Physica 149A (1988) 551–560
North-Holland, Amsterdam

ON THE DERIVATION OF THE BURNETT HYDRODYNAMIC EQUATIONS FROM THE HILBERT EXPANSION

V. GARZÓ
Departament de Termología, Facultat de Física, Universitat de València, 46100-Burjassot, Spain

Received 13 May 1987
Revised manuscript received 1 October 1987

We have derived the Burnett-order transport equations by applying the Hilbert expansion to the steady BGK equation. The pressure tensor and heat flux vector are computed for potentials of the form $r^{-\mu}$. The calculated transport coefficients are compared with those obtained by means of the Chapman–Enskog expansion.

1. Introduction

The kinetic theory of dilute gas is based on the Boltzmann equation[1]) (BE). The standard method of solving it is the Chapman–Enskog (C–E) expansion. This method relies upon two assumptions: (a) in the hydrodynamic regime the entire time dependence of the one-particle distribution function f is through the locally conserved variables, density $n(\boldsymbol{r}; t)$, velocity $\boldsymbol{u}(\boldsymbol{r}; t)$ and temperature $T(\boldsymbol{r}; t)$ (normal solution); (b) f is expandable in a power series of the gradients of the thermodynamic fields. When this power expansion is substituted in the expressions of the average fluxes (momentum, heat, . . .), we may derive the Euler, Navier–Stokes, Burnett, . . . hydrodynamic equations in successive approximation. The coefficients that appear in this expansion are the so-called transport coefficients.

However, historically the C–E method was preceded by another expansion due to Hilbert[2]). In this method, the local variables n, $\boldsymbol{u}$ and T are expanded in powers of an auxiliar parameter instead of expanding the transport equations as in the C–E solution. Thus, the formal expressions of the momentum and heat fluxes will be given in terms of the variables which characterize the power expansions of the hydrodynamic fields n, $\boldsymbol{u}$ and T, and they are not in powers of the exact (unexpanded) hydrodynamic variables as in the C–E solution.

Nevertheless, although the Hilbert method is previous to the C–E method, the latter has been used more than the Hilbert one in the last few years. Perhaps, this may be due to the fact that the Hilbert method is principally a formal tool that is only slightly used for explicit calculations. The only solution we are aware of that has applied the Hilbert expansion for solving the BE, has been that of Delale[3]). In this study, the expansion has been done up to the Navier–Stokes approximation. It shows that the thermal conductivity and viscosity coefficients that appear in the steady transport equations are the same as those obtained by the C–E expansion.

But, in general, due to the complex structure of the Boltzmann collision integral it is a very hard task to find explicit results. This problem stimulates the search for simplified models. One expects the most relevant features of the BE to be reasonably well mimicked by other types of equations. Therefore, explicit expressions for the transport coefficients can be obtained using a model of the Boltzmann collision operator, the nonlinear Bhatnagar–Gross–Krook (BGK) model[4]). The BGK equation has been resolved by Cha and McCoy[5]) using the C–E method. Detailed calculations for the pressure tensor and the heat flux vector have been carried out by the super-Burnett hydrodynamic order (third approximation).

The aim of this work is to analyze the Hilbert expansion of the BGK equation. In a previous paper[6]), we studied the Navier–Stokes order in the same way as Delale's work. Now, our purpose is to evaluate the pressure tensor and the heat flux vector to the nonlinear Burnett order. In doing so, the expansion is restricted to steady state to avoid the problem of the initial conditions. On the other hand, we want to note that in this paper we are not interested in questions about the convergence of the Hilbert series. An adequate discussion of this problem can be found in the work of Cercignani[2]).

