{V}_1) \rightarrow f_{2,M} \left[\frac{\rho}{T n_2 T_2} D' \mathbf{V}_1 + c_{2,2} \mathbf{S}_2(\mathbf{V}_1) \right],$$

where

$$\mathbf{S}_i(\mathbf{V}_1) = \left(\frac{1}{2} m_i V_1^2 - \frac{5}{2} T_i \right) \mathbf{V}_1. \tag{88}$$

In Eqs. (85)–(88), it is understood that $D, D_p,$ and D' are given by Eqs. (75), (76) and (77), respectively. The coefficients $a_{i,2}, b_{i,2},$ and $c_{i,2}$ are defined as

$$\begin{pmatrix} a_{i,2} \\ b_{i,2} \\ c_{i,2} \end{pmatrix} = \frac{2}{15} \frac{m_i}{n_i T_i^3} \int d\mathbf{v}_1 \mathbf{S}_i(\mathbf{V}_1) \cdot \begin{pmatrix} \mathcal{A}_i \\ \mathcal{B}_i \\ \mathcal{C}_i \end{pmatrix}. \tag{89}$$

Consequently, the transport coefficients appearing in the heat flux, $D'', L,$ and λ are given by

$$D'' = - \frac{5}{2} T \left(\frac{n_1 \gamma_1^3}{m_1} a_{1,2} + \frac{n_2 \gamma_2^3}{m_2} a_{2,2} \right) + \frac{5}{2} \frac{n m_1 m_2}{\rho T} \left(\frac{\gamma_1}{m_1} - \frac{\gamma_2}{m_2} \right) D, \tag{90}$$

$$L = - \frac{5}{2} T^3 \left(\frac{n_1 \gamma_1^3}{m_1} b_{1,2} + \frac{n_2 \gamma_2^3}{m_2} b_{2,2} \right) + \frac{5}{2} \frac{\rho}{n} \left(\frac{\gamma_1}{m_1} - \frac{\gamma_2}{m_2} \right) D_p, \tag{91}$$

$$\lambda = - \frac{5}{2} T^3 \left(\frac{n_1 \gamma_1^3}{m_1} c_{1,2} + \frac{n_2 \gamma_2^3}{m_2} c_{2,2} \right) + \frac{5}{2} \rho \left(\frac{\gamma_1}{m_1} - \frac{\gamma_2}{m_2} \right) D'. \tag{92}$$

The computation of the coefficients $D'', L,$ and λ is also carried out in Appendix A. By using matrix notation, the coupled set of six equations for the unknowns

$$\{a_{1,2}; a_{2,2}; b_{1,2}; b_{2,2}; c_{1,2}; c_{2,2}\} \tag{93}$$

can be written as

$$\Lambda_{\sigma\sigma'} X_{\sigma'} = Y_{\sigma}. \tag{94}$$

Here, $X_{\sigma'}$ is the column matrix defined by the set (93) and $\Lambda_{\sigma\sigma'}$ is the matrix

$$\Lambda = \begin{pmatrix} \nu_{11} - \frac{3}{2}\zeta^{(0)} & \nu_{12} & -p\left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T} & 0 & -T\left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T} & 0 \\ \nu_{21} & \nu_{22} - \frac{3}{2}\zeta^{(0)} & 0 & p\left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T} & 0 & T\left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T} \\ 0 & 0 & \nu_{11} - \frac{5}{2}\zeta^{(0)} & \nu_{12} & -T\zeta^{(0)}/p & 0 \\ 0 & 0 & \nu_{21} & \nu_{22} - \frac{5}{2}\zeta^{(0)} & 0 & -T\zeta^{(0)}/p \\ 0 & 0 & p\zeta^{(0)}/2T & 0 & \nu_{11} - \zeta^{(0)} & \nu_{12} \\ 0 & 0 & 0 & p\zeta^{(0)}/2T & \nu_{21} & \nu_{22} - \zeta^{(0)} \end{pmatrix}. \tag{95}$$

Here, we have introduced the collision frequencies

$$\nu_{ii} = \frac{2}{15} \frac{m_i}{n_i T_i^3} \int d\mathbf{v}_1 \mathbf{S}_i \cdot \mathcal{L}_i(f_{i,M} \mathbf{S}_i), \tag{96}$$

$$\nu_{ij} = \frac{2}{15} \frac{m_i}{n_i T_i^3} \int d\mathbf{v}_1 \mathbf{S}_i \cdot \mathcal{M}_i(f_{j,M} \mathbf{S}_j), \quad i \neq j. \tag{97}$$

The column matrix \mathbf{Y} is

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix}, \tag{98}$$

where

$$Y_1 = -\frac{\zeta^{(0)} m_1 m_2 n D}{\rho n_1 T_1^2} - \frac{1}{2} \frac{T^2}{n_1 T_1^3} \frac{\partial}{\partial x_1} (n_1 \gamma_1^2 c_1) + \frac{2}{15} \frac{m_1^2 m_2 n D}{\rho n_1^2 T_1^4} \left[\int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{L}_1(f_{1,M} \mathbf{V}_1) - \delta\gamma \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{M}_1(f_{2,M} \mathbf{V}_2) \right], \tag{99}$$

$$Y_3 = -\frac{\zeta^{(0)} \rho D_p}{\rho n_1 T_1^2} - \frac{1}{2} \frac{c_1}{p T_1} + \frac{2}{15} \frac{m_1 \rho D_p}{\rho n_1^2 T_1^4} \left[\int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{L}_1(f_{1,M} \mathbf{V}_1) - \delta\gamma \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{M}_1(f_{2,M} \mathbf{V}_2) \right], \tag{100}$$

$$Y_5 = -\frac{\zeta^{(0)} \rho D'}{T n_1 T_1^2} - \frac{2 + c_1}{2 T T_1} + \frac{2}{15} \frac{m_1 \rho D'}{T n_1^2 T_1^4} \left[\int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{L}_1(f_{1,M} \mathbf{V}_1) - \delta\gamma \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathcal{M}_1(f_{2,M} \mathbf{V}_2) \right], \tag{101}$$

where

$$c_i = \frac{8}{15} \left[\frac{m_i^2}{4 n_i T_i^2} \int d\mathbf{v}_1 V_1^4 f_i^{(0)} - \frac{15}{4} \right]. \tag{102}$$

The corresponding expressions of the elements $Y_2, Y_4,$ and Y_6 can be deduced from Eqs. (99), (100), and (101), respectively, by interchanging $1 \leftrightarrow 2$ and setting $D \rightarrow D, D_p \rightarrow -D_p,$ and $D' \rightarrow -D'$. The evaluation of the collision integrals (96) and (97) is given in Appendix D.

The solution to Eq. (94) is

$$X_\sigma = (\Lambda^{-1})_{\sigma\sigma'} Y_{\sigma'}. \tag{103}$$

This relation provides an explicit expression for the coefficients $a_{i,2}, b_{i,2},$ and $c_{i,2}$ in terms of the restitution coefficients and the parameters of the mixture. From these expressions one can easily get the transport coefficients $D', L,$ and λ from Eqs. (90), (91), and (92), respectively.