The Burnett hydrodynamic equations have been extensively dealt with in the literature from the macroscopic point of view[7]) as well as from the kinetic theory[8]) viewpoint. The results have generally the same form, with terms which are linear in second derivatives or quadratic in first derivatives of the thermodynamic variables. According to the proposed Hilbert expansion, the derived Burnett-order relations are analogous to those given by the C–E method. Besides, the transport coefficients that appear in the expansion of the average fluxes happen to be identical to the coefficients given by Cha and McCoy[5]) for the hard spheres interaction. In this way, the transport equations derived from the Hilbert or C–E expansions are in agreement with the conventional results of macroscopic theory[7]).

2. Hilbert expansion of the BGK equation

The BGK model kinetic equation is a version of the nonlinear Boltzmann equation in which the collision integral is replaced by a simple approximation. In terms of an auxiliar parameter ε, which may be set equal to unity at the end of the calculations, the BGK equation can be written as

$$\left(\frac{\partial}{\partial t}+v_j\nabla_j\right)f=-\varepsilon^{-1}\zeta(f-f_{\mathrm{LE}})\,, \tag{1}$$

where $f(\boldsymbol{r},\boldsymbol{v};t)$ is the one-particle distribution function, $\zeta(\boldsymbol{r};t)$ is the collision frequency and $f_{\mathrm{LE}}(\boldsymbol{r},\boldsymbol{v};t)$ is the local equilibrium distribution function defined in the usual form as

$$f_{\mathrm{LE}}=n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{m}{2k_{\mathrm{B}}T}\,(\boldsymbol{v}-\boldsymbol{u})^2\right). \tag{2}$$

Here k_{B} is the Boltzmann constant, m is the mass of a particle, and $n(\boldsymbol{r};t)$, $\boldsymbol{u}(\boldsymbol{r};t)$ and $T(\boldsymbol{r};t)$ are the local density, velocity and temperature, respectively. From f, they are given by

$$n(\boldsymbol{r};t)=\int \mathrm{d}\boldsymbol{v}\; f(\boldsymbol{r},\boldsymbol{v};t)\,, \tag{3}$$

$$n(\boldsymbol{r};t)\boldsymbol{u}(\boldsymbol{r};t)=\int \mathrm{d}\boldsymbol{v}\; \boldsymbol{v}f(\boldsymbol{r},\boldsymbol{v};t)\,, \tag{4}$$

$$\tfrac{3}{2}n(\boldsymbol{r};t)k_{\mathrm{B}}T(\boldsymbol{r};t)=\int \mathrm{d}\boldsymbol{v}\;\frac{m}{2}\,(\boldsymbol{v}-\boldsymbol{u}(\boldsymbol{r};t))^2 f(\boldsymbol{r},\boldsymbol{v};t)\,. \tag{5}$$

The collision frequency ζ is velocity independent but depends on space and time through its dependence on the density and temperatur . In order to carry out explicit calculations, we consider generic interaction potentials of the form $r^{-\mu}$ for which

$$\zeta \propto nT^{\alpha}\,, \tag{6}$$

where $\alpha=\frac{1}{2}-2/\mu$.

The BGK equation is a model that preserves the most relevant properties of the BE. It has five collision invariants (mass, momentum and energy) and verifies an H theorem. Then, by taking moments in velocity space the BGK equation leads to familiar transport equations,

$$\frac{\partial n}{\partial t} + u_j \nabla_j n = -n \nabla_j u_j \,, \tag{7}$$

$$mn\left[\frac{\partial u_i}{\partial t} + u_j \nabla_j u_i\right] = -\nabla_j P_{ij} \,, \tag{8}$$

$$\frac{3}{2} nk_{\mathrm{B}}\left[\frac{\partial T}{\partial t} + u_j \nabla_j T\right] = -\nabla_j J_j - \tfrac{1}{2} P_{ij}(\nabla_i u_j + \nabla_j u_i) \,, \tag{9}$$

where the summation convention has been used and we have introduced the pressure tensor P_{ij} and the heat flux vector J_i given respectively by

$$P_{ij} = \int \mathrm{d}\boldsymbol{v}\, m(v_i - u_i)(v_j - u_j) f \,, \tag{10}$$

$$J_i = \int \mathrm{d}\boldsymbol{v}\, \frac{m}{2}\, (\boldsymbol{v} - \boldsymbol{u})^2 (v_i - u_i) f \,. \tag{11}$$

Assuming that f does not explicitly depends on $\boldsymbol{r}$ and t, we construct a normal solution of the BGK equation in the form of a series,

$$f = \sum_{k=0}^{\infty} \varepsilon^k f^{(k)} \,. \tag{12}$$

Similarly, in the Hilbert theory the hydrodynamic fields n, u_j and T can be expanded in powers of ε:

$$\begin{Bmatrix} n \\ u_j \\ T \end{Bmatrix} = \sum_{k=0}^{\infty} \varepsilon^k \begin{Bmatrix} n^{(k)} \\ u_j^{(k)} \\ T^{(k)} \end{Bmatrix} , \tag{13}$$

where the parameters denoted by $\xi^{(k)} \equiv \{n^{(k)}, u_j^{(k)}, T^{(k)}\}$ are for the moment left arbitrary. Clearly, every function that depends on these variables will be able to be expanded in an analogous way.

In order to get information about the terms appearing in eq. (13) we use the hydrodynamic balance equations (7)–(9). Thus, we must consider the corresponding expansions in (7)–(9) and collect the terms of equal power in ε together. In this way, we obtain a self-consistent solution of the BGK equation since $f(\boldsymbol{r}, \boldsymbol{v}; t)$ reproduces the five hydrodynamic moments n, nu_j and $\frac{3}{2}nk_{\mathrm{B}}T$, for each approximation. Therefore, we obtain for the $f^{(k)}$ approximations the so-called solvability conditions.

When we insert (12), (13) into the BGK equation (1) and separate terms for each order, we obtain the equations

$$
\begin{aligned}
f^{(0)} &= f^{(0)}_{\mathrm{LE}} \,, \\
f^{(1)} &= f^{(1)}_{\mathrm{LE}} - \frac{1}{\zeta^{(0)}} \left(\frac{\partial}{\partial t} + v_j \nabla_j \right) f^{(0)} \,, \\
f^{(2)} &= f^{(2)}_{\mathrm{LE}} - \frac{1}{\zeta^{(0)}} \left(\frac{\partial}{\partial t} + v_j \nabla_j \right) f^{(1)} - \frac{\zeta^{(1)}}{\zeta^{(0)}} \left(f^{(1)} - f^{(1)}_{\mathrm{LE}} \right) , \\
&\;\;\vdots \\
f^{(k)} &= f^{(k)}_{\mathrm{LE}} - \frac{1}{\zeta^{(0)}} \left(\frac{\partial}{\partial t} + v_j \nabla_j \right) f^{(k-1)} - \frac{1}{\zeta^{(0)}} \sum_{l=1}^{k-1} \zeta^{(l)} \left(f^{(k-l)} - f^{(k-l)}_{\mathrm{LE}} \right) , \\
&\;\;\vdots
\end{aligned}
\tag{14}
$$

where now $f^{(0)}$ is zeroth order in gradients of the variables $\xi^{(0)}$ (the local equilibrium approximation), $f^{(1)}$ is zeroth order in gradients of the variables $\xi^{(1)}$ and first order in gradients of the variables $\xi^{(0)}$, etc. Eq. (14) are algebraic and can be solved sequentially. At each stage, five arbitrary functions appear in the solution. These functions satisfy certain partial differential equations that correspond to the well-known Euler, Navier–Stokes, . . . transport equations. So, in contrast to what happens in the BE, from the BGK equation we can evaluate exactly the transport coefficients at each order.

In order to neglect the effects of the initial distribution function, our discussion is limited to the steady state. Thus, in the previous expressions all the partial time derivatives disappear.