V. DISCUSSION

The primary objective of this work has been to obtain the hydrodynamic description of a binary mixture of granular gases from an underlying kinetic theory. The derivation of the hydrodynamic equations consists of two steps. First, the macroscopic balance equations (9)–(11) for number densities, total momentum, and energy are obtained from the corresponding kinetic equation. Next, the fluxes and cooling rate in these equations are determined from a solution of the coupled Boltzmann equations given in terms of the hydrodynamic fields and their spatial gradients. The corresponding constitutive equations for the mass, heat, and momentum fluxes are given to Navier–Stokes order by Eqs. (15)–(17), and the associated transport coefficients are given by Eqs. (52)–(54), (55)–(57), and (58)–(59), respectively. These results are exact within the context of the Boltzmann kinetic equation. A practical evaluation of these coefficients is possible by means of a Sonine polynomial approximation and the derivation and approximate results are *not* limited to weak inelasticity. An exploration of the full parameter space (mass ratio, diameters, concentrations, inelasticity parameters) is straightforward but beyond the scope of this presentation. We intend to create a library of programs for calculation of distribution functions and transport coefficients in one- and two-component granular systems, available to the public on our web site.

The Chapman–Enskog method provides an expansion of the distribution function for weak spatial inhomogeneity. This means that the relative spatial variation of all hydrodynamic fields must be small over distances of order of the

mean free path. This encompasses a wide range of interesting phenomena, but excludes strongly driven systems such as those under steady shear. For such states it is not profitable to go beyond the order considered here in the Chapman–Enskog expansion, but rather to use other methods not based on small gradients. Previous applications of the Chapman–Enskog method have typically introduced additional assumptions for convenience that are not internally consistent with constructing a solution to the Boltzmann equation. In most of these cases the reference state was chosen to be a Maxwellian, presumed to give accurate results at weak dissipation. This is not the case, as has been demonstrated in the one-component case.¹ In addition, the effects of cooling in the lowest order time derivative of the temperature and pressure were neglected, again under the assumption of weak dissipation. Here, the reference state has been taken to be an exact solution to the uniform Boltzmann equation and consequently, there is no limitation on the parameters of the system (mass ratio, size ratio, concentrations, restitution coefficients). An interesting and important result of this exact analysis is that the temperatures of each species must be different. Furthermore, the time derivatives are calculated at each order in the gradient expansion without restriction on the restitution coefficients. This assures that a consistent solution to the Boltzmann equation is constructed at each order in the gradients, without any restrictions on the system parameters. This consistency is reflected in the verification of solubility conditions for the integral equations determining the transport coefficients. These are the primary new features of this work.

The evaluation of the transport coefficients for practical results introduces a new approximation, truncation of an expansion for the solutions to the integral equations in polynomials. The leading order truncation is known to be accurate to approximately 5% in the case of elastic collisions. Exceptions are extreme mass ratios (e.g., electron–proton systems). Its validity for inelastic systems has been recently checked by Monte Carlo simulation for shear viscosity. The accuracy is found to be similar to that for elastic collisions.

Recently, a seemingly similar analysis for granular binary mixtures has been given in Ref. 16, also based on a two temperature description. However, this work is phenomenological with no attempt to solve the kinetic equation. Instead, it is assumed that the distribution function is a local Maxwellian. This is reasonable for estimating the dense gas collisional transfer contributions to the fluxes, but it predicts that all transport coefficients calculated here at low density should vanish. Clearly, the phenomenology is flawed.

In a subsequent paper, we will evaluate the expressions for the different transport coefficients for a variety of mass and diameter ratios. Here, as an illustration and to give some insight into the influence of dissipation on transport, we consider the pressure diffusion coefficient D_p and the diffusion coefficient D . For the sake of concreteness, consider the case $\alpha_{11} = \alpha_{22} = \alpha_{12} \equiv \alpha$. Figure 1 shows the reduced pressure diffusion coefficient $D_p(\alpha)/D_p(1)$ as a function of the restitution coefficient for $\sigma_{11} = \sigma_{22}$, $n_1/n_2 = 0.25$, and for two values of the mass ratio ($m_1/m_2 = 0.5$ and 4). Here, $D_p(1) = (n_1 T / v_e \rho)(1 - m_1 n / \rho)$ is the pressure diffusion coefficient

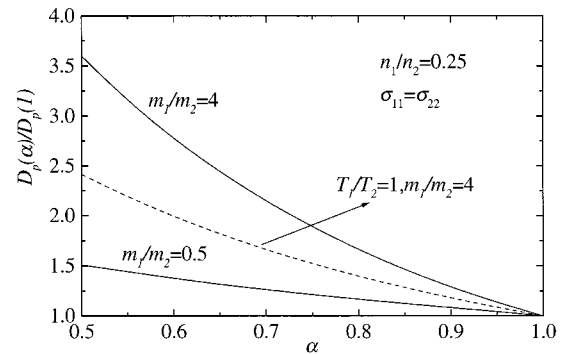


FIG. 1. Plot of the reduced pressure diffusion coefficient $D_p(\alpha)/D_p(1)$ as a function of the restitution coefficient $\alpha \equiv \alpha_{11} = \alpha_{22} = \alpha_{12}$ for $\sigma_{11} = \sigma_{22} = \sigma_{12}$, a concentration ratio $n_1/n_2 = 0.25$, and two different values of the mass ratio: $m_1/m_2 = 0.5$ and $m_1/m_2 = 4$. The dashed line refers to the case $m_1/m_2 = 4$ by assuming the equality of the partial temperatures $\gamma = T_1/T_2 = 1$.

for elastic collisions. We see that the deviation from the functional form for elastic collisions is quite important even for moderate dissipation (say $\alpha \approx 0.9$). This tendency becomes more significant as the mass of the defect particles is larger than that of the excess particles. Also shown for comparison in Fig. 1 is the result for $m_1/m_2 = 4$ with $T_1 = T_2 = 1$, which would be obtained if the differences in the partial temperatures were neglected. Clearly, inclusion of this effect makes a significant difference over the whole range of dissipation shown (the actual value is $\gamma = T_1/T_2 \approx 1.36$ for $\alpha = 0.8$). At the level of the mass flux, the main transport coefficient is the diffusion coefficient D . In Fig. 2 we plot the dependence of the ratio $D(\alpha)/D(1)$ on α for $\sigma_{11} = \sigma_{22}$, $n_1/n_2 = 0.25$, and for several values of the mass ratio ($m_1/m_2 = 0.5, 1$, and 4). As before, $D(1) = (\rho T / m_1 m_2 v_e)$ is the diffusion coefficient for elastic collisions. The shape of these curves is very similar to those presented for the coefficient D_p , although the influence of dissipation on D is a bit stronger than the one observed in the case of D_p . According to the behavior of D and D_p one can conclude that the main effect of inelasticity in collisions is to enhance the mass transport with respect to the elastic collisions case.