3. Burnett hydrodynamic order

According to eqs. (14), at the zeroth order the distribution function corresponds to the local equilibrium function defined from the variables $n^{(0)}$, $\boldsymbol{u}^{(0)}$ and $T^{(0)}$,

$$
f^{(0)} = f^{(0)}_{\mathrm{LE}} = n^{(0)} \left(\frac{m}{2\pi k_{\mathrm{B}} T^{(0)}} \right)^{3/2} \exp\left(- \frac{m}{2 k_{\mathrm{B}} T^{(0)}} (v - \boldsymbol{u}^{(0)})^2 \right) . \tag{15}
$$

This characteristic happens to be a basic difference from the C–E expansion, because in the C–E method the fluid variables appearing in the Maxwellian (zeroth order) are exact and hence no corrections are permitted. By using eqs. (7)–(9), we obtain easily the Euler steady transport equations:

$$\nabla_j(n^{(0)}u_j^{(0)}) = 0\,, \tag{16}$$

$$\rho^{(0)}u_j^{(0)}\nabla_j u_i^{(0)} = -\nabla_i p^{(0)}\,, \tag{17}$$

$$u_j^{(0)}\nabla_j T^{(0)} = -\tfrac{2}{3}T^{(0)}\nabla_k u_k^{(0)}\,, \tag{18}$$

where $\boldsymbol{J}^{(0)} = \boldsymbol{0}$, and we have introduced the density $\rho^{(0)} = mn^{(0)}$ and the pressure $p^{(0)} = n^{(0)}k_B T^{(0)}$ in the zeroth approximation.

Solving sequentially, and after some manipulations, we find that the first-order approximation (Navier–Stokes order) to f is given by

$$\begin{aligned} f^{(1)} = f_{\rm LE}^{(0)}\Big[&\frac{n^{(1)}}{n^{(0)}} + \left(\frac{mV^{(0)\,2}}{2k_B T^{(0)}} - \frac{3}{2}\right)\frac{T^{(1)}}{T^{(0)}} + \frac{m}{k_B T^{(0)}}\,V_j^{(0)}u_j^{(1)} \\ &- \frac{V_j^{(0)}}{\zeta^{(0)}}\left(\frac{mV^{(0)2}}{2k_B T^{(0)}} - \frac{5}{2}\right)\frac{\nabla_j T^{(0)}}{T^{(0)}} \\ &- \frac{m}{k_B T^{(0)}\zeta^{(0)}}\,(V_j^{(0)}V_i^{(0)} - \tfrac{1}{3}V^{(0)2}\delta_{ij})\nabla_j u_i^{(0)}\Big]\,, \end{aligned} \tag{19}$$

where $V_i^{(0)} = v_i - u_i^{(0)}$. In (19) we have introduced the quantities $n^{(1)}$, $\boldsymbol{u}^{(1)}$ and $T^{(1)}$ defined respectively by

$$n^{(1)} = \int \mathrm{d}\boldsymbol{v}\, f^{(1)}\,, \tag{20}$$

$$n^{(0)}u_i^{(1)} = \int \mathrm{d}\boldsymbol{v}\, V_i^{(0)} f^{(1)}\,, \tag{21}$$

$$\tfrac{3}{2}p^{(1)} = \tfrac{3}{2}(n^{(0)}k_B T^{(1)} + n^{(1)}k_B T^{(0)}) = \int \mathrm{d}\boldsymbol{v}\,\frac{m}{2}\,V^{(0)\,2}f^{(1)}\,. \tag{22}$$

From the expression for $f^{(1)}$, we can compute the pressure tensor and the heat flux vector in that order. Taking into account only first order terms, it is straightforward to show that[6])

$$\begin{aligned} P_{ij}^{(1)} &= \int \mathrm{d}\boldsymbol{v}\, mV_i^{(0)}V_j^{(0)} f^{(1)} \\ &= p^{(1)}\delta_{ij} - \eta(\nabla_i u_j^{(0)} + \nabla_j u_i^{(0)} - \tfrac{2}{3}\delta_{ij}\nabla_k u_k^{(0)})\,, \end{aligned} \tag{23}$$

$$\begin{aligned} J_i^{(1)} &= \int \mathrm{d}\boldsymbol{v}\,\frac{m}{2}\,V^{(0)\,2}V_i^{(0)} f^{(1)} - \tfrac{5}{2}p^{(0)}u_i^{(1)} \\ &= -\lambda\nabla_i T^{(0)}\,, \end{aligned} \tag{24}$$

where we have introduced the shear viscosity η and the thermal conductivity λ coefficients given by

$$\eta = \frac{n^{(0)} k_B T^{(0)}}{\zeta^{(0)}} , \tag{25}$$

$$\lambda = \frac{5}{2} \frac{n^{(0)} k_B^2 T^{(0)}}{m\zeta^{(0)}} . \tag{26}$$

The expressions of these coefficients are similar to those given by the Chapman–Enskog theory[5]). Its form, as expected, does not depend on the particular choice for the collision frequency ζ. When (23), (24) are introduced into the transport equations (7)–(9), it is a matter of simple manipulation to find the Navier–Stokes steady equations[6]):

$$\nabla_j (n^{(0)} u_j^{(1)} + n^{(1)} u_j^{(0)}) = 0 , \tag{27}$$

$$\nabla_j [\rho^{(0)} u_j^{(0)} u_i^{(1)} + \rho^{(0)} u_i^{(0)} u_j^{(1)} + \rho^{(1)} u_i^{(0)} u_j^{(0)} + p^{(1)} \delta_{ij} - \eta (\nabla_i u_j^{(0)} + \nabla_j u_i^{(0)} - \tfrac{2}{3} \delta_{ij} \nabla_k u_k^{(0)})] = 0 , \tag{28}$$

$$\tfrac{3}{2} n^{(0)} k_B (u_j^{(0)} \nabla_j T^{(1)} + u_j^{(1)} \nabla_j T^{(0)}) = \nabla_j (\lambda \nabla_j T^{(0)}) - p^{(0)} \nabla_k u_k^{(0)} - n^{(0)} k_B T^{(1)} \nabla_k u_k^{(0)} + \eta (\nabla_i u_j^{(0)} \nabla_j u_i^{(0)} + \nabla_j u_i^{(0)} \nabla_j u_i^{(0)}) - \tfrac{2}{3} \eta [\nabla_k u_k^{(0)}]^2 , \tag{29}$$

where $\rho^{(1)} = mn^{(1)}$. These equations are the same as those obtained by Delale[4]) from the BE. The steady field equations are linear in $n^{(1)}$, $\boldsymbol{u}^{(1)}$ and $T^{(1)}$ and can be solved from the Euler solution.

Nevertheless, as noted in the introduction, we are interested in the second-order (Burnett) transport constitutive relations for the momentum and heat fluxes. Therefore, it will be possible to compare the transport coefficients obtained from the Hilbert expansion with those given in the C–E theory.

At the Burnett order, by utilizing (3)–(5), we introduce the hydrodynamic variables $n^{(2)}$, $\boldsymbol{u}^{(2)}$ and $T^{(2)}$ through the relations

$$n^{(2)} = \int \mathrm{d}\boldsymbol{v}\, f^{(2)} , \tag{30}$$

$$n^{(0)} u_i^{(2)} = \int \mathrm{d}\boldsymbol{v}\, V_i^{(0)} f^{(2)} - n^{(1)} u_i^{(1)} , \tag{31}$$

$$\tfrac{3}{2} p^{(2)} = \tfrac{3}{2} (n^{(0)} k_B T^{(2)} + n^{(2)} k_B T^{(0)} + n^{(1)} k_B T^{(1)})$$

$$= \int \mathrm{d}\boldsymbol{v}\, \frac{m}{2}\, V^{(0)\,2} f^{(2)} - \tfrac{1}{2}\rho^{(0)} u_j^{(1)} u_j^{(1)} . \tag{32}$$