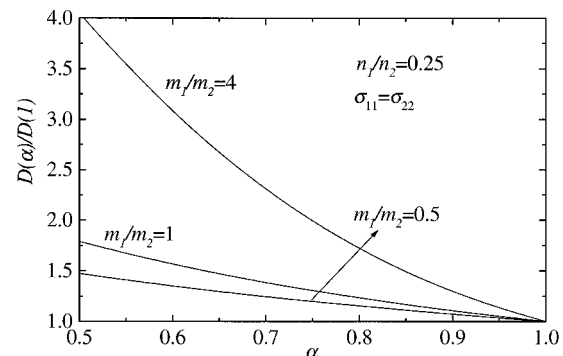


FIG. 2. Plot of the reduced diffusion coefficient $D(\alpha)/D(1)$ as a function of the restitution coefficient $\alpha \equiv \alpha_{11} = \alpha_{22} = \alpha_{12}$ for $\sigma_{11} = \sigma_{22} = \sigma_{12}$, a concentration ratio $n_1/n_2 = 0.25$, and three different values of the mass ratio: $m_1/m_2 = 0.5$, $m_1/m_2 = 1$, and $m_1/m_2 = 4$.

The application of the Chapman–Enskog procedure here closely follows recent derivations of hydrodynamics and transport coefficients for one-component granular gases.^{1,5} An important difference in the mixture is the need for two different temperatures in the reference local homogeneous cooling states, leading to qualitative differences in the concentration dependence of the transport coefficients relative to the elastic limit. This reference state is discussed in more detail in Ref. 10. Effectively, the absence of energy conservation for granular gases leads to a failure of detailed balance between the velocity distributions in the homogeneous state. One consequence is non-Maxwellian distributions observed already for one-component systems. For multicomponent systems a second consequence is different covariances of the distributions for different species, although the cooling rates are all equal. The latter property implies that the different partial temperatures can be expressed in terms of the global temperature (as is required for a hydrodynamic description) although the functional relationships defining such partial temperatures introduce a new dependence on the concentrations. This leads to additional spatial gradients at first order in the Chapman–Enskog expansion and consequently additional contributions to the transport coefficients. The effect of these terms can be significant, as illustrated in Fig. 1. Previous work on granular mixtures^{6–9} is restricted by weak dissipation approximations. In addition, these studies do not include contributions from the partial temperature differences, which are significant even for weak dissipation.

One interesting question is whether the mixture hydrodynamics is more or less stable than that of the one-component case, and if such phenomena as phase separation or segregation can occur. Another direction of study is the extension of the present simple hydrodynamic state to higher densities based on the revised Enskog kinetic equation, following recent results for the one-component fluid.⁵ In this case the new complexity is due to the dependence of the collision operator on concentration through the pair correlation functions for the different species. Finally, we hope that the present results stimulate the performance of computer simulations (molecular dynamics and/or by using the direct simulation Monte Carlo method¹⁷) to study hydrodynamics in granular binary mixtures.

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APPENDIX A: LEADING SONINE APPROXIMATIONS

In this appendix the coefficients $\{a_{i,j}; b_{i,j}; c_{i,j}\}$ in the leading Sonine approximations are evaluated. First, let us consider the coefficients $\{a_{1,1}; b_{1,1}; c_{1,1}\}$ determining the mass flux. Substitution of Eq. (65) into the integral equations (46)–(48) gives

$$\begin{aligned}
 & -\zeta^{(0)}(T\partial_T + p\partial_p)a_{1,1}f_{1,M}\mathbf{V}_1 + a_{1,1}[\mathcal{L}_1 f_{1,M}\mathbf{V}_1 \\
 & - \delta\gamma\mathcal{M}_1 f_{2,M}\mathbf{V}_1] \\
 & = \mathbf{A}_1 + \left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T} f_{1,M}(pb_{1,1} + Tc_{1,1})\mathbf{V}_1, \tag{A1}
 \end{aligned}$$

$$\begin{aligned}
 & [-\zeta^{(0)}(T\partial_T + p\partial_p) - 2\zeta^{(0)}]b_{1,1}f_{1,M}\mathbf{V}_1 + b_{1,1}[\mathcal{L}_1 f_{1,M}\mathbf{V}_1 \\
 & - \delta\gamma\mathcal{M}_1 f_{2,M}\mathbf{V}_1] \\
 & = \mathbf{B}_1 + \frac{T\zeta^{(0)}}{p} f_{1,M}c_{1,1}\mathbf{V}_1, \tag{A2}
 \end{aligned}$$

$$\begin{aligned}
 & [-\zeta^{(0)}(T\partial_T + p\partial_p) - \frac{1}{2}\zeta^{(0)}]c_{1,1}f_{1,M}\mathbf{V}_1 + c_{1,1}[\mathcal{L}_1 f_{1,M}\mathbf{V}_1 \\
 & - \delta\gamma\mathcal{M}_1 f_{2,M}\mathbf{V}_1] = \mathbf{C}_1 - \frac{p\zeta^{(0)}}{2T} f_{1,M}b_{1,1}\mathbf{V}_1. \tag{A3}
 \end{aligned}$$

Next, multiplication by $m_1\mathbf{V}_1$ and integrating over the velocity leads to

$$\begin{aligned}
 & [-\zeta^{(0)}(T\partial_T + p\partial_p) + \nu]n_1T_1a_{1,1} \\
 & = \frac{1}{3} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot \mathbf{A}_1 + \left(\frac{\partial\zeta^{(0)}}{\partial x_1}\right)_{p,T} n_1T_1(pb_{1,1} + Tc_{1,1}), \tag{A4}
 \end{aligned}$$

$$\begin{aligned}
 & [-\zeta^{(0)}(T\partial_T + p\partial_p) - 2\zeta^{(0)} + \nu]n_1T_1b_{1,1} \\
 & = \frac{1}{3} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot \mathbf{B}_1 + \frac{T\zeta^{(0)}}{p} n_1T_1c_{1,1}, \tag{A5}
 \end{aligned}$$

$$\begin{aligned}
 & \left[-\zeta^{(0)}(T\partial_T + p\partial_p) - \frac{1}{2}\zeta^{(0)} + \nu\right]n_1T_1c_{1,1} \\
 & = \frac{1}{3} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot \mathbf{C}_1 - \frac{p\zeta^{(0)}}{2T} n_1T_1b_{1,1}. \tag{A6}
 \end{aligned}$$

Here ν is a collision frequency defined by

$$\begin{aligned}
 \nu & = \frac{1}{3n_1T_1} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot [\mathcal{L}_1 f_{1,M}\mathbf{V}_1 - \delta\gamma\mathcal{M}_1 f_{2,M}\mathbf{V}_2] \\
 & = -\frac{1}{3n_1T_1} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot (J_{12}[\mathbf{v}_1|f_{1,M}\mathbf{V}_1, f_2^{(0)}] \\
 & - \delta\gamma J_{12}[\mathbf{v}_1|f_1^{(0)}, f_{2,M}\mathbf{V}_2]). \tag{A7}
 \end{aligned}$$

The self-collision terms of \mathcal{L}_i arising from J_{11} do not occur in Eq. (A7) since these conserve momentum for species 1. The velocity integrals appearing in Eqs. (A4)–(A6) can be performed using Eqs. (39)–(41),

$$\frac{1}{3} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot \mathbf{A}_1 = -\left(\frac{\partial}{\partial x_1} n_1T_1\right)_{p,T}, \tag{A8}$$

$$\frac{1}{3} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot \mathbf{B}_1 = -\frac{n_1T_1}{p} \left(1 - \frac{m_1nT}{\rho T_1}\right), \tag{A9}$$

$$\frac{1}{3} \int d\mathbf{V}_1 m_1\mathbf{V}_1 \cdot \mathbf{C}_1 = 0. \tag{A10}$$

From dimensional analysis $n_1 T_1 a_{1,1} \sim T^{1/2}$, $n_1 T_1 b_{1,1} \sim T^{1/2}/p$, and $n_1 T_1 c_{1,1} \sim T^{-1/2}$ so the temperature derivatives can be performed in Eqs. (A4)–(A6) and the result is

$$\left(\nu - \frac{1}{2}\zeta^{(0)}\right)a_{1,1} = -\left(\frac{\partial}{\partial x_1} \ln n_1 T_1\right)_{p,T} + \left(\frac{\partial \zeta^{(0)}}{\partial x_1}\right)_{p,T} \times (pb_{1,1} + Tc_{1,1}), \quad (A11)$$

$$\left(\nu - \frac{3}{2}\zeta^{(0)}\right)b_{1,1} = -\frac{1}{p}\left(1 - \frac{m_1 n T}{\rho T_1}\right) + \frac{T\zeta^{(0)}}{p}c_{1,1}, \quad (A12)$$

$$c_{1,1} = -\frac{p\zeta^{(0)}}{2T\nu}b_{1,1}. \quad (A13)$$

The solutions are given by Eqs. (69)–(71).

The analysis for the coefficients $\{a_{1,2}; b_{1,2}; c_{1,2}\}$ (Dufour coefficient, pressure energy coefficient, and thermal conductivity) is similar to that given previously for the mass flux. The result is

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) + \nu_{11}]T^3 a_{1,2} + T^3 \nu_{12} a_{2,2} = \frac{2}{15} \frac{m_1 T^3}{n_1 T_1^3} \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{A}_{12} + T^3 \left(\frac{\partial \zeta^{(0)}}{\partial x_1}\right)_{p,T} (pb_{1,2} + Tc_{1,2}), \quad (A14)$$

$$[-\zeta^{(0)}(T\partial_T + p\partial_p) - 2\zeta^{(0)} + \nu_{11}]T^3 b_{1,2} + T^3 \nu_{12} b_{2,2} = \frac{2}{15} \frac{m_1 T^3}{n_1 T_1^3} \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{B}_{12} + \frac{T^4 \zeta^{(0)}}{p} c_{1,2}, \quad (A15)$$

$$\left[-\zeta^{(0)}(T\partial_T + p\partial_p) - \frac{1}{2}\zeta^{(0)} + \nu_{11}\right]T^3 c_{1,2} + T^3 \nu_{12} c_{2,2} = \frac{2}{15} \frac{m_1 T^3}{n_1 T_1^3} \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{C}_{12} - \frac{1}{2} p T^2 \zeta^{(0)} c_{1,2}. \quad (A16)$$

Here, the collision frequencies ν_{ii} and ν_{ij} are defined by Eqs. (96) and (97), respectively, and

$$\mathbf{A}_{12} = \mathbf{A}_1 - \frac{m_1 m_2 n}{\rho n_1 \gamma_1} \zeta^{(0)} (T\partial_T + p\partial_p) \frac{D}{T} \mathbf{V}_1 f_{1,M} + \frac{m_1 m_2 n}{\rho n_1 T_1} D [\mathcal{L}_1(f_{1,M} \mathbf{V}_1) - \delta\gamma \mathcal{M}_1(f_{2,M} \mathbf{V}_1)] - \left(\frac{\partial \zeta^{(0)}}{\partial x_1}\right)_{p,T} \frac{\rho}{n_1 T_1} (D_p + D') f_{1,M} \mathbf{V}_1, \quad (A17)$$

$$\mathbf{B}_{12} = \mathbf{B}_1 - \frac{\rho}{n_1 \gamma_1} \zeta^{(0)} (T\partial_T + p\partial_p) \frac{D_p}{pT} \mathbf{V}_1 f_{1,M} + \frac{\rho}{p n_1 T_1} D_p \times [\mathcal{L}_1(f_{1,M} \mathbf{V}_1) - \delta\gamma \mathcal{M}_1(f_{2,M} \mathbf{V}_1)] - \zeta^{(0)} \frac{\rho}{p n_1 T_1} (D' - 2D_p) f_{1,M} \mathbf{V}_1, \quad (A18)$$

$$\mathbf{C}_{12} = \mathbf{C}_1 - \frac{\rho}{n_1 \gamma_1} \zeta^{(0)} (T\partial_T + p\partial_p) \frac{D'}{T^2} \mathbf{V}_1 f_{1,M} + \frac{\rho}{T n_1 T_1} D' \times [\mathcal{L}_1(f_{1,M} \mathbf{V}_1) - \delta\gamma \mathcal{M}_1(f_{2,M} \mathbf{V}_1)] - \frac{1}{2} \zeta^{(0)} \frac{\rho}{T n_1 T_1} (D' - D_p) f_{1,M} \mathbf{V}_1, \quad (A19)$$

where $\gamma_i = T_i/T$. The corresponding integral equations for $a_{2,2}$, $b_{2,2}$, and $c_{2,2}$ can be obtained from Eqs. (A14) to (A16) by just making the change $1 \leftrightarrow 2$. Dimensional analysis requires that $a_{1,2} \sim T^{-3/2}$, $b_{1,2} \sim T^{-3/2}/p$, and $c_{1,2} \sim T^{-5/2}$. When one takes into account this dependence and the forms of \mathbf{A}_{ij} , \mathbf{B}_{ij} , and \mathbf{C}_{ij} , one finally arrives to Eq. (94).