Therefore, considering only second-order terms in (10), (11), we get the following relations for the pressure tensor and heat flux vector:

$$P_{ij}^{(2)} = \int \mathrm{d}\boldsymbol{v}\, m V_i^{(0)} V_j^{(0)} f^{(2)} - \rho^{(0)} u_i^{(1)} u_j^{(1)} , \tag{33}$$

$$J_i^{(2)} = \int \mathrm{d}\boldsymbol{v}\, \frac{m}{2}\, V^{(0)\,2} V_i^{(0)} f^{(2)} - \tfrac{3}{2} p^{(1)} u_i^{(1)} - \tfrac{5}{2} p^{(0)} u_i^{(2)} - u_j^{(1)} P_{ij}^{(1)} . \tag{34}$$

The evaluation of these expressions is straightforward but tedious. In order to obtain (33), (34) in a compact form, we have considered adequate to express the results in terms of well-known quantities as

$$D_{ij}^{(k)} = \tfrac{1}{2}(\nabla_i u_j^{(k)} + \nabla_j u_i^{(k)}) \quad \text{is the strain rate at } k\text{th order} , \tag{35}$$

$$D^{(k)} = D_{ii}^{(k)} = \nabla \cdot \boldsymbol{u}^{(k)} , \tag{36}$$

$$\omega_{ij}^{(k)} = \tfrac{1}{2}(\nabla_j u_i^{(k)} - \nabla_i u_j^{(k)}) \quad \text{is the vorticity at } k\text{th order} . \tag{37}$$

In this way, when the extensive manipulations required to obtain the fluxes are performed, one obtains finally the following expressions:

$$\begin{aligned}
P_{ij}^{(2)} = {} & p^{(2)}\delta_{ij} - 2\eta(p^{(1)} p^{(0)\,-1} - \zeta^{(1)}\zeta^{(0)\,-1})(D_{ij}^{(0)} - \tfrac{1}{3}\delta_{ij} D^{(0)}) \\
& -2\eta(D_{ij}^{(1)} - \tfrac{1}{3}\delta_{ij} D^{(1)}) \\
& + \eta\zeta^{(0)\,-1}\delta_{ij}\Big\{\frac{2}{3}\,\frac{k_{\mathrm{B}}}{mT^{(0)}}\,[(\alpha - 1)(\nabla T^{(0)})^2 - T^{(0)}\nabla^2 T^{(0)}] \\
& + \tfrac{2}{9}(1 - 2\alpha)D^{(0)\,2} - \tfrac{2}{3}\rho^{(0)\,-1} p^{(0)\,-1}(\nabla p^{(0)})^2 + \tfrac{2}{3}\rho^{(0)\,-1}\nabla^2 p^{(0)} \\
& - \tfrac{2}{3}(D_{kl}^{(0)} D_{kl}^{(0)} + \omega_{kl}^{(0)}\omega_{kl}^{(0)}) + \tfrac{2}{3}\rho^{(0)\,-1} T^{(0)\,-1}\nabla p^{(0)} \cdot \nabla T^{(0)}\Big\} \\
& + \tfrac{4}{5}\lambda\zeta^{(0)\,-1}(\nabla_i\nabla_j T^{(0)} - (\alpha - 1)T^{(0)\,-1}\nabla_i T^{(0)}\nabla_j T^{(0)}) \\
& - \eta\zeta^{(0)\,-1}\rho^{(0)\,-1} T^{(0)\,-1}(\nabla_i p^{(0)}\nabla_j T^{(0)} + \nabla_j p^{(0)}\nabla_i T^{(0)}) \\
& - 2\eta\zeta^{(0)\,-1}\rho^{(0)\,-1}(\nabla_i\nabla_j p^{(0)} - p^{(0)\,-1}\nabla_i p^{(0)}\nabla_j p^{(0)}) \\
& + 2\eta\zeta^{(0)\,-1}(D_{ik}^{(0)} D_{kj}^{(0)} + \omega_{ki}^{(0)}\omega_{kj}^{(0)}) - \tfrac{2}{3}(1 - 2\alpha)\eta\zeta^{(0)\,-1} D^{(0)} D_{ij}^{(0)} \\
& - 2\eta\zeta^{(0)\,-1}(D_{ik}^{(0)}\omega_{kj}^{(0)} + D_{jk}^{(0)}\omega_{ki}^{(0)}) ,
\end{aligned} \tag{38}$$