APPENDIX B: EVALUATION OF ν

The collision frequency ν is defined by the collision integrals in Eq. (A7). To simplify the integrals, a useful identity for an arbitrary function $h(\mathbf{v}_1)$ is given by

$$\int d\mathbf{v}_1 h(\mathbf{v}_1) J_{ij}[\mathbf{v}_1 | f_i, f_j] = \sigma_{ij}^2 \int d\mathbf{v}_1 \int d\mathbf{v}_2 f_i(\mathbf{r}, \mathbf{v}_1, t) f_j(\mathbf{r}, \mathbf{v}_2, t) \times \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) [h(\mathbf{v}'_1) - h(\mathbf{v}_1)], \quad (B1)$$

with

$$\mathbf{v}'_1 = \mathbf{v}_1 - \mu_{ji} (1 + \alpha_{ij}) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \hat{\boldsymbol{\sigma}}. \quad (B2)$$

This result applies for both $i = j$ and $i \neq j$. Use of this in Eq. (A7) gives

$$\nu = -\frac{m_1}{3n_1 T_1} \int d\mathbf{V}_1 \mathbf{V}_1 \cdot (J_{12}[f_{1,M} \mathbf{V}_1, f_2^{(0)}] - \delta\gamma J_{12}[f_1^{(0)}, f_{2,M} \mathbf{V}_2]) = \frac{1}{6} \pi \sigma_{12}^2 \mu_{21} (1 + \alpha_{12}) \frac{m_1}{n_1 T_1} \int d\mathbf{V}_1 \int d\mathbf{V}_2 \times g_{12} [f_{1,M}(\mathbf{V}_1) f_2^{(0)}(\mathbf{V}_2) (\mathbf{V}_1 \cdot \mathbf{g}_{12}) - \delta\gamma f_1^{(0)}(\mathbf{V}_1) f_{2,M}(\mathbf{V}_2) (\mathbf{V}_2 \cdot \mathbf{g}_{12})], \quad (B3)$$

where $\gamma = T_1/T_2$ and $\delta = n_1/n_2$. Substitution of the distribution functions from Eqs. (63) and (64) gives

$$\nu = \frac{n_2}{3\pi^2(1+\mu)} (1 + \alpha_{12}) \sigma_{12}^2 \theta_1^{5/2} \theta_2^{3/2} v_0 \times \int d\mathbf{V}_1^* \int d\mathbf{V}_2^* e^{-\theta_1 V_1^{*2}} e^{-\theta_2 V_2^{*2}} g_{12}^* \times \left\{ \left[1 + \frac{c_2}{4} \left(\theta_2^2 V_2^{*4} - 5\theta_2 V_2^{*2} + \frac{15}{4} \right) \right] (\mathbf{V}_1^* \cdot \mathbf{g}_{12}^*) - \delta\gamma \left[1 + \frac{c_1}{4} \left(\theta_1^2 V_1^{*4} - 5\theta_1 V_1^{*2} + \frac{15}{4} \right) \right] (\mathbf{V}_2^* \cdot \mathbf{g}_{12}^*) \right\}$$

$$= \frac{(1 + \alpha_{12})\sigma_{12}^2}{3\pi^2\gamma_1\gamma_2} (\theta_1\theta_2)^{3/2} v_0 [n_2\gamma_2 X(c_2, \theta_1, \theta_2) + n_1\gamma_1 X(c_1, \theta_2, \theta_1)]. \tag{B4}$$

Here, $\mu = m_1/m_2$, $\mathbf{V}_i^* = \mathbf{V}_i/v_0$, $\mathbf{g}_{12}^* = \mathbf{g}_{12}/v_0$, $\theta_i = (\mu_{ji}\gamma_i)^{-1}$, and $v_0 = \sqrt{2T(m_1+m_2)/m_1m_2}$. In addition, the quantities $X(c_2, \theta_1, \theta_2)$ and $I(\theta_1, \theta_2)$ are given by

$$X(c_2, \theta_1, \theta_2) = \left[1 + \frac{c_2}{4} \left(\theta_2^2 \frac{d^2}{d\theta_2^2} + 5\theta_2 \frac{d}{d\theta_2} + \frac{15}{4} \right) \right] \times I(\theta_1, \theta_2), \tag{B5}$$

$$I(\theta_1, \theta_2) = \int d\mathbf{V}_1^* \int d\mathbf{V}_2^* e^{-\theta_1 V_1^{*2}} e^{-\theta_2 V_2^{*2}} g_{12}^*(\mathbf{V}_1^* \cdot \mathbf{g}_{12}^*). \tag{B6}$$

The integral $I(\theta_1, \theta_2)$ can be performed by the change of variables

$$\mathbf{x} = \mathbf{V}_1^* - \mathbf{V}_2^*, \quad \mathbf{y} = \theta_1 \mathbf{V}_1^* + \theta_2 \mathbf{V}_2^*, \tag{B7}$$

with the Jacobian $(\theta_1 + \theta_2)^{-3}$. The integral becomes

$$I(\theta_1, \theta_2) = 4\pi^{5/2} \frac{(\theta_1 + \theta_2)^{1/2}}{\theta_2^2 \theta_1^3}. \tag{B8}$$

Use of this result in (B5) and (B3) gives the desired result

$$\nu = \frac{4}{3} \sqrt{\pi} \sigma_{12}^2 v_0 \frac{1 + \alpha_{12}}{\gamma_1 \gamma_2} (\theta_1 \theta_2 (\theta_1 + \theta_2))^{-3/2} \times \left[(n_2 \gamma_2 \theta_2 + n_1 \gamma_1 \theta_1) (\theta_1 + \theta_2)^2 - \frac{1}{16} (c_1 n_1 \gamma_1 \theta_1 \theta_2^2 + c_2 n_2 \gamma_2 \theta_2 \theta_1^2) \right]. \tag{B9}$$

This leads directly to the result given by Eq. (74).

APPENDIX C: EVALUATION OF τ_{ij} AND τ_{ij}

The collision frequencies τ_{ii} and τ_{ij} appearing in the evaluation of the shear viscosity are defined by Eqs. (83) and (84). All of these have the form

$$\int d\mathbf{v}_1 \mathbf{V}_1 \mathbf{V}_1 J_{ij}[\mathbf{v}_1 | f_i, f_j] = \sigma_{ij}^2 \int d\mathbf{v}_1 \int d\mathbf{v}_2 f_i(\mathbf{v}_1) f_j(\mathbf{v}_2) \int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \times [\mathbf{V}_1'' \mathbf{V}_1'' - \mathbf{V}_1 \mathbf{V}_1], \tag{C1}$$

where the identity (B1) has been used. This result applies for both $i=j$ and $i \neq j$. Using the scattering rule (B2), the last term on the right-hand side can be explicitly computed as

$$\mathbf{V}_1'' \mathbf{V}_1'' - \mathbf{V}_1 \mathbf{V}_1 = -\mu_{ji}(1 + \alpha_{ij})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) [\mathbf{G}_{ij} \boldsymbol{\sigma} + \boldsymbol{\sigma} \mathbf{G}_{ij} + \mu_{ji}(\mathbf{g}_{12} \boldsymbol{\sigma} + \boldsymbol{\sigma} \mathbf{g}_{12}) - \mu_{ji}(1 + \alpha_{ij}) \times (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) \boldsymbol{\sigma} \boldsymbol{\sigma}]. \tag{C2}$$

Here, $\mathbf{G}_{ij} = \mu_{ij} \mathbf{V}_1 + \mu_{ji} \mathbf{V}_2$. Substitution of (C2) into Eq. (C1) allows the angular integral to be performed with the result

$$\int d\hat{\boldsymbol{\sigma}} \Theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) [\mathbf{V}_1'' \mathbf{V}_1'' - \mathbf{V}_1 \mathbf{V}_1] = -\frac{m_i}{2} \pi \mu_{ji} (1 + \alpha_{ij}) \left[g_{12} (\mathbf{G}_{ij} \mathbf{g}_{12} + \mathbf{g}_{12} \mathbf{G}_{ij}) + \frac{\mu_{ji}}{2} (3 - \alpha_{ij}) g_{12} \mathbf{g}_{12} \mathbf{g}_{12} - \frac{\mu_{ji}}{6} (1 + \alpha_{ij}) g^3 \mathbf{1} \right]. \tag{C3}$$