$$J_i^{(2)} = -\lambda(p^{(1)}p^{(0)^{-1}} - \zeta^{(1)}\zeta^{(0)^{-1}})\nabla_i T^{(0)} - \lambda\nabla_i T^{(1)}$$
$$+ \lambda\zeta^{(0)^{-1}}[\tfrac{2}{5}(11-2\alpha)D_{ik}^{(0)}\nabla_k T^{(0)} + 2\omega_{ik}^{(0)}\nabla_k T^{(0)}$$
$$- \tfrac{2}{5}(\tfrac{17}{6} - \tfrac{7}{3}\alpha)D^{(0)}\nabla_i T^{(0)} - \tfrac{8}{15}T^{(0)}\nabla_i D^{(0)} + \tfrac{2}{5}T^{(0)}\nabla^2 u_i^{(0)}]$$
$$- 2\eta\rho^{(0)^{-1}}\zeta^{(0)^{-1}}(D_{ik}^{(0)} - \tfrac{1}{3}\delta_{ik}D^{(0)})\nabla_k p^{(0)} , \qquad (39)$$

where α is the interaction parameter introduced in (6).

As expected, in the Hilbert theory the Burnett equations are formed by terms up to second derivatives in $\boldsymbol{u}^{(0)}$ and $T^{(0)}$ (linear terms in second derivatives and quadratic terms in first derivatives), up to first derivatives in $\boldsymbol{u}^{(1)}$ and $T^{(1)}$, combinations of them plus a linear form of the pressure $p^{(2)}$.

In order to carry out a closer comparison with the results given by Cha and McCoy[5]) from the C–E expansion we have considered the hard sphere model ($\alpha = \frac{1}{2}$). However, the Burnett equations derived from the Hilbert theory cannot be compared directly with the results obtained by means of the C–E expansion since the latter has been defined in terms of the exact (unexpanded) hydrodynamic variables. The transport equations derived from the C–E method must be expressed in terms of the $\xi^{(k)}$ parameters (eq. (13)). Then, in an attempt to carry out a comparison we have expanded in powers of ε the hydrodynamic variables that appear in the results of Cha and McCoy (relations (9), (12) and (13) of ref. 5 corresponding to first and second order) and we have collected the terms that belong to Burnett order according to the re-ordering scheme introduced before. In this way, the transport equations obtained with the C–E method are the same as those obtained by the Hilbert method. Besides, the transport coefficients coincide exactly with the coefficients given by eqs. (38) and (39). Therefore, we may conclude that the transport equations derived by both expansions are equivalent and they are in agreement with the conventional results of the macroscopic theory.

Finally, introducing the corresponding expansions in the balance equations (7)–(9), we get the Burnett steady transport equations:

$$\nabla_j[n^{(2)}u_j^{(0)} + n^{(1)}u_j^{(1)} + n^{(0)}u_j^{(2)}] = 0 , \qquad (40)$$

$$\nabla_j[\rho^{(0)}u_j^{(1)}u_i^{(1)} + \rho^{(0)}u_j^{(2)}u_i^{(0)} + \rho^{(0)}u_j^{(0)}u_i^{(2)} + \rho^{(1)}u_j^{(1)}u_i^{(0)} + \rho^{(1)}u_j^{(0)}u_i^{(1)}$$
$$+ \rho^{(2)}u_j^{(0)}u_i^{(0)} + P_{ij}^{(2)}] = 0 , \qquad (41)$$