Notice that the last term in (C3) vanishes when it is contracted with the traceless tensor \mathbf{R}_i . Now, the different collision integrals can be easily calculated by the same method as described in Appendix B. After a lengthy calculation, one gets

$$\mathcal{A}_{12} = \int d\mathbf{v}_1 R_{1,\alpha\beta} J_{12} [f_1^{(0)}, f_{2,M} R_{2,\alpha\beta}] = -\frac{4}{3} m_1 m_2 n_1 n_2 \sqrt{\pi} \mu_{21} (1 + \alpha_{12}) \sigma_{12}^2 v_0^5 (\theta_1 \theta_2)^{-1/2} \times \left[6\theta_2^{-2} (\mu_{12} \theta_2 - \mu_{21} \theta_1) (\theta_1 + \theta_2)^{-1/2} + \frac{3}{2} \mu_{21} \theta_2^{-2} (\theta_1 + \theta_2)^{1/2} (3 - \alpha_{12}) - 5\theta_2^{-1} (\theta_1 + \theta_2)^{-1/2} + \frac{c_1}{16} \frac{2\theta_1(10 - 12\mu_{12} - 9\mu_{21}) + \theta_2(5 - 6\mu_{12}) - \frac{3}{2} \mu_{21}(3 - \alpha_{12})(\theta_1 + \theta_2)}{(\theta_1 + \theta_2)^{5/2}} \right], \tag{C4}$$

$$\begin{aligned}
 \mathcal{B}_{12} &= \int d\mathbf{v}_1 R_{1,\alpha\beta} J_{12} [f_{1,M} R_{1,\alpha\beta} f_2^{(0)}] \\
 &= -\frac{4}{3} m_1^2 n_1 n_2 \sqrt{\pi} \mu_{21} (1 + \alpha_{12}) \sigma_{12}^2 v_0^5 (\theta_1 \theta_2)^{-1/2} \\
 &\quad \times \left[6 \theta_1^{-2} (\mu_{12} \theta_2 - \mu_{21} \theta_1) (\theta_1 + \theta_2)^{-1/2} + \frac{3}{2} \mu_{21} \theta_1^{-2} (\theta_1 + \theta_2)^{1/2} (3 - \alpha_{12}) \right. \\
 &\quad \left. + 5 \theta_1^{-1} (\theta_1 + \theta_2)^{-1/2} + \frac{c_2}{16} \frac{2 \theta_2 (12 \mu_{21} + 9 \mu_{12} - 10) - \theta_1 (5 - 6 \mu_{21}) - \frac{3}{2} \mu_{21} (3 - \alpha_{12}) (\theta_1 + \theta_2)}{(\theta_1 + \theta_2)^{5/2}} \right], \tag{C5}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_{11} + \mathcal{B}_{11} &= \int d\mathbf{v}_1 R_{1,\alpha\beta} \{ J_{11} [f_1^{(0)}, f_{1,M} R_{1,\alpha\beta}] + J_{11} [f_{1,M} R_{1,\alpha\beta}, f_1^{(0)}] \} \\
 &= -32 m_1^2 n_1^2 \sqrt{\pi} (1 + \alpha_{11}) \sigma_1^2 (T_1 / m_1)^{5/2} \left[1 - \frac{1}{4} (1 - \alpha_{11})^2 \right] \left(1 - \frac{c_1}{64} \right). \tag{C6}
 \end{aligned}$$

In the case of mechanically equivalent particles ($m_1 = m_2 = m$, $\sigma_{ij} = \sigma$, $\alpha_{ij} = \alpha$, $c_i = c$), Eqs. (C4)–(C6) reduce to those previously calculated in the single gas case in the determination of the shear viscosity.¹

This completely determines τ_{11} and τ_{12} . The corresponding expressions for τ_{22} and τ_{21} can be inferred from Eqs. (C4), (C5), and (C6) by interchanging $1 \leftrightarrow 2$.

APPENDIX D: EVALUATION OF Y_i , ν_{ii} , AND ν_{ij}

The collision frequencies ν_{ii} , ν_{ij} and Y_i ($i = 1, \dots, 6$) that determine the coefficients $a_{i,2}$, $b_{i,2}$, and $c_{i,2}$ for the heat flux

are defined by the collision integrals (96)–(101). All of these have the form

$$\begin{aligned}
 &\int d\mathbf{v}_1 \mathbf{S}_i(\mathbf{V}_1) J_{ij} [\mathbf{v}_1 | f_i, f_j] \\
 &= \sigma_{ij}^2 \int d\mathbf{v}_1 \int d\mathbf{v}_2 f_i(\mathbf{v}_1) f_j(\mathbf{v}_2) \\
 &\quad \times \int d\hat{\boldsymbol{\theta}} \Theta(\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) [\mathbf{S}_i(\mathbf{V}_i'') - \mathbf{S}_i(\mathbf{V}_1)]. \tag{D1}
 \end{aligned}$$

Using the scattering rule (B2), the last term on the right-hand side can be explicitly computed as

$$\begin{aligned}
 \mathbf{S}_i(\mathbf{V}_i'') - \mathbf{S}_i(\mathbf{V}_1) &= \frac{m_i}{2} (1 + \alpha_{ij}) \mu_{ji} (\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) \left\{ \left[(1 - \alpha_{ij}^2) \mu_{ji}^2 (\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12})^2 - G_{ij}^2 - \mu_{ji}^2 g_{12}^2 - 2 \mu_{ji} (g_{12} \cdot \mathbf{G}_{ij}) + 2(1 + \alpha_{ij}) \mu_{ji} (\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) \right. \right. \\
 &\quad \left. \left. \times (\hat{\boldsymbol{\theta}} \cdot \mathbf{G}_{ij}) - \frac{5T_i}{m_i} \right] \hat{\boldsymbol{\theta}} - [(1 - \alpha_{ij}) \mu_{ji} (\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) + 2(\hat{\boldsymbol{\theta}} \cdot \mathbf{G}_{ij})] \mathbf{G}_{ij} - \mu_{ji} [(1 - \alpha_{ij}) \mu_{ji} (\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) + 2(\hat{\boldsymbol{\theta}} \cdot \mathbf{G}_{ij})] \mathbf{g}_{12} \right\}. \tag{D2}
 \end{aligned}$$

Substitution of (D2) into (D1) allows the angular integral to be performed with the result

$$\begin{aligned}
 &\int d\hat{\boldsymbol{\theta}} \Theta(\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\theta}} \cdot \mathbf{g}_{12}) [\mathbf{S}_i(\mathbf{V}_i'') - \mathbf{S}_i(\mathbf{V}_1)] \\
 &= -\frac{m_i}{2} \pi (1 + \alpha_{ij}) \mu_{ji} \left\{ \left[\frac{1}{2} g_{12} G_{ij}^2 + \frac{1}{6} \mu_{ji}^2 (2\alpha_{ij}^2 - 3\alpha_{ij} + 4) g_{12}^3 - \frac{1}{2} \mu_{ji} (\alpha_{ij} - 3) g_{12} (\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) - \frac{5T_i}{2m_i} g_{12} \right] \mathbf{g}_{12} \right. \\
 &\quad \left. + \left[g_{12} (\mathbf{g}_{12} \cdot \mathbf{G}_{ij}) - \frac{1}{3} \mu_{ji} (2\alpha_{ij} - 1) g_{12}^3 \right] \mathbf{G}_{ij} \right\}. \tag{D3}
 \end{aligned}$$

Now the different collision integrals can be evaluated.

1. Evaluation of Y_i

The coefficients $Y_{1,3,5}$ are obtained from Eqs. (99) to (101). The collision integrals appearing in these expressions can be evaluated directly by using identical mathematical steps as before. After some algebra, the result is

$$\begin{aligned}
 C_{12} &= \int d\mathbf{v}_1 \mathbf{S}_1 \cdot J_{12}[f_1^{(0)}, f_{2,M} \mathbf{V}_2] \\
 &= -\frac{1}{2} m_1 n_1 n_2 \sqrt{\pi} \mu_{21} (1 + \alpha_{12}) \sigma_{12}^2 v_0^5 (\theta_1 + \theta_2)^{-1/2} (\theta_1 \theta_2)^{-3/2} \\
 &\quad \times \left\{ 5(2\beta_{12} - \theta_1) + \mu_{21}(\theta_1 + \theta_2)[5(1 - \alpha_{12}) + 2(7\alpha_{12} - 11)\beta_{12}\theta_2^{-1}] - 18\beta_{12}^2\theta_2^{-1} - 2\mu_{21}^2(2\alpha_{12}^2 - 3\alpha_{12} + 4) \right. \\
 &\quad \times \theta_2^{-1}(\theta_1 + \theta_2)^2 + 5(\theta_1 + \theta_2) - \frac{c_1}{16}(\theta_1 + \theta_2)^{-2}\theta_2\{4\theta_1^2[-5 + 36\mu_{12}^2 + 5\mu_{21}(\alpha_{12} - 3) + \mu_{21}^2(25 - 8\alpha_{12} + 3\alpha_{12}^2)] \\
 &\quad - 2\mu_{12}(10 + \mu_{21}(7\alpha_{12} - 29))\} + \theta_2^2[5 + 54\mu_{12}^2 + 15\mu_{21}(\alpha_{12} - 1) + 6\mu_{21}^2(4 - 3\alpha_{12} + 2\alpha_{12}^2) - 6\mu_{12}(5 + \mu_{21}(7\alpha_{12} - 11))] \\
 &\quad \left. - \theta_1\theta_2[-144\mu_{12}^2 + 2\mu_{12}(40 + \mu_{21}(49\alpha_{12} - 95)) + \mu_{21}(45 - 35\alpha_2 - 2\mu_{21}(35 - 25\alpha_{12} + 12\alpha_{12}^2))] \right\}, \tag{D4}
 \end{aligned}$$

$$\begin{aligned}
 D &= \int d\mathbf{v}_1 \mathbf{S}_1 \cdot J_{12}[f_{1,M} \mathbf{V}_1, f_2^{(0)}] \\
 &= -\frac{1}{2} m_1 n_1 n_2 \sqrt{\pi} \mu_{21} (1 + \alpha_{12}) \sigma_{12}^2 v_0^5 (\theta_1 + \theta_2)^{-12} (\theta_1 \theta_2)^{-3/2} \\
 &\quad \times \left\{ 5(2\beta_{12} + \theta_2) + \mu_{21}(\theta_1 + \theta_2)[5(1 - \alpha_{12}) - 2(7\alpha_{12} - 11)\beta_{12}\theta_1^{-1}] + 18\beta_{12}^2\theta_1^{-1} + 2\mu_{21}^2(2\alpha_{21}^2 - 3\alpha_{12} + 4)\theta_1^{-1}(\theta_1 + \theta_2)^2 \right. \\
 &\quad - 5\theta_2\theta_1^{-1}(\theta_1 + \theta_2) + \frac{c_2}{16}(\theta_1 + \theta_2)^{-2}\theta_1\{3\theta_1^2\mu_{21}(1 + \alpha_{12})[4\mu_{21}(1 + \alpha_{12}) - 5] + \theta_2^2[-15 + 54\mu_{12}^2 - 20\mu_{21}(3 + \alpha_{12}) \\
 &\quad + 2\mu_{21}^2(40 + 19\alpha_{12} + 6\alpha_{12}^2) + 2\mu_{12}(-20 + \mu_{21}(61 + 7\alpha_{12}))\} + \theta_1\theta_2[2\mu_{12}(-5 + 7\mu_{21}(1 + \alpha_{12})) + \mu_{21}(-5(9 + 7\alpha_{12}) \\
 &\quad \left. + \mu_{21}(38 + 62\alpha_{12} + 24\alpha_{12}^2))] \right\}, \tag{D5}
 \end{aligned}$$

$$\begin{aligned}
 C_{11} + D_{11} &= \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \{J_{11}[f_1^{(0)}, f_{1,M} \mathbf{V}_2] + J_{11}[f_{1,M} \mathbf{V}_1, f_1^{(0)}]\} \\
 &= -10\sqrt{\pi} m_1 \sigma_{11}^2 n_1^2 (T_1 / m_1)^{5/2} (1 + \alpha_{11}) \left[1 - \alpha_{11} + \frac{1}{320} c_1 (21\alpha_{11} - 53) \right]. \tag{D6}
 \end{aligned}$$

In the above-given expressions we have introduced the quantity

$$\beta_{12} = \mu_{12}\theta_2 - \mu_{21}\theta_1. \tag{D7}$$

This completely determines the coefficients $Y_{1,3,5}$. The corresponding expressions for $Y_{2,4,6}$ can be inferred from Eqs. (D5) to (D7) by interchanging $1 \leftrightarrow 2$.