$$\tfrac{3}{2}n^{(0)}k_B[u_j^{(0)}\nabla_j T^{(2)} + u_j^{(1)}\nabla_j T^{(1)} + u_j^{(2)}\nabla_j T^{(0)}]$$
$$= -\nabla_j J_j^{(2)} - P_{ij}^{(2)}D_{ij}^{(0)} - p^{(0)}D^{(2)}$$
$$- n^{(0)}k_B T^{(1)}D^{(1)} + n^{(1)}k_B T^{(1)}D^{(0)} - n^{(1)}n^{(0)^{-1}}\nabla_j(\lambda\nabla_j T^{(0)})$$
$$- 2\eta n^{(1)}n^{(0)^{-1}}(D_{ij}^{(0)}D_{ij}^{(0)} - \tfrac{1}{3}D^{(0)^2}) + 2\eta(D_{ij}^{(0)}D_{ij}^{(1)} - \tfrac{1}{3}D^{(0)}D^{(1)}) . \qquad (42)$$

The solution of the steady field equations can be obtained provided that the solution from the above approximations is known.

4. Remarks

We have resolved the BGK equation using the Hilbert perturbative expansion. Constitutive equations for the pressure tensor and heat flux vector have been worked out to the order corresponding to the Burnett approximation. In the first approximation (Navier–Stokes), we have obtained expressions for the viscosity and thermal conductivity coefficients that are similar to those given by the C–E method. The terms that appear in the second-order (Burnett) relations are analogous to results obtained in the C–E method when this latter is expressed in terms of expanded hydrodynamic variables. The calculated transport coefficients are also identical to those obtained from the C–E method in the hard sphere case.

Despite the fact that the C–E method provides an elegant treatment for the derivation of the transport equations for a simple fluid, we think that this alternative treatment by Hilbert's method may be suitable for further applications in some physical problems. Then, recently we have obtained a solution of the BGK equation by means of a perturbative expansion analogous to the Hilbert method in order to study the transport properties in a gas with large shear rate[9]).

References

1) J.R. Dorfman and H. van Beijeren, in: Statistical Mechanics, part B: Time-Dependent Processes, B.J. Berne, ed. (Plenum, New York, 1977), chap. 3.
2) C. Cercignani, Nonequilibrium Phenomena I: The Boltzmann Equation, J.L. Lebowitz and E.W. Montroll, eds. (North-Holland, Amsterdam, 1983), chap. 4.
3) C.F. Delale, J. Stat. Phys. **28** (1982) 589; and references therein; in the case of multiple gas mixtures see: J. Chem. Phys. **83** (1985) 3069.
4) J. Ferziger and H. Kaper, Mathematical Theory of Transport Processes in Gases (North-Holland, Amsterdam, 1972).
5) C.Y. Cha and B.J. McCoy, J. Chem. Phys. **54** (1971) 4369.
6) V. Garzó and J. de la Rubia, Chem. Phys. Lett. **135** (1987) 143, **138** (1987) 283 (erratum).
7) J.A. Robles-Dominguez, B. Silva and L.S. García-Colin, Physica **106A** (1981) 539.
L.S. García-Colin and M. López de Haro, J. Nonequilib. Thermodyn. **7** (1982) 95.
G. Lebon, D. Jou and J. Casas-Vázquez, J. Phys. A **13** (1980) 25.
8) C.K. Wong, J.A. McLennan, M. Lindenfeld and J.W. Dufty, J. Chem. Phys. **63** (1978) 1563.
D. Ronis, Physica **99A** (1979) 403.
J.J. Brey, R. Zwanzig and J.R. Dorfman, Physica **109A** (1981) 425.
9) V. Garzó, Mol. Phys. **61** (1987) 421.