2. Evaluation of ν_{ii} and ν_{ij}

The collision frequencies ν_{ii} and ν_{ij} are defined by Eqs. (96) and (97). These collision integrals are evaluated in the same manner as those for Y_i . The result is

$$\begin{aligned}
 \int d\mathbf{v}_1 \mathbf{S}_1 \cdot J_{12}[f_1^{(0)}, f_{2,M} \mathbf{S}_2] &= \int d\mathbf{v}_1 \mathbf{S}_1 \cdot J_{12} \left[f_1^{(0)}, f_{2,M} \frac{m_2}{2} V_2^2 \mathbf{V}_2 - \frac{5}{2} T_2 \mathbf{V}_2 \right] = \mathcal{F}_{12} - \frac{5}{2} T_2 C_{12}, \tag{D8} \\
 \mathcal{F}_{12} &= \int d\mathbf{v}_1 \mathbf{S}_1 \cdot J_{12} \left[f_1^{(0)}, f_{2,M} \frac{m_2}{2} V_2^2 \mathbf{V}_2 \right] \\
 &= \frac{1}{8} m_1 n_1 n_2 \sqrt{\pi} \mu_{21} (1 + \alpha_{12}) \sigma_{12}^2 v_0^7 (\theta_1 + \theta_2)^{-3/2} (\theta_1 \theta_2)^{-3/2} \\
 &\quad \times \left\{ 2\mu_{21}^2\theta_2^{-2}(\theta_1 + \theta_2)^2(2\alpha_{12}^2 - 3\alpha_{12} + 4)(8\theta_1 + 5\theta_2) - \mu_{21}(\theta_1 + \theta_2)[2\beta_{12}\theta_2^{-2}(8\theta_1 + 5\theta_2)(7\alpha_{12} - 11) \right. \\
 &\quad \left. - 2\theta_1\theta_2^{-1}(29\alpha_{12} - 37) + 25(1 - \alpha_{12})\right] + 18\beta_{12}^2\theta_2^{-2}(8\theta_1 + 5\theta_2) - 2\beta_{12}\theta_2^{-1}(66\theta_1 + 25\theta_2) + 5\theta_1\theta_2^{-1}(6\theta_1 + 11\theta_2)
 \end{aligned}$$

$$\begin{aligned}
& -5(\theta_1 + \theta_2)\theta_2^{-1}(6\theta_1 + 5\theta_2) + \frac{c_1}{16}(\theta_1 + \theta_2)^{-2}\{16\theta_1^2[5 + 72\mu_{12}^2 + \mu_{12}(-66 + \mu_{21}(137 - 7\alpha_{12})) - 2\mu_{21}(34 + \alpha_{12}) \\
& + \mu_{21}^2(68 - \alpha_{12} + 3\alpha_{12}^2)] + 5\theta_2^3[5 + 54\mu_{12}^2 - 15\mu_{21}(1 - \alpha_{12}) + 6\mu_{21}^2(4 - 3\alpha_{12} + 2\alpha_{12}^2) - 6\mu_{12}(5 + \mu_{21}(7\alpha_{12} - 11))] \\
& + 2\theta_1^2\theta_2[-170 + 504\mu_{12}^2 + \mu_{21}(55\alpha_{12} - 17) + 2\mu_{21}^2(151 - 62\alpha_{12} + 39\alpha_{12}^2) - 8\mu_{12}(9 + 7\mu_{21}(5\alpha_{12} - 13))] \\
& - \theta_1\theta_2^2[20 - 936\mu_{12}^2 + \mu_{21}(251 - 217\alpha_{12}) + \mu_{21}^2(-446 + 332\alpha_{12} - 168\alpha_{12}^2) \\
& + 2\mu_{12}(234 + \mu_{21}(329\alpha_{12} - 607))]\}, \tag{D9}
\end{aligned}$$

$$\int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{J}_{12} [f_{1,M} \mathbf{S}_1, f_2^{(0)}] = \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{J}_{12} \left[f_{1,M} \frac{m_1}{2} V_1^2 \mathbf{V}_1 - \frac{5}{2} T_1 \mathbf{V}_1, f_2^{(0)} \right] = \mathcal{H}_{12} - \frac{5}{2} T_1 \mathcal{D}_{12}, \tag{D10}$$

$$\begin{aligned}
\mathcal{H}_{12} &= \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \mathbf{J}_{12} \left[f_{1,M} \frac{m_1}{2} V_1^2 \mathbf{V}_1, f_2^{(0)} \right] \\
&= -\frac{1}{8} m_1 n_1 n_2 \sqrt{\pi} \mu_{21} (1 + \alpha_{12}) \sigma_{12}^2 v_0^7 (\theta_1 + \theta_2)^{-3/2} (\theta_1 \theta_2)^{-3/2} \\
&\times \left\{ 2\mu_{21}^2 \theta_1^{-2} (\theta_1 + \theta_2)^2 (2\alpha_{12}^2 - 3\alpha_{12} + 4) (5\theta_1 + 8\theta_2) - \mu_{21} (\theta_1 + \theta_2) [2\beta_{12} \theta_1^{-2} (5\theta_1 + 8\theta_2) (7\alpha_{12} - 11) \right. \\
&+ 2\theta_2 \theta_1^{-1} (29\alpha_{12} - 37) - 25(1 - \alpha_{12})] + 18\beta_{12}^2 \theta_1^{-2} (5\theta_1 + 8\theta_2) + 2\beta_{12} \theta_1^{-1} (66\theta_1 + 25\theta_2) + 5\theta_2 \theta_1^{-1} (11\theta_1 + 6\theta_2) \\
&- 5(\theta_1 + \theta_2) \theta_1^{-2} \theta_2 (6\theta_1 + 5\theta_2) + \frac{c_2}{16} (\theta_1 + \theta_2)^{-2} \{15\theta_1^3 \mu_{21} (1 + \alpha_{12}) (4\mu_{21} (1 + \alpha_{12}) - 5) + 2\theta_2^3 [45 + 540\mu_{12}^2 \\
&+ 16\mu_{21} (\alpha_{12} - 36) + 4\mu_{21}^2 (134 + 5\alpha_{12} + 6\alpha_{12}^2) - 4\mu_{12} (148 + \mu_{21} (7\alpha_{12} - 263))] + \theta_1^2 \theta_2 [-30 - \mu_{21} (267 + 217\alpha_{12}) \\
&+ 14\mu_{21}^2 (17 + 29\alpha_{12} + 12\alpha_{12}^2) + 10\mu_{12} (7\mu_{21} (1 + \alpha_{12} - 5))] + \theta_1 \theta_2^2 [-315 + 270\mu_{12}^2 - 2\mu_{21} (55\alpha_{12} + 57) \\
&+ \mu_{21}^2 (440 + 326\alpha_{12} + 156\alpha_{12}^2) + 2\mu_{12} (-2 + \mu_{21} (7\alpha_{12} + 277))] \left. \right\}, \tag{D11}
\end{aligned}$$

$$\begin{aligned}
& \int d\mathbf{v}_1 \mathbf{S}_1 \cdot \{J_{11} [f_1^{(0)}, f_{1,M} \mathbf{S}_1] + J_{11} [f_{1,M} \mathbf{S}_1, f_1^{(0)}]\} \\
&= -8\sqrt{\pi} n_1^2 \sigma_{12}^2 m_1 T_1 (T_1/m_1)^{5/2} (1 + \alpha_{11}) \left[1 + \frac{33}{16} (1 - \alpha_{11}) + \frac{1}{1024} c_1 (19 - 3\alpha_{11}) \right]. \tag{D12}
\end{aligned}$$

In these expressions, \mathcal{C}_{12} and \mathcal{D}_{12} are given by Eqs. (D4) and (D5), respectively.

In the case of mechanically equivalent particles, expression (D12) coincides with the one previously obtained in the context of determining the thermal conductivity in a one-component granular gas.¹ From Eqs. (D9) to (D12), one easily gets the expressions for ν_{22} and ν_{21} by interchanging $1 \leftrightarrow 2$.

